ANOTHER CHARACTERISTIC CONJUGACY CLASS OF SUBGROUPS OF FINITE SOLUBLE GROUPS

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To Bernhard Hermann Neumann on his 60th birthday

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1. Introduction

Let $\mathfrak{F}$ be a class of finite soluble groups with the properties: (1) $\mathfrak{F}$ is a Fitting class (i.e. normal subgroup closed and normal product closed) and (2) if $N \leqq H \leqq G \in \mathfrak{F}$, $N \lhd G$ and H/N is a p-group for some prime p, then $H \in \mathfrak{F}$. Then $\mathfrak{F}$ is called a *Fischer class*. In any finite soluble group G, there exists a unique conjugacy class of maximal $\mathfrak{F}$-subgroups V called the $\mathfrak{F}$-injectors which have the property that for every $N \lhd\lhd G$, $N \cap V$ is a maximal $\mathfrak{F}$-subgroup of N [3]. By Lemma 1(4) [7] an $\mathfrak{F}$-injector V of G covers or avoids a chief factor of G. As in [7] we will call a chief factor $\mathfrak{F}$-covered or $\mathfrak{F}$-avoided according as V covers or avoids it and $\mathfrak{F}$-complemented if it is complemented and each of its complements contains some $\mathfrak{F}$-injector. Furthermore we will call a chief factor *partially $\mathfrak{F}$-complemented* if it is complemented and at least one of its complements contains some $\mathfrak{F}$-injector of G.

For a finite soluble group G, W. Gaschütz [4] has constructed a characteristic conjugacy class of subgroups of G called the Prefrattini subgroups, which avoid all complemented chief factors and cover the rest. Also, given a formation f locally defined by $\{f(p)\}$ (see [5]), T. Hawkes [8] has constructed, using Sylow systems [6], another characteristic conjugacy class of subgroups of G which avoid all complemented f-eccentric [1] chief factors and cover the rest. If $f(p) = \phi$ for each prime p then the latter reduce to Prefrattini subgroups. In this note we show:

THEOREM. *Let G be a finite soluble group. Then there exists a characteristic class of conjugate subgroups of G which avoid all partially $\mathfrak{F}$-complemented chief factors and cover the rest.*

All groups are assumed to be finite and soluble.

2. Partially $\mathfrak{F}$-complemented chief factors

In this section we will give simple characterizations of partially $\mathfrak{F}$-complemented chief factors of a group G, which we shall need in the next section. We begin with two examples. That a chief factor of G may be partially $\mathfrak{F}$-complemented without being $\mathfrak{F}$-complemented is shown by the following example.

EXAMPLE. Consider the dihedral group

$$D_{20} \cong C_2 \times D_{10} = \langle x, y, z | x^2 = y^5 = z^2 = 1, z^{-1}yz = y^{-1}, xy = yx, xz = zx \rangle,$$

and take $\mathfrak{F}$ to be the Fischer class $\mathfrak{N}$ of all nilpotent groups. Clearly $\langle x, y \rangle$ is the $\mathfrak{N}$-injector of D_{20}. The chief factor $\langle y, z \rangle / \langle y \rangle$ is complemented by $\langle x, y \rangle$ as well as $\langle y, xz \rangle$; however only one of these contains the $\mathfrak{N}$-injector of D_{20}.

Every partially $\mathfrak{F}$-complemented chief factor of G is clearly $\mathfrak{F}$-avoided. But every $\mathfrak{F}$-avoided complemented chief factor may not be partially $\mathfrak{F}$-complemented as the following example shows.

EXAMPLE. Let $H = \langle x \rangle \times \langle y \rangle$ be the direct product of a 2-cycle by a 4-cycle respectively. No complement of $\langle x \rangle$ in H contains the 2-cycle $\langle xy^2 \rangle$, since the only 4-cycles of H complementing $\langle x \rangle$ are $\langle xy \rangle = \langle xy^3 \rangle$ and $\langle y \rangle$, and none of these contains xy^2.

