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ON EXPRESSIBLE SETS AND p-ADIC NUMBERS

JAROSLAV HANČL[1], RADHAKRISHNAN NAIR[2],
SIMONA PULCEROVA[3] AND JAN ŠUSTEK[1]

[1] *Department of Mathematics and Institute for Research and Applications of Fuzzy Modelling, University of Ostrava, 30 dubna 22, 701 03 Ostrava 1, Czech Republic* (hancl@osu.cz; jan.sustek@seznam.cz)

[2] *Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool L69 7ZL, UK* (nair@liverpool.ac.uk)

[3] *Department of Mathematical Methods in Economics, Faculty of Economics, VŠB—Technical University of Ostrava, Sokolská třída 33, 701 21 Ostrava 1, Czech Republic* (simona.sobkova@vsb.cz)

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Abstract Continuing earlier studies over the real numbers, we study the expressible set of a sequence $\mathcal{A} = (a_n)_{n\geqslant 1}$ of p-adic numbers, which we define to be the set $E^p_{\mathcal{A}} = \{\sum_{n\geqslant 1} a_n c_n : c_n \in \mathbb{N}\}$. We show that in certain circumstances we can calculate the Haar measure of $E^p_{\mathcal{A}}$ exactly. It turns out that our results extend to sequences of matrices with p-adic entries, so this is the setting in which we work.

Keywords: expressible set; p-adic numbers; Khinchin–Lutz Theorem

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1. Introduction

A long-standing class of problems in number theory is establishing the rationality or otherwise of particular infinite series. Very occasionally, spectacular special results like Apéry's proof of the irrationality of

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

come along [**1**]. General methods are, however, very rare. Motivated by investigations in this vein, Erdős [**7**] defined a sequence of real numbers $\mathcal{A} = (a_n)_{n=1}^{\infty}$ to be *irrational* if the set

$$E_{\mathcal{A}} = \left\{ \sum_{n\geqslant 1} \frac{1}{a_n c_n} : c_n \in \mathbb{N} \right\},$$

which we henceforth refer to as the *expressible set of* $\mathcal{A}$, contains no rational numbers. Sequences $\mathcal{A}$ that are not irrational are called *rational* sequences. In Theorem 2 of [**7**]

Erdős shows that if $\lim_{n\to\infty} a_n^{1/2^n} = \infty$ and $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$, then $\sum_{n\geqslant 1} a_n^{-1}$ is an irrational number. In Theorem 3 of [**7**] Erdős proves that $\mathcal{A}$ with $a_n = 2^{2^n}$ $(n \in \mathbb{N})$ is an irrational sequence. To do this he uses Theorem 2 of [**7**], though this is evidently not an immediate corollary as $\lim_{n\to\infty}(2^{2^n})^{1/2^n} = 2$. In [**8**], on the other hand, it is shown that if, for given $\epsilon > 0$, we have $a_n < 2^{(2-\epsilon)^n}$ $(n \in \mathbb{N})$, then $\mathcal{A}$ is rational and in fact $E_{\mathcal{A}}$ contains an interval. Furthermore, it is shown that if $\mathcal{A}$ is a sequence of non-zero numbers such that $\sum_{n\geqslant 1} 1/a_n$ is conditionally convergent, then $E_{\mathcal{A}} = \mathbb{R}$ [**5**]. At the same time it is possible to give conditions on $\mathcal{A}$ for $E_{\mathcal{A}}$ to have zero measure [**12**], and even conditions for $E_{\mathcal{A}}$ to have zero Hausdorff dimension [**13**]. All this shows that the structure of $E_{\mathcal{A}}$ depends on $\mathcal{A}$ in an interesting and complex fashion. While our ultimate goal may be to decide the rationality or transcendence of individual elements in $E_{\mathcal{A}}$, a more realistic goal, given our current state of knowledge, is to calculate the measure of $E_{\mathcal{A}}$, or say something about its structure.

The purpose of this paper is to extend our study of expressible sets to the setting of the p-adic field for the rational prime p. To make this discussion meaningful and to fix ideas we need some definitions. For $r = p^{v_p}(u/v)$ in $\mathbb{Q}$ with u and v coprime to p and to each other, let $|r|_p = p^{-v_p}$. Then $d_p(r,r') = |r-r'|_p$ defines a metric on $\mathbb{Q}$ and the completion of $\mathbb{Q}$ with respect to the metric d_p is denoted $\mathbb{Q}_p$ and referred to as the set of p-adic numbers. We also use $\mathbb{Z}_p$ to denote $\{x \in \mathbb{Q}_p \colon |x|_p \leqslant 1\}$: the ring of p-adic integers. It is worth keeping in mind that the metric d_p has the ultrametric property: namely, that $d_p(r,r'') \leqslant \max(d_p(r,r'), d_p(r',r''))$. A very basic and easily verified property of $\mathbb{Q}_p$ is that each element α of $\mathbb{Q}_p$ has a 'p-adic expansion' $\alpha = \sum_{n=n_0}^{\infty} k_n p^n$ for $n_0 \in \mathbb{Z}$ with $k_n \in \{0,1,\dots,p-1\}$ $(n \in \mathbb{Z})$ and $k_{n_0} \neq 0$. Furthermore, this p-adic expansion is unique, by which we mean that the pair $(n_0, (k_n)_{n=n_0}^{\infty})$ is uniquely determined by α. From this we note that $|\alpha|_p = p^{-n_0}$ and that the equivalence relation $\alpha \equiv \beta \bmod p^k$, for a non-negative integer k may also be stated as the inequality $d_p(\alpha,\beta) \leqslant p^{-k}$. The main characteristics of $\mathbb{Q}_p$ that distinguish it from $\mathbb{R}$ stem from the ultrametric property. It turns out that $\mathbb{Q}_p$ is a locally compact abelian field and hence comes endowed with a translation-invariant Haar measure which we refer to as λ. A detailed introduction to this subject appears in [**6**]; see also [**2**] for an alternative construction.

