Proceedings of the Edinburgh Mathematical Society (1982) **25**, 173–181 ©

THE APPROXIMATION OF BIVARIATE FUNCTIONS BY SUMS OF UNIVARIATE ONES USING THE L_1-METRIC

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(Received 1st December 1980)

1. Introduction

We shall study a special case of the following abstract approximation problem: given a normed linear space E and two subspaces, M_1 and M_2, of E, we seek to approximate $f \in E$ by elements in the sum of M_1 and M_2. In particular, we might ask whether closest points to f from $M = M_1 + M_2$ exist, and if so, how they are characterised. If we can define proximity maps p_1 and p_2 for M_1 and M_2, respectively, then an algorithm analogous to the one given by Diliberto and Straus [4] can be defined by the formulae

(i) $p_i: E \to M_i$ is such that $\|f - p_i f\| = \operatorname{dist}(f, M_i)$ for $f \in E$ and $i = 1, 2$, where $\operatorname{dist}(f, M_i) = \inf_{m \in M_i} \|f - m\|$.

(ii) The sequence $\{f_n\}$ is defined by $f_0 = f$, $f_{n+1} = f_n - p_1 f_n - p_2(f_n - p_1 f_n)$ for $n = 0, 1, 2 \ldots$.

With these definitions it is easy to establish that all f_n are *equivalent* to f, in the sense that $f_n - f \in M$, and that $\|f_n\|$ converges monotonically downward to a number satisfying

$$\lim_{n \to \infty} \|f_n\| \geqq \operatorname{dist}(f, M). \tag{1}$$

Two interesting questions are whether equality holds in (1), and whether the sequence $\{f_n\}$ converges. In the case $E = C(X \times Y)$ and $M = C(X) + C(Y)$, these questions have been answered affirmatively by Diliberto and Straus [3] and Aumann [2], respectively, for the supremum norm. Golomb [5] gives an abstract account of this algorithm and establishes equality in (1) under the assumptions that M_1 and M_2 are complemented, and that

$$\|f - p_i f + n\| = \|f - p_i f - n\| \quad \text{for all} \quad n \in M_i \quad \text{and} \quad i = 1, 2.$$

There are two papers, Sullivan [8] and Attlestan and Sullivan [1], which deal with the

*The authors were supported by the U.S. Army Research Office at the University of Texas at Austin. In addition, Drs. McCabe and Phillips were supported by the Carnegie Trust for the Universities of Scotland.

same algorithm in a rotund or uniformly rotund Banach space which is either smooth or reflexive. The second of these papers gives some convergence results for the algorithm.

The situation we shall consider is as follows. Let (X, Σ, μ) and (Y, Θ, ν) be two measure spaces of finite measure. It is convenient, and does not sacrifice generality, to suppose that $\mu(X)=\nu(Y)=1$. Let $(Z, \Phi, \sigma)=(X, \Sigma, \mu)\times(Y, \Theta, \nu)$. By identifying an element $g\in L_1(X)$ with a function $\bar{g}(x,y)=g(x)$ we imbed $L_1(X)$ in $L_1(Z)$. Since $\mu(Y)=1$ this imbedding is isometric. Henceforth we ignore the distinction between g and $\bar{g}$. In the same way, $L_1(Y)$ is imbedded in $L_1(Z)$, and we set $M=L_1(X)+L_1(Y)$. Now we inquire into the approximability of elements of $E=L_1(Z)$ by elements of M, and also for the case when $E=C(X\times Y)$ and $M=C(X)+C(Y)$, using the L_1 norm. Our investigations are concerned mainly with the existence of best approximations.

2. Existence of best L_1 approximations

An important question is whether best approximations to f from M exist for every $f\in L_1(Z)$. A trivial answer could be given if the subspace M failed to be closed, so we verify first that M is indeed closed.

Lemma 2.1. *The subspace $M=L_1(X)+L_1(Y)$ is closed in $L_1(Z)$.*

Proof. Let $\{m_n\}$ be a sequence of elements in M with limit m.