Now let G be the semi-direct product of a 5-cycle $\langle z \rangle$ by H, with the action of x and y on z given by $x^{-1}zx = z^{-1}$ and $y^{-1}zy = z^2$. Clearly xy^2 acts trivially on z. Also $\langle z \rangle \times \langle xy^2 \rangle$ is the $\mathfrak{N}$-injector of G. Now the chief factor $\langle z, x \rangle / \langle z \rangle$ is $\mathfrak{N}$-avoided and complemented in G but it is certainly not partially $\mathfrak{N}$-complemented.

Let H/K be a partially $\mathfrak{F}$-complemented chief factor of G and M a complement containing an $\mathfrak{F}$-injector of G, then we will say that H/K is partially $\mathfrak{F}$-complemented by M.

PROPOSITION 2.1. *Let V be an $\mathfrak{F}$-injector of G and P a Sylow p-subgroup of V. A p-chief factor of G is partially $\mathfrak{F}$-complemented by M iff P^G lies in M.*

REMARK. By Corollary to Lemma 3 [7], P is a Sylow p-subgroup of P^G. Hence it follows that a p-chief factor of G is $\mathfrak{F}$-avoided iff it is avoided by P^G. Henceforth we will be using these facts without further mention. Also V will always denote an $\mathfrak{F}$-injector of G and P a Sylow p-subgroup of V.

PROOF. Let H/K be a p-chief factor of G partially $\mathfrak{F}$-complemented by M. Let $D = \mathrm{Core}_G M$ and A/D the unique minimal normal subgroup of G/D. By Theorem 3.1 [4], $A = C_G(H/K)$. Since H/K is $\mathfrak{F}$-avoided, $[H, P^G] \leqq H \cap P^G = K \cap P^G \leqq K$. Thus $P^G \leqq A$. But A/D is also $\mathfrak{F}$-avoided since it is partially $\mathfrak{F}$-complemented by M. Hence $P^G = P^G \cap A = P^G \cap D \leqq D \leqq M$.

Conversely, let H/K be a complemented p-chief factor of G and M a complement containing P^G. If A and D are as before, then A/D is $\mathfrak{F}$-avoided as P^G avoids it. Since A/D is self-centralizing, by Lemma 4 [7] M contains an $\mathfrak{F}$-injector of G. Hence H/K is partially $\mathfrak{F}$-complemented by M.

COROLLARY 2.2. *G has no partially $\mathfrak{F}$-complemented p-chief factors iff all of its p-chief factors are $\mathfrak{F}$-covered.*

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PROOF. Assume first of all that G has no partially $\mathfrak{F}$-complemented p-chief factors. Then we claim that P is a Sylow p-subgroup of G. For otherwise $P^G S^p < G$, where S^p is a Sylow p-complement of G. Let M be a maximal subgroup of G containing $P^G S^p$. Let A and D be as in Proposition 2.1. Then by the latter, A/D is partially $\mathfrak{F}$-complemented by M, a contradiction. Thus P is a Sylow p-subgroup of G which means all p-chief factors of G are $\mathfrak{F}$-covered. The converse is trivial, since a partially $\mathfrak{F}$-complemented p-chief factor is necessarily $\mathfrak{F}$-avoided.

COROLLARY 2.3. *Every complemented p-chief factor of G/P^G is an $\mathfrak{F}$-complemented chief factor of G.*

Let $\mathfrak{A}$ be the set of all $\mathfrak{F}$-avoided complemented chief factors of G with the following property : If $H/K \in \mathfrak{A}$ and for some prime $p, p \mid \, |H/K|$, then HP^G/KP^G is a complemented chief factor of G.

PROPOSITION 2.4. *A chief factor of G belongs to $\mathfrak{A}$ iff it is partially $\mathfrak{F}$-complemented.*

PROOF. Let H/K be archief factor of G which belongs to $\mathfrak{A}$ and let $p \mid \, |H/K|$. By hypotheses HP^G/KP^G is complemented in G by M, say. Clearly M contains P^G and complements H/K. Hence by Proposition 2.1 H/K is partially $\mathfrak{F}$-complemented.