One of the consequences of the ultrametric inequality is the fact that in $\mathbb{Q}_p$ a series $\sum_{n\geqslant 1} \beta_n$ converges if and only if $\lim_{n\to\infty} |\beta_n|_p = 0$. This leads to some striking series converging to perfectly well-defined p-adic numbers. For instance, $\phi_p = \sum_{n=1}^{\infty} n!$ is a convergent series in $\mathbb{Q}_p$. This is because, as a standard undergraduate calculation shows, $|n!|_p = p^{-n_p}$, where

$$n_p = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor,$$

which tends to infinity with n. Here, of course, for a real number x we have used $\lfloor x \rfloor$ to denote its integer part. This example, ϕ_p, whose diophantine properties are still a mystery, illustrates the fact that p-adic numbers with series representations are very different from those on the real line. For instance, it is not known whether ϕ_p is rational or not. Showing that $e = \sum_{n=1}^{\infty} 1/n!$ is irrational, which might be considered an analogous question for $\mathbb{R}$, is a routine matter.

One thing that is immediately clear is that the definition of the expressible set of a sequence on $\mathbb{Q}_p$ must be different from that on $\mathbb{R}$. For a sequence of p-adic numbers $\mathcal{A}$, a natural p-adic analogue of the expressible set is $E^p_{\mathcal{A}} = \{\sum_{n=1}^{\infty} a_n c_n \colon c_n \in \mathbb{N}\}$. It turns out that our results also work in the more general context of sequences of matrices with p-adic entries. In this more general context let $\boldsymbol{\mathcal{A}} = \{(\boldsymbol{A}_k)\}_{k=1}^{\infty} = \{(a_{m,n,k})\}_{k=1}^{\infty}$ be the sequence of $M \times N$ matrices of positive integers. The analogues of expressible sets for sequences of matrices over $\mathbb{R}$ have yet to be properly addressed in the literature. For a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ with entries that are p-adic numbers, as is standard, we let $|\alpha|_p = \max(|\alpha_1|_p, \dots, |\alpha_n|_p)$. For convergence with respect to this metric we call

$$\mathbb{E}^p_{\boldsymbol{\mathcal{A}}} = \left\{ \sum_{k=1}^{\infty} (a_{m,n,k} c_{m,n,k}) \colon (c_{m,n,k}) \in \mathbb{N}^{M \times N} \text{ for each } k \in \mathbb{N} \right\}$$

the *expressible set of the sequence* $\mathcal{A}$. Let $B(a, r)$ denote the open ball of centre a and radius r in $\mathbb{Q}_p$.

Theorem 1.1. *We have*

$$\mathbb{E}^p_{\boldsymbol{\mathcal{A}}} = \prod_{m=1}^{M} \prod_{n=1}^{N} B\Big(0, \max_{k \in \mathbb{N}} |a_{m,n,k}|_p\Big)$$

and, in particular,

$$\lambda(\mathbb{E}^p_{\boldsymbol{\mathcal{A}}}) = \prod_{m=1}^{M} \prod_{n=1}^{N} \max_{k \in \mathbb{N}} |a_{m,n,k}|_p.$$

As an illustration, if, for every $m = 1, \dots, M$ and $n = 1, \dots, N$, the number $a_{m,n,1}$ is not divisible by p (that is, if $|a_{m,n,1}|_p = 1$), then $\lambda(\mathbb{E}^p_{\mathcal{A}}) = 1$, which of course means that $\mathbb{E}^p_{\mathcal{A}}$ has full measure.

When working over $\mathbb{R}$ [**12**] for technical reasons, it is necessary to make the restriction that $\mathcal{A} \subseteq \mathbb{N}$. By analogy, when working over $\mathbb{Q}_p$, it is necessary to restrict elements of $\mathcal{A}$ to being members of $\mathbb{S}^*$: a special subset of the space sequences in $\mathbb{Q}_p^{MN}$. We now describe this subset $\mathbb{S}^*$. For a rational number $r = a/b$ with a and b coprime, we use $H(r)$ to denotes its height $\max(|a|, |b|)$. Assume that α is a positive real number. Let $\mathbb{S} = \mathbb{Z}_p^{MN} \cap \mathbb{Q}^{MN}$. Also let $\mathbb{S}^{\mathbb{N}}$ denote the set of all sequences of elements from $\mathbb{S}$. We now use $\mathbb{S}^* = \mathbb{S}^*(\alpha)$ to denote the subset of $\mathbb{S}^{\mathbb{N}}$ consisting of elements $\{\boldsymbol{C}_k\}_{k=1}^{\infty} = \{(c_{m,n,k})\}_{k=1}^{\infty}$ with the property that there exist real numbers β and d with $0 < \beta < 1$ and $d > 0$ such that for each $k \in \mathbb{N}$ we have