Write $m_n(x,y)=g_n(x)+h_n(y)$ where $g_n\in L_1(X)$, $h_n\in L_1(Y)$. This representation is unique only up to an additive constant and so we can insist that $\int_X g_n\,d\mu=0$. Now we have, by the Fubini Theorem,

$$\begin{aligned}\|m_n\|_1=\iint|g_n+h_n|\,d\sigma &\geqq \iint(g_n+h_n)\operatorname{sgn} h_n\,d\sigma\\ &=\mu(X)\int_Y|h_n|\,d\nu\\ &=\|h_n\|_1 \quad\text{since}\quad \mu(X)=1.\end{aligned}$$

This argument, when applied to m_n-m_k, shows that the sequence $\{h_n\}$ is Cauchy in $L_1(Y)$ and so converges to some $h\in L_1(Y)$. The convergence $m_n\to m$ implies that $g_n\to m-h$, and since $g_n\in L_1(X)$, we must have $m-h\in L_1(X)$, which concludes the proof.

For completeness, we state the following result of R. C. James [**6**], which characterises best L_1-approximations.

Theorem 2.1. *In order that 0 be a best L_1-approximation to an $f\in L_1(X)$ from some linear subspace Φ, it is necessary and sufficient that $\int\phi\operatorname{sgn} f\leqq\int_{Z(f)}|\phi|$ for all $\phi\in\Phi$. Here $Z(f)$ denotes the set of points where $f(x)=0$.*

Lemma 2.2. *In order that a real number r be a best approximation to $f\in L_1(X)$ by a*

https://doi.org/10.1017/S0013091500016655 Published online by Cambridge University Press

constant, it is necessary and sufficient that $\max\{\mu N(f-r), \mu P(f-r)\} \leqq \frac{1}{2}\mu(X)$. *Here N and P denote the sets where the indicated function is negative or positive, respectively.*

Proof. By Theorem 2.1, and obvious deductions, the following are equivalent:

(a) r is a best approximation to f.

(b) $|\int c \operatorname{sgn}(f-r)| \leqq \int_{Z(f-r)} |c|$ for all $c \in \mathbb{R}$.

(c) $|\int \operatorname{sgn}(f-r)| \leqq \int_{Z(f-r)} 1$.

(d) $|\mu P(f-r) - \mu N(f-r)| \leqq \mu Z(f-r) = \mu(X) - \mu P(f-r) - \mu N(f-r)$.

(e) $\begin{cases} \mu P(f-r) - \mu N(f-r) \leqq \mu(X) - \mu P(f-r) - \mu N(f-r), \\ \mu N(f-r) - \mu P(f-r) \leqq \mu(X) - \mu P(f-r) - \mu N(f-r). \end{cases}$

(*f*) $\mu P(f-r) \leqq \frac{1}{2}\mu(X)$ and $\mu N(f-r) \leqq \frac{1}{2}\mu(X)$.

In the space $L_1(X)$ we define an operator A which produces best approximations by constants. Since best approximations are not unique, we let $I(f)$ denote the *interval* of all best constant approximations to f. Formally,

$$r \in I(f) \quad \text{iff} \quad \|f-r\|_1 \leqq \|f-c\|_1 \quad \text{for all} \quad c \in \mathbb{R}.$$

Then Af is defined as the midpoint of $I(f)$.

Lemma 2.3. *If* $I(f) = [\alpha, \beta]$, *then*

$$\alpha = \inf\{r : \mu P(f-r) \leqq 1/2\},$$

$$\beta = \sup\{r : \mu N(f-r) \leqq 1/2\}.$$

Proof. It suffices to prove the equation for β, as the other is similar. By Lemma 2.2,

$$\beta = \sup\left\{r : \|f-r\|_1 = \inf_c \|f-c\|_1\right\}$$

$$= \sup\{r : \mu P(f-r) \leqq 1/2 \quad \text{and} \quad \mu N(f-r) \leqq 1/2\}$$

$$\leqq \sup\{r : \mu N(f-r) \leqq 1/2\}$$

If the last inequality is a strict inequality, select $c > \beta$ such that $\mu N(f-r) \leqq 1/2$ for some $r > c$. Since $c \notin I(f)$, either $\mu N(f-c) > 1/2$ or $\mu P(f-c) > 1/2$. Since $c > \beta$ and $\mu P(f-\beta) \leqq 1/2$, we have $\mu P(f-c) \leqq 1/2$ and $\mu N(f-c) > 1/2$. This, however, contradicts the inequalities $c < r$ and $\mu N(f-r) \leqq 1/2$.

Lemma 2.4. *The map* $A : L_1(X) \to \mathbb{R}$ *has these properties:*

(a) $A(f+c) = Af + c$ *for all* $c \in \mathbb{R}$, $f \in L_1(X)$.