Conversely, if H/K is a p-chief factor of G partially $\mathfrak{F}$-complemented by M, say, then by Proposition 2.1 $M \geqq P^G$. Now clearly M complements HP^G/KP^G. Hence H/K belongs to $\mathfrak{A}$.

The following Proposition is a consequence of Proposition 2.4.

PROPOSITION 2.5. *Given any two chief series of G there is a* 1-1 *correspondence between partially $\mathfrak{F}$-complemented chief factors in the two series, the corresponding chief factors being G-isomorphic.*

PROOF. Consider any two chief series of G/P^G. By Lemma 2.6 [2] there is a 1-1 correspondence between complemented chief factors in the above series, corresponding chief factors being G-isomorphic. In particular there is such correspondence between complemented p-chief factors of G/P^G. Now let $(*)$ and $(**)$ be two arbitrary chief series of G. Consider the chief series $(*)'$ and $(**)'$ of G/P^G obtained by multiplying each member of $(*)$ and $(**)$ respectively by P^G. From what has been just said above and from Proposition 2.4 the result follows for partially $\mathfrak{F}$-complemented p-chief factors of G. Since p was an arbitrary prime we are done.

3. Proof of the main theorem

In order to prove the main theorem we will need the following two lemmas.

LEMMA 3.1. *Let $\mathfrak{S}$ be a Sylow system of a group G and $H \leqq G$. Then there is a conjugate H^g, $g \in G$, of H in G such that $\mathfrak{S}$ reduces into H^g.*

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PROOF. This is a consequence of a result of P. Hall [6] which states that any two Sylow systems of G are conjugate.

LEMMA 3.2. *Let $\mathfrak{S}$ be a Sylow system of G which reduces into V. If $\mathfrak{S}$ also reduces into a subgroup M of G which contains some $\mathfrak{F}$-injector of G, then $V \leqq M$.*

PROOF. By hypotheses $\mathfrak{S} \cap M = \mathfrak{T}$ is a Sylow system of M. Let W be an $\mathfrak{F}$-injector of G contained in M. By Lemma 3.1 there is a conjugate W^m, $m \in M$, of W in M into which $\mathfrak{T}$ reduces. In particular $\mathfrak{S}$ reduces into W^m. Now W^m is an $\mathfrak{F}$-injector of G and hence is conjugate to V. But by Lemma 1(2) [7] $\mathfrak{F}$-injectors of G are pronormal in G. Hence by the corollary to the Theorem in [9], $W^m = V$ and therefore $V \leqq M$, as required.

We now prove the main theorem.

PROOF. Let $\mathfrak{S}$ be a Sylow system of G. By Lemma 3.1 there is some $\mathfrak{F}$-injector V of G into which $\mathfrak{S}$ reduces. Consider a chief series of G/P^G, and denote by B_p the intersection of exactly one complement of each complemented p-chief factor of G/P^G, into which $\mathfrak{S}$ reduces. By Corollary 2.3 each of these complements contains some $\mathfrak{F}$-injector of G, and therefore by Lemma 3.2 each contains V. Thus $V \leqq B_p$. Also let W be a Prefrattini subgroup of G corresponding to $\mathfrak{S}$, then $W \leqq B_p$ (See [8]). By Theorem 3.1 and Corollary 3.6 [8] with $f(p) = \phi$, B_p/P^G avoids all complemented p-chief factors of G/P^G and covers the rest. Thus let H/K be an $\mathfrak{F}$-avoided complemented p-chief factor of G. If H/K is partially $\mathfrak{F}$-complemented in G, then by Proposition 2.4 HP^G/KP^G is complemented in G/P^G so that $B_p \cap HP^G = B_p \cap KP^G$ i.e. $P^G(B_p \cap H) = P^G(B_p \cap K)$ using the modular law. Also $P^G \cap (B_p \cap H) = P^G \cap H = P^G \cap K = P^G \cap (B_p \cap K)$; since $B_p \cap H \geqq B_p \cap K$ it follows that $B_p \cap H = B_p \cap K$ which means B_p avoids H/K. However if H/K is not partially $\mathfrak{F}$-complemented then HP^G/KP^G is a Frattini chief factor of G and so covered by B_p which means B_p also covers H/K since $B_p \geqq P^G$. Finally since B_p covers all $\mathfrak{F}$-covered and all Frattini p-chief factors it follows that B_p avoids just the partially $\mathfrak{F}$-complemented p-chief factors of G, and covers the rest.