$$d \cdot |c_{1,1,k}|_p^{-1} \cdot 2^{(\log_2(|c_{1,1,k}|_p^{-1}))^{\beta}} \geqslant |c_{m,n,k}|_p^{-1} \geqslant d^{-1} \cdot |c_{1,1}|_p^{-1} \cdot 2^{-(\log_2(|c_{1,1,k}|_p^{-1}))^{\beta}} \tag{1.1}$$

and

$$H(c_{m,n,k}) \leqslant d \cdot |c_{m,n,k}|_p^{-\alpha}. \tag{1.2}$$

Condition (1.1) ensures that the entries of the sequence of matrices in an element of $\mathbb{S}^*$ are of the same order and condition (1.2) arises from the special character of diophantine approximation on the p-adic field. The following is our main result; it is a p-adic analogue of Theorem 1 from [**12**].

Theorem 1.2. *Let* $\boldsymbol{\mathcal{A}} = \{\boldsymbol{A}_k\}_{k=1}^{\infty} = \{(a_{m,n,k})\}_{k=1}^{\infty} \in \mathbb{S}^*$ *such that the sequence* $\{|a_{1,1,k}|_p\}_{k=1}^{\infty}$ *is non-increasing. Suppose that*

$$\limsup_{k\to\infty} |a_{1,1,k}|_p^{-1/(\alpha(M+N)+1)^k} = \infty. \tag{1.3}$$

Set

$$\mathbb{S}_{\boldsymbol{\mathcal{A}}} = \left\{ \sum_{k=1}^{\infty} \boldsymbol{B}_k : \boldsymbol{B}_k = (a_{m,n,k} \cdot c_{m,n,k}) \text{ with } \{(c_{m,n,k})\}_{k=1}^{\infty} \in \mathbb{S}^* \right\}.$$

Then the measure $\lambda(\mathbb{S}_{\boldsymbol{\mathcal{A}}}) = 0$.

The underlying idea of the proof of Theorem 1.2 is stability under perturbation. By this we mean that for a suitably chosen sequence $(a_n)_{n=1}^{\infty}$ taken from an additive coset of $\mathbb{Z}$ in $\mathbb{R}$, if the real number $\sum_{n=1}^{\infty} 1/a_n$ has a particular diophantine property—whether that is being transcendental [**9**], being Liouville [**10**] or having a particular irrationality measure [**11**], for instance—then it is likely, for any sequence of natural numbers $(c_n)_{n=1}^{\infty}$, that the sequence $\sum_{n=1}^{\infty} 1/a_n c_n$ will have the same or similar properties. The link between the series $\sum_{n=1}^{\infty} 1/a_n$ and $\sum_{n=1}^{\infty} 1/a_n c_n$ is achieved by diophantine approximation. In [**12**], Khinchin's Theorem on metric diophantine approximation is used [**12**, Lemma 7]. In this paper the link between the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n c_n$ is achieved via the p-adic analogue of Khinchin's Theorem [**14**].

2. Proof of Theorem 1.1

Let m and n be positive integers such that $0 \leqslant m \leqslant M$ and $0 \leqslant n \leqslant N$. Then, for every $K_1 \in \mathbb{N}$, the number a_{m,n,K_1} is divisible by $(\max_{k\in\mathbb{N}} |a_{m,n,k}|_p)^{-1}$ and therefore belongs to $B(0, \max_{k\in\mathbb{N}} |a_{m,n,k}|_p)$. Therefore,

$$\mathbb{E}_{\boldsymbol{\mathcal{A}}}^p \subset \prod_{m=1}^{M} \prod_{n=1}^{N} B\Big(0, \max_{k\in\mathbb{N}} |a_{m,n,k}|_p\Big).$$

Recall from elementary number theory the fact that the least common multiple of two non-zero integers can be expressed as an integer linear combination of the two integers. This yields, for all integers m, n, K_2 and s with $0 \leqslant m \leqslant M$, $0 \leqslant n \leqslant N$, $K_2 > 0$ and $s > 0$, that the set $\{a_{m,n,K_2} |a_{m,n,K_2}|_p c;\ c \in \mathbb{N}\}$ contains elements of all residue classes modulo p^s. It follows that each number $v \in B(0, \max_{k\in\mathbb{N}} |a_{m,n,k}|_p)$ can be expressed as $v = \sum_{k=1}^{\infty} a_{m,n,k} c_{m,n,k}$, where $c_{m,n,k} \in \mathbb{N}$. From this we obtain that

$$\prod_{m=1}^{M} \prod_{n=1}^{N} B\Big(0, \max_{k\in\mathbb{N}} |a_{m,n,k}|_p\Big) \subset \mathbb{E}_{\boldsymbol{\mathcal{A}}}^p$$

and the proof of Theorem 1.1 is complete.

3. An auxiliary result

We deduce Theorem 1.2 from the following more general auxiliary result, which we prove in this section.