(b) *$Af \geqq Ag$ whenever $f \geqq g, f, g \in L_1(X)$.*

(c) $\|f - Af\|_1 \leqq \|f\|_1$ *for all* $f \in L_1(X)$.

(d) *If f and g belong to $L_\infty(X)$ then $|Af - Ag| \leqq \|f - g\|_\infty$.*

(e) *In general, A is discontinuous.*

Proof. It is elementary to prove that $I(f+c) = I(f) + c$. From this, part (a) follows at once.

In order to prove (b), assume that $f \geqq g$. Let $I(f) = [a, b]$ and let $I(g) = [\alpha, \beta]$. We will show that $a \geqq \alpha$ and $b \geqq \beta$, from which assertion (b) will follow at once. Suppose that $a < \alpha$. Then $a \notin I(g)$, and by Lemma 2.2 either $\mu P(g - a) > \frac{1}{2}$ or $\mu N(g - a) > \frac{1}{2}$. In the first case we conclude that $\mu P(f - a) > \frac{1}{2}$ (because $f \geqq g$). By Lemma 2.2, $a \notin I(f)$, a contradiction. In the second case, $\mu N(g - \alpha) > \frac{1}{2}$ (because $a < \alpha$). By Lemma 2.2, $\alpha \notin I(g)$, a contradiction. A completely analogous proof shows that $b \geqq \beta$.

Part (c) is immediate from the fact that Af is a best approximation to f and is therefore as good an approximation as 0.

For part (d), start with the pointwise inequality (valid almost everywhere)

$$-\|f - g\|_\infty \leqq f - g \leqq \|f - g\|_\infty.$$

Then

$$g - \|f - g\|_\infty \leqq f \leqq g + \|f - g\|_\infty.$$

Using parts (a) and (b) of this lemma, we obtain

$$Ag - \|f - g\|_\infty \leqq Af \leqq Ag + \|f - g\|_\infty$$

and

$$-\|f - g\|_\infty \leqq Af - Ag \leqq \|f - g\|_\infty.$$

An example in support of part (e) is obtained by letting $g(x) = f(x + c)$, where

$$f(x) = \begin{cases} \operatorname{sgn} x & |x| > \varepsilon \\ x/\varepsilon & |x| \leqq \varepsilon \end{cases}$$

On the interval $[-1, 1]$ we have then $Af = 0$, $Ag = 1$, $\|f - g\|_1 = 2\varepsilon$.

The discontinuous nature of L_1-approximation was pointed out previously by K. Usow in [**10**]. See also Lazar, Morris and Wulbert [**12**].

If f is a function of two variables, $f(x, y)$, we define $A_x f$ to be the function of y which results upon applying A to $(f(\cdot, y)$. Similarly, A_y is defined. Thus we have, for all $y \in Y$ and for all $h \in L_1(Y)$,

$$\int |f(x, y) - (A_x f)(y)| d\mu \leqq \int |f(x, y) - h(y)| \, d\mu.$$

Lemma 2.5. *The operator A_x maps $L_1(X\times Y)$ into $L_1(Y)$.*

Proof. Let $f\in L_1(X\times Y)$ and $h=A_xf$. Modifying slightly the notation used above, let $I(y)$ denote the interval of all best constant approximations to $f(\cdot,y)$. Let $I(y)=[\alpha(y),\beta(y)]$. Then $h(y)=\frac{1}{2}[\alpha(y)+\beta(y)]$. In order to show that h is measurable, it suffices to prove that α and β are measurable. These two proofs are similar, and only the proof for β is given.

For $y\in Y$ and r an extended-real number, let

$$\psi_r(y)=\mu\{x:f(x,y)<r\}.$$

By Theorem 7.6 of Rudin's book [11], ψ_r is measurable. Now let $r_1,r_2,\ldots,$ be an enumeration of the rational numbers. Define ϕ_n on Y by

$$\phi_n(y)=\begin{cases} r_n & \text{if} \quad \psi_{r_n}\leqq 1/2 \\ -\infty & \text{otherwise.}\end{cases}$$

Then ϕ_n is measurable. Indeed, the set $A_\lambda=\{y:\phi_n(y)\leqq\lambda\}$ is either Y (when $\lambda\geqq r_n$) or $\{y:\psi_{r_n}(y)>1/2\}$. Thus A_λ is measurable for all extended-real numbers λ. By Lemma 2.3, $\