Next assume that for each prime p B_p has been constructed as above corresponding to the same Sylow $\mathfrak{S}$. Let $Z(\mathfrak{S}) = \bigcap_p B_p$. Then clearly $Z(\mathfrak{S})$ avoids all partially $\mathfrak{F}$-complemented chief factors of G. Also $|G : Z(\mathfrak{S})| = \prod_p |G : B_p|$. Thus $|Z(\mathfrak{S})|$ is the product of the orders of all chief factors of G which are not partially $\mathfrak{F}$-complemented. Hence $Z(\mathfrak{S})$ has the required covering/avoidance property.

Finally since Sylow systems of G are transitively permuted by the inner automorphism of G, $Z(\mathfrak{S})$ as $\mathfrak{S}$ runs through the Sylow systems of G, form a characteristic class of conjugate subgroups.

We will refer to these subgroups as $\mathfrak{F}_\Phi$-subgroups.

COROLLARY 3.3. *For each prime p and each Sylow p-subgroup Z_p of an $\mathfrak{F}_\Phi$-subgroup $Z(\mathfrak{S})$ of G, Z_pP^G/P^G covers all Frattini p-chief factors of G/P^G and avoids the rest.*

COROLLARY 3.4. *Let $Z(\mathfrak{S})$ be an $\mathfrak{F}_\Phi$-subgroup of G corresponding to a Sylow system $\mathfrak{S}$ and let $\mathfrak{S}$ reduce into V. If W is a Prefrattini subgroup of G corresponding to $\mathfrak{S}$, then $\langle V, W\rangle \leqq Z(\mathfrak{S})$.*

Next we show that one can say even more, with the help of the following Lemma.

LEMMA 3.5. *Let $\mathfrak{S} = \{S^p\}$ be any Sylow system of G and W a Prefrattini subgroup of G corresponding to $\mathfrak{S}$, then $\mathfrak{S}$ reduces into W.*

PROOF. By definition (See [8]), $W = \bigcap_{p\,|\,|G|} M^p$ where M^p is the intersection of exactly one complement of each complemented p-chief factor in a given chief series of G, into which $\mathfrak{S}$ reduces. The Lemma now follows from the following simple facts, namely (a) that if X and Y are subgroups of G with coprime indices and if $\mathfrak{S}$ reduces into each of them then $\mathfrak{S}$ reduces into $X \cap Y$, and (b) that $|G : X \cap Y| = |G : X| \cdot |G : Y|$.

THEOREM 3.6. *Let $Z(\mathfrak{S})$, V and W be as in corollary 3.4. Then $Z(\mathfrak{S}) = VW$.*

PROOF. By Corollary 3.4, $\langle V, W\rangle \leqq Z(\mathfrak{S})$. Let $S_p \in \mathfrak{S}$ be a Sylow p-subgroup of G. Then by Lemma 3.5, $S_p \cap W = W_p$ is a Sylow p-subgroup of W. Since by hypotheses, $\mathfrak{S}$ reduces into V, $V_p = S_p \cap P^G$ is a Sylow p-subgroup of V. Now $S_p \cap W_p P^G = W_p(S_p \cap P^G) = W_p V_p$, by the modular law; thus W_p and V_p commute. Let Z_p be a Sylow p-subgroup of $Z(\mathfrak{S})$ containing $W_p V_p$. By Corollary 3.3, $P^G W_p/P^G = P^G Z_p/P^G$ i.e. $Z_p = W_p(Z_p \cap P^G) = W_p V_p$; thus $Z_p = W_p V_p \leqq WV \subseteq \langle W, V\rangle \leqq Z(\mathfrak{S})$. Since p is an arbitrary prime, using order argument it follows that $Z(\mathfrak{S}) \leqq W V$. For $|W_p \cap V_p|$ is the product of orders of all $\mathfrak{F}$-covered and Frattini p-chief factors in any chief series of G through P^G whereas $|(W \cap V)_p| \leqq$ the product of orders of such p-chief factors in any given chief series. Thus

$$\begin{aligned}|Z(\mathfrak{S})| = \Pi_p|Z_p| = \Pi_p|W_p V_p| &= \Pi_p|W_p| \cdot \Pi_p|V_p|/\Pi_p|W_p \cap V_p| \\ &\leqq \Pi_p|W_p| \cdot \Pi_p|V_p|/\Pi_p|(W \cap V)_p| = |W V|.\end{aligned}$$