Theorem 3.1. *Assume that $\mathbb{Y} \subset \mathbb{Z}_p^{MN}$ is such that, for every $\boldsymbol{x} \in \mathbb{Y}$, there exists an infinite sequence of $M \times N$ matrices $\boldsymbol{\mathcal{A}} = \{\boldsymbol{A}_k\}_{k=1}^{\infty} = \{(a_{m,n,k})\}_{k=1}^{\infty} \in \mathbb{S}^*$ such that $\boldsymbol{x} = \sum_{k=1}^{\infty} \boldsymbol{A}_k$ with convergence in the metric $|\cdot|_p$. Suppose that the sequence $\{|a_{1,1,k}|_p\}_{k=1}^{\infty}$ is non-increasing and that*

$$\limsup_{k\to\infty} |a_{1,1,k}|_p^{-1/(\alpha(M+N)+1)^k} = \infty. \tag{3.1}$$

Then the measure $\lambda(\mathbb{Y}) = 0$.

For the proof of our auxiliary theorem we need the following result from metric number theory on the p-adic numbers. This is a corollary of the p-adic version of a theorem of Khinchin, a proof of which can be found in [**3**] or [**14**] (see also [**4**, Theorem 6.3, p. 127]). To keep our exposition uncluttered, we postpone to § 5 the derivation of Theorem 3.2 from the p-adic version of Khinchin's Theorem.

Theorem 3.2. *Let M and N be positive integers and assume that $\boldsymbol{x} = (a_{m,n}) \in \mathbb{Z}_p^{MN}$, $\boldsymbol{q} = (q_1, \dots, q_N) \in \mathbb{Z}^N$ and $\boldsymbol{r} = (r_1, \dots, r_M) \in \mathbb{Z}^M$. Suppose that*

$$|\boldsymbol{q}\cdot\boldsymbol{x} - \boldsymbol{r}|_p = \max_{1\leqslant m\leqslant M}\biggl(\biggl|-r_m + \sum_{n=1}^{N} a_{m,n}q_n\biggr|_p\biggr)$$

and that

$$|(\boldsymbol{q},\boldsymbol{r})| = \max\Bigl(\max_{1\leqslant n\leqslant N}|q_n|, \max_{1\leqslant m\leqslant M}|r_m|\Bigr).$$

Set $\tau = (M+N)/M$. Then, for almost all numbers $\boldsymbol{x}$, the inequality

$$|(\boldsymbol{q},\boldsymbol{r})|^{\tau}\cdot\log^3|(\boldsymbol{q},\boldsymbol{r})| < \frac{1}{|\boldsymbol{q}\cdot\boldsymbol{x} - \boldsymbol{r}|_p}$$

has finitely many solutions in unknowns $\boldsymbol{q}$ and $\boldsymbol{r}$.

To complete the proof of Theorem 3.1 we establish that, for $\boldsymbol{x} \in \mathbb{Y}$,

$$|(\boldsymbol{q},\boldsymbol{r})|^{\tau}\cdot\log^3|(\boldsymbol{q},\boldsymbol{r})| < \frac{1}{|\boldsymbol{q}\cdot\boldsymbol{x} - \boldsymbol{r}|_p} \tag{3.2}$$

for infinitely many $(\boldsymbol{q},\boldsymbol{r}) \in \mathbb{Z}^N \times \mathbb{Z}^M$.

Let $\operatorname{den}(y)$ be the denominator of the rational number y in reduced form. Let K be a large positive integer and set $\boldsymbol{q} = (q_1, \dots, q_N)$, where $q_n = \prod_{k=1}^{K}\prod_{m=1}^{M}\operatorname{den}(a_{m,n,k})$ for $n = 1, \dots, N$. Also set $\boldsymbol{r} = \boldsymbol{q}\cdot\sum_{k=1}^{K}\boldsymbol{A}_k$. Condition (1.1) and the fact that the sequence

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$\{|a_{1,1,k}|_p\}_{k=1}^{\infty}$ is non-increasing imply that

$$\begin{aligned}|\boldsymbol{q}\cdot\boldsymbol{x}-\boldsymbol{r}|_p &= \left|\boldsymbol{q}\cdot\sum_{k=K+1}^{\infty}\boldsymbol{A}_k\right|_p \\ &\leqslant \max_{\substack{m=1,\dots,M,\\ n=1,\dots,N,\\ k\in\{K+1,K+2,\dots\}}} |q_n a_{m,n,k}|_p \\ &= \max_{\substack{m=1,\dots,M,\\ n=1,\dots,N,\\ k\in\{K+1,K+2,\dots\}}} |a_{m,n,k}|_p \\ &\leqslant d\cdot|a_{1,1,K+1}|_p\cdot 2^{(\log_2|a_{1,1,K+1}|_p^{-1})^{\beta}}.\end{aligned}$$

This implies that

$$\frac{1}{|\boldsymbol{q}\cdot\boldsymbol{x}-\boldsymbol{r}|_p}\geqslant d^{-1}\cdot|a_{1,1,K+1}|_p^{-1}\cdot 2^{-(\log_2|a_{1,1,K+1}|_p^{-1})^{\beta}}. \tag{3.3}$$

From the definitions of $\boldsymbol{q}$ and $\boldsymbol{r}$, and from the inequalities (1.1) and (1.2), we obtain that there exists a positive real number W that does not depend on K and such that

$$\begin{aligned}|(\boldsymbol{q},\boldsymbol{r})| &\leqslant N\cdot K\cdot\max_{n=1,\dots,N}\prod_{k=1}^{K}\prod_{m=1}^{M}H(a_{m,n,k}) \\ &\leqslant W^{M\cdot K}\cdot N\cdot K\cdot\max_{n=1,\dots,N}\prod_{k=1}^{K}\prod_{m=1}^{M}|a_{m,n,k}|_p^{-\alpha} \\ &\leqslant W^{M\cdot K}\cdot N\cdot K\cdot\left(\prod_{k=1}^{K}\prod_{m=1}^{M}d\cdot|a_{1,1,k}|_p^{-1}\cdot 2^{(\log_2(|a_{1,1,k}|_p^{-1}))^{\beta}}\right)^{\alpha} \\ &= (d^{\alpha}\cdot W)^{M\cdot K}\cdot N\cdot K\cdot\left(\prod_{k=1}^{K}|a_{1,1,k}|_p^{-1}\cdot 2^{(\log_2(|a_{1,1,k}|_p^{-1}))^{\beta}}\right)^{M\alpha} \\ &= (d^{\alpha}\cdot W)^{M\cdot K}\cdot N\cdot K\cdot\left(\prod_{k=1}^{K}|a_{1,1,k}|_p^{-1}\right)^{M\alpha}\cdot 2^{M\alpha\sum_{k=1}^{K}(\log_2(|a_{1,1,k}|_p^{-1}))^{\beta}}.\end{aligned}$$

This implies that, for all sufficiently large $|(\boldsymbol{q},\boldsymbol{r})|$,

$$\begin{aligned}&|(\boldsymbol{q},\boldsymbol{r})|^{\tau}\cdot\log^3|(\boldsymbol{q},\boldsymbol{r})| \\ &\quad\leqslant (d^{\alpha}\cdot W)^{M\cdot K\cdot\tau}\cdot(N\cdot K)^{\tau}\cdot\left(\prod_{k=1}^{K}|a_{1,1,k}|_p^{-1}\right)^{M\alpha\tau}\cdot 2^{M\alpha\tau\sum_{k=1}^{K}(\log_2(|a_{1,1,k}|_p^{-1}))^{\beta}} \\ &\qquad\times\log^3\left((d^{\alpha}\cdot W)^{M\cdot K\cdot}N\cdot K\cdot\left(\prod_{k=1}^{K}|a_{1,1,k}|_p^{-1}\right)^{M\alpha}\cdot 2^{M\alpha\sum_{k=1}^{K}(\log_2(|a_{1,1,k}|_p^{-1}))^{\beta}}\right) \\ &\quad\leqslant (d^{\alpha}\cdot W)^{M\cdot K\cdot\tau}\cdot(N\cdot K)^{\tau}\cdot\left(\prod_{k=1}^{K}|a_{1,1,k}|_p^{-1}\right)^{M\alpha\tau}\cdot 2^{2M\alpha\tau\sum_{k=1}^{K}(\log_2(|a_{1,1,k}|_p^{-1}))^{\beta}}.\end{aligned} \tag{3.4}$$

Set $R = M\alpha\tau + 1$. We now consider two cases.

Case 1. First assume that there exists $\varepsilon > 0$ such that

$$\limsup_{k\to\infty} |a_{1,1,k}|_p^{-1/(R+\varepsilon)^k} = \infty. \tag{3.5}$$

From this and the fact that $\{|a_{1,1,k}|_p\}_{k=1}^{\infty}$ is non-increasing we obtain, for infinitely many K_3, that

$$|a_{1,1,K_3+1}|_p^{-1/(R+\varepsilon)^{K_3+1}} > \left(1+\frac{1}{K_3^2}\right)\cdot\left(\max_{k=1,\dots,K_3} |a_{1,1,k}|_p^{-1/(R+\varepsilon)^k}\right). \tag{3.6}$$

This is because otherwise there would exist k_0 such that, for every $k_1 \in \mathbb{N}$ with $k_1 > k_0$,

$$|a_{1,1,k_1+1}|_p^{-1/(R+\varepsilon)^{k_1+1}} \leqslant \left(1+\frac{1}{k_1^2}\right)\cdot\left(\max_{l=1,\dots,k_1} |a_{1,1,l}|_p^{-1/(R+\varepsilon)^l}\right).$$

This would mean that

$$\begin{aligned}
&|a_{1,1,k_1+1}|_p^{-1/(R+\varepsilon)^{k_1+1}} \\
&\quad\leqslant \left(1+\frac{1}{k_1^2}\right)\cdot\left(\max_{l=1,\dots,k_1} |a_{1,1,l}|_p^{-1/(R+\varepsilon)^l}\right) \\
&\quad\leqslant \left(1+\frac{1}{k_1^2}\right)\cdot\left(1+\frac{1}{(k_1-1)^2}\right)\cdot\left(\max_{l=1,\dots,k_1-1} |a_{1,1,l}|_p^{-1/(R+\varepsilon)^l}\right) \\
&\quad\vdots \\
&\quad\leqslant \left(1+\frac{1}{k_1^2}\right)\cdot\left(1+\frac{1}{(k_1-1)^2}\right)\cdots\left(1+\frac{1}{k_0^2}\right)\cdot\left(\max_{l=1,\dots,k_0} |a_{1,1,l}|_p^{-1/(R+\varepsilon)^l}\right) \\
&\quad\leqslant \left(\prod_{l=k_0}^{\infty}\left(1+\frac{1}{l^2}\right)\right)\cdot\left(\max_{l=1,\dots,k_0} |a_{1,1,l}|_p^{-1/(R+\varepsilon)^l}\right) \\
&\quad= \text{const}.
\end{aligned}$$