Hence $Z(\mathfrak{S}) = WV$.

COROLLARY 3.7. *Let $Z(\mathfrak{S}) = VW$ be an $\mathfrak{F}_\Phi$-subgroup of G and let W_p and V_p be permutable Sylow p-subgroups of W and V respectively. Then $W_p \cap V_p$ is a Sylow p-subgroup of $W \cap V$ as well as of $V_p^G \cap W$.*

Now let $\mathfrak{H}$ be the set of all $\mathfrak{F}$-covered, Frattini chief factors H/K of G with the property that if $p \mid |H/K|$, then $H \cap P^G/K \cap P^G$ is a Frattini p-chief factor of G.

COROLLARY 3.8. *$V \cap W$ covers all chief factors of G which belong to $\mathfrak{H}$ and avoids the rest.*

PROOF. Clearly $V \cap W$ avoids all $\mathfrak{F}$-avoided and all complemented chief factors of G. Let H/K be an $\mathfrak{F}$-covered and Frattini chief factor of G and assume H/K is a p-chief factor. If $H/K \notin \mathfrak{H}$, then $H \cap P^G/K \cap P^G$ is a complemented chief

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factor of G, whence $H \cap P^G \cap W = K \cap P^G \cap W$, i.e. $P^G \cap W$ avoids H/K. By Corollary 3.7, it follows that H/K is avoided by $V \cap W$. On the other hand, if $H/K \in \mathfrak{H}$ and if $p \mid |H/K|$, then by hypotheses $H \cap P^G/K \cap P^G$ is covered by W, i.e. $(H \cap P^G \cap W)(K \cap P^G) = H \cap P^G$, i.e. $(H \cap P^G \cap W)K = (H \cap P^G)K = H$, since H/K is $\mathfrak{F}$-covered. Thus H/K is covered by $P^G \cap W$ and hence by $V \cap W$, by corollary 3.7.

Finally we remark that if $\mathfrak{F}$ is a trivial Fischer class then an $\mathfrak{F}_\Phi$-subgroup coincides with a Prefrattini subgroup of G.

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References

[1] R. W. Carter and T. O. Hawkes, 'The $\mathfrak{F}$-normalizers of a finite soluble group', *J. Algebra* 5 (1967), 175–202.

[2] R. W. Carter, B. Fischer and T. O. Hawkes, 'Extreme classes of finite soluble groups', *J. Algebra* 9 (1968), 285–313.

[3] B. Fischer, W. Gaschütz and B. Hartley, 'Injektoren endlicher auflösbarer Gruppen', *Math. Z.* 102 (1967) 337–339.

[4] W. Gaschütz, 'Praefrattinigruppen', *Arch. Math.* 13 (1962), 418–426.

[5] W. Gaschütz, 'Zur Theorie der endlicher auflösbarer Gruppen', *Math. Z.* 80 (1963), 300–305.

[6] P. Hall, 'On the Sylow system of a soluble group', *Proc. London Math. Soc.* (2) 43 (1937), 316–323.

[7] B. Hartley, 'On Fischer's dualization of Formation Theory', *Proc. London Math. Soc.* (3) 19 (1969), 193–207.

[8] T. O. Hawkes, 'Analogues of Prefrattini subgroups', *Proc. Internat. Conf. Theory of Groups, Austral. Nat. Univ. Canberra*, August 1965, pp. 145–150.

[9] A. Mann, 'A Criterion for Pronormality', *J. London Math. Soc.* 44 (1969), 175–176.

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