This is in contradiction to (3.1). Thus inequality (3.6) holds for infinitely many K_3. From (3.6) we obtain that

$$\begin{aligned}
|a_{1,1,K_3+1}|_p^{-1} &> \left(1+\frac{1}{K_3^2}\right)^{(R+\varepsilon)^{K_3+1}}\cdot\left(\max_{k=1,\dots,K_3} |a_{1,1,k}|_p^{-1/(R+\varepsilon)^k}\right)^{(R+\varepsilon)^{K_3+1}} \\
&> \left(1+\frac{1}{K_3^2}\right)^{(R+\varepsilon)^{K_3+1}}\cdot\left(\max_{k=1,\dots,K_3} |a_{1,1,k}|_p^{-1/(R+\varepsilon)^k}\right)^{(R+\varepsilon)^{K_3+1}-(R+\varepsilon)} \\
&= \left(1+\frac{1}{K_3^2}\right)^{(R+\varepsilon)^{K_3+1}}\left(\max_{k=1,\dots,K_3} |a_{1,1,k}|_p^{-1/(R+\varepsilon)^k}\right)^{T(\varepsilon,R,K_3)},
\end{aligned}$$

where

$$T(\varepsilon, R, K) = (R+\varepsilon-1)((R+\varepsilon)^K + (R+\varepsilon)^{K-1} + \cdots + (R+\varepsilon))$$

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and this is greater than or equal to

$$\left(1+\frac{1}{K_3^2}\right)^{(R+\varepsilon)^{K_3+1}} \cdot \prod_{k=1}^{K_3} |a_{1,1,k}|_p^{-(R+\varepsilon-1)}. \tag{3.7}$$

Inequalities (3.3), (3.4) and (3.7) yield, for infinitely many large K_4, that

$$\begin{aligned}
&\frac{1}{|\boldsymbol{q}\cdot\boldsymbol{x}-\boldsymbol{r}|_p}\\
&\quad\geqslant d^{-1}\cdot|a_{1,1,K_4+1}|_p^{-1}\cdot 2^{-(\log_2|a_{1,1,K_4+1}|_p^{-1})^{\beta}}\\
&\quad= |a_{1,1,K_4+1}|_p^{-(R+\varepsilon/2-1)/(R+\varepsilon-1)}\cdot d^{-1}\cdot|a_{1,1,K_4+1}|_p^{-(\varepsilon/2)/(R+\varepsilon-1)}\cdot 2^{-(\log_2|a_{1,1,K_4+1}|_p^{-1})^{\beta}}\\
&\quad\geqslant |a_{1,1,K_4+1}|_p^{-(R+\varepsilon/2-1)/(R+\varepsilon-1)}\\
&\quad\geqslant \left(1+\frac{1}{K_4^2}\right)^{((R+\varepsilon/2-1)/(R+\varepsilon-1))(R+\varepsilon)^{K_4+1}}\cdot\prod_{k=1}^{K_4}|a_{1,1,k}|_p^{-(R+\varepsilon/2-1)}\\
&\quad\geqslant (d^{\alpha}\cdot W)^{M\cdot K_4\cdot\tau}\cdot(N\cdot K_4)^{\tau}\cdot\left(\prod_{k=1}^{K_4}|a_{1,1,k}|_p^{-1}\right)^{(R-1)}\prod_{k=1}^{K_4}|a_{1,1,k}|_p^{-\varepsilon/2}\\
&\quad\geqslant (d^{\alpha}\cdot W)^{M\cdot K_4\cdot\tau}\cdot(N\cdot K_4)^{\tau}\cdot\left(\prod_{k=1}^{K_4}|a_{1,1,k}|_p^{-1}\right)^{(R-1)}\prod_{k=1}^{K_4}2^{2M\alpha\tau(\log_2(|a_{1,1,k}|_p^{-1}))^{\beta}}\\
&\quad= (d^{\alpha}\cdot W)^{M\cdot K_4\cdot\tau}\cdot(N\cdot K_4)^{\tau}\cdot\left(\prod_{k=1}^{K_4}|a_{1,1,k}|_p^{-1}\right)^{M\alpha\tau}\cdot 2^{2M\alpha\tau\sum_{k=1}^{K_4}(\log_2(|a_{1,1,k}|_p^{-1}))^{\beta}}\\
&\quad\geqslant |(\boldsymbol{q},\boldsymbol{r})|^{\tau}\log^3|(\boldsymbol{q},\boldsymbol{r})|
\end{aligned}$$

and (3.2) follows.

Case 2. Now assume that, for every $\delta > 0$,

$$\limsup_{k\to\infty}|a_{1,1,k}|_p^{-1/(R+\delta)^k} < \infty.$$

There is then an appropriate choice of $\delta = \xi$ (say) such that

$$\limsup_{k\to\infty}|a_{1,1,k}|_p^{-1/(R+\xi)^k} = 1. \tag{3.8}$$

From (3.8) we obtain that, for every sufficiently large k_2,

$$|a_{1,1,k_2}|_p^{-1} < 2^{(R+\xi)^{k_2}}. \tag{3.9}$$

This implies that there exists a constant $C = C(\xi)$ such that, for every $k_3 \in \mathbb{N}$,

$$\begin{aligned}
2^{2M\alpha\tau\sum_{l=1}^{k_3}(\log_2(|a_{1,1,l}|_p^{-1}))^{\beta}} &< C\cdot 2^{2M\alpha\tau\sum_{l=1}^{k_3}(\log_2 2^{(R+\xi)^l})^{\beta}}\\
&\leqslant C\cdot 2^{2M\alpha\tau((R+\xi)^{\beta(k_3+1)}/(R+\xi)^{\beta}-1)}.
\end{aligned} \tag{3.10}$$

From (3.1) and the fact that $\{|a_{1,1,k}|_p\}_{k=1}^{\infty}$ is non-increasing we obtain, for infinitely many K_5, that

$$|a_{1,1,K_5+1}|_p^{-1} > \left(1+\frac{1}{K_5^2}\right)^{R^{K_5+1}} \cdot \prod_{k=1}^{K_5} |a_{1,1,k}|_p^{-(R-1)}, \tag{3.11}$$

where we have used the same procedure as in the first case but used R instead of $R+\varepsilon$. Inequalities (3.3), (3.4), (3.9)–(3.11) and the fact that ξ is sufficiently small yield, for infinitely many sufficiently large K_6, that

$$\begin{aligned}
&\frac{1}{|\boldsymbol{q}\cdot\boldsymbol{x}-\boldsymbol{r}|_p}\\
&\quad\geqslant d^{-1}\cdot|a_{1,1,K_6+1}|_p^{-1}\cdot 2^{-(\log_2|a_{1,1,K_6+1}|_p^{-1})^{\beta}}\\
&\quad> d^{-1}\cdot 2^{-(\log_2|a_{1,1,K_6+1}|_p^{-1})^{\beta}}\cdot\left(1+\frac{1}{K_6^2}\right)^{R^{K_6+1}}\cdot\prod_{k=1}^{K_6}|a_{1,1,k}|_p^{-(R-1)}\\
&\quad> d^{-1}\cdot 2^{-(\log_2(2^{(R+\xi)^{K_6+1}}))^{\beta}}\cdot\left(1+\frac{1}{K_6^2}\right)^{R^{K_6+1}}\cdot\prod_{k=1}^{K_6}|a_{1,1,k}|_p^{-(R-1)}\\
&\quad> \left(1+\frac{1}{K_6^2}\right)^{R^{K_6+1}/2}\cdot\prod_{k=1}^{K_6}|a_{1,1,k}|_p^{-(R-1)}\\
&\quad> (d^{\alpha}\cdot W)^{M\cdot K_6\cdot\tau}\cdot(N\cdot K_6)^{\tau}\cdot\left(\prod_{k=1}^{K_6}|a_{1,1,k}|_p^{-1}\right)^{M\alpha\tau}\cdot 2^{2M\alpha\tau\sum_{k=1}^{K_6}(\log_2(|a_{1,1,k}|_p^{-1}))^{\beta}}\\
&\quad\geqslant |(\boldsymbol{q},\boldsymbol{r})|^{\tau}\cdot\log^3|(\boldsymbol{q},\boldsymbol{r})|,
\end{aligned}$$

and (3.2) follows.

4. Proof of Theorem 1.2

In this section we prove that $\mathbb{S}_{\boldsymbol{\mathcal{A}}}$ is null by showing that $\mathbb{S}_{\boldsymbol{\mathcal{A}}} \subset \mathbb{Y}$ and then using Theorem 3.1. Let

$$y = \sum_{k=1}^{\infty} \boldsymbol{B}_k = \sum_{k=1}^{\infty}(a_{m,n,k}\cdot c_{m,n,k}) \in \mathbb{S}_{\boldsymbol{\mathcal{A}}}.$$

To prove Theorem 1.2 we have to prove that y satisfies the conditions of Theorem 3.1.

Let $\phi:\mathbb{N}\to\mathbb{N}$ be a bijection such that, for $e_{m,n,k} = a_{m,n,\phi(k)}c_{m,n,\phi(k)}$, the sequence $\{e_{1,1,k}\}_{k=1}^{\infty}$ is non-decreasing. Set $\gamma = \frac{1}{2}(1+\beta)$. Then the number of solutions in non-negative integers a and b of the inequality

$$(\log_2 p^a)^{\beta} + (\log_2 p^b)^{\beta} \geqslant (\log_2 p^a + \log_2 p^b)^{\gamma}$$

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is finite. From this fact and (1.1) we obtain that there exists a positive real number D such that

$$
\begin{aligned}
D \cdot |e_{1,1,k}|_p^{-1} \cdot 2^{(\log_2(|e_{1,1,k}|_p^{-1}))^{\gamma}} \\
&\geqslant d^2 |a_{1,1,\phi(k)}|_p^{-1} \cdot 2^{(\log_2(|a_{1,1,\phi(k)}|_p^{-1}))^{\beta}} |c_{1,1,\phi(k)}|_p^{-1} \cdot 2^{(\log_2(|c_{1,1,\phi(k)}|_p^{-1}))^{\beta}} \\
&\geqslant |c_{m,n,\phi(k)} \cdot a_{m,n,\phi(k)}|_p^{-1} \\
&= |e_{m,n,\phi(k)}|_p^{-1} \\
&\geqslant d^{-2} \cdot |c_{1,1,\phi(k)}|_p^{-1} \cdot 2^{-(\log_2(|c_{1,1,\phi(k)}|_p^{-1}))^{\beta}} \cdot |a_{1,1,\phi(k)}|_p^{-1} \cdot 2^{-(\log_2(|a_{1,1,\phi(k)}|_p^{-1}))^{\beta}} \\
&\geqslant D^{-1} \cdot |e_{1,1,k}|_p^{-1} \cdot 2^{-(\log_2(|e_{1,1,k}|_p^{-1}))^{\gamma}}
\end{aligned}
$$

and inequality (1.1) follows when instead of β and $a_{m,n,k}$ we have γ and $e_{m,n,k}$ respectively. From (1.2) we obtain that

$$
\begin{aligned}
H(e_{m,n,k}) &= H(a_{m,n,\phi(k)} \cdot c_{m,n,\phi(k)}) \\
&\leqslant H(a_{m,n,\phi(k)}) \cdot H(c_{m,n,\phi(k)}) \\
&\leqslant d^2 \cdot |a_{m,n,\phi(k)}|_p^{-\alpha} \cdot |c_{m,n,\phi(k)}|_p^{-\alpha} \\
&= d^2 \cdot |e_{m,n,\phi(k)}|_p^{-\alpha}
\end{aligned}
$$

and so (1.2) follows when instead of $a_{m,n,k}$ we have $e_{m,n,k}$. The fact that the sequences $\{|a_{1,1,k}|_p^{-1}\}_{k=1}^{\infty}$ and $\{|e_{1,1,k}|_p^{-1}\}_{k=1}^{\infty}$ are non-decreasing and the definition of $e_{1,1,k}$ imply that $|e_{1,1,k}|_p^{-1}$ is greater than or equal to the first $k-1$ terms of $\{|e_{1,1,k}|_p^{-1}\}_{k=1}^{\infty}$. Hence $|e_{1,1,k}|_p^{-1} \geqslant |a_{1,1,k}|_p^{-1}$ and (3.1) follows.

5. Proof of Theorem 3.2

In this section we deduce Theorem 3.2 from the p-adic analogue of the convergence part of a well-known theorem of Khinchin's on metric Diophantine approximation. See Theorem 15 in [**3**] for an up-to-date version of this theorem or see [**14**, p. 93] if you want the original reference in which a result of this type was first proved.

For $\boldsymbol{u} = (u_1, \dots, u_n)$ in $\mathbb{Z}^n$ let $H(\boldsymbol{u}) = \max_{1 \leqslant i \leqslant n} |u_i|$. To $\boldsymbol{v} = (v_1, \dots, v_n)$ in $\mathbb{Q}_p^n$ and $\boldsymbol{a} = (a_{ij})$ in $\mathbb{Q}_p^{st}$, where $n = s + t$ and $1 \leqslant s < n$, we associate the affine form

$$
L(\boldsymbol{v}, \boldsymbol{a}) = \max_{1 \leqslant i \leqslant s} \left| v_j + \sum_{i=1}^{t} a_{ij} v_{s+i} \right|_p .
$$

The p-adic analogue of the convergence part of Khinchin's result is the following theorem.

Theorem 5.1. *Assume that the real function $f(h)$ is positive for all natural numbers h, and assume that $f(h)$ decreases to 0 as h tends to ∞. Also, for each pair of natural numbers (s, n) with $1 \leqslant s < n$ assume that*

$$
\sum_{h=1}^{\infty} h^{n-1} f^s(h) < \infty.
$$

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Then, for almost all $\boldsymbol{a} \in \mathbb{Q}_p^{st}$*, the diophantine inequality*

$$L(\boldsymbol{u}, \boldsymbol{a}) \leqslant f(H(\boldsymbol{u})) \tag{5.1}$$

admits only finitely many solutions $\boldsymbol{u} \in \mathbb{Z}^n$*.*

To prove Theorem 1.2 we use only Theorem 3.2, which is a consequence of Theorem 5.1 (a) and is derived as follows. First we make a series of choices. Set $f(h) = 1/h^\tau \log^3 h$ ($h \in \mathbb{N}$), $\boldsymbol{u} = (-\boldsymbol{r}, \boldsymbol{q}) = (-r_1, -r_2, \dots, -r_M, q_1, q_2, \dots, q_N)$, $\boldsymbol{x} = \boldsymbol{a} = (a_{m,n})$, $n = M + N$, $\tau = (M + N)/M$, $t = N$ and $s = M$. Then $L(\boldsymbol{u}, \boldsymbol{a}) = |\boldsymbol{q} \cdot \boldsymbol{x} - \boldsymbol{r}|_p$ and $H(\boldsymbol{u}) = |(-\boldsymbol{r}, \boldsymbol{q})|$. Now we have

$$\sum_{h=1}^{\infty} h^{n-1} f^s(h) = \sum_{h=1}^{\infty} \frac{1}{h \log^{3M} h} < \infty.$$

Theorem 5.1 (a) tells us that for almost all $\boldsymbol{x} = (a_{m,n})$ the inequality

$$|\boldsymbol{q} \cdot \boldsymbol{x} - \boldsymbol{r}|_p < \frac{1}{|(\boldsymbol{q}, \boldsymbol{r})|^\tau \log^3 |(\boldsymbol{q}, \boldsymbol{r})|}$$

has only finitely many solutions in unknown pairs $\boldsymbol{u} = (\boldsymbol{r}, \boldsymbol{q})$. This is Theorem 3.2, as required.

Note that our choice $\tau = (M + N)/M$ is the critical exponent. By this we mean that if $\tau < (M + N)/M$ in our choice of $f(h)$, the corresponding diophantine inequality has infinitely many solutions.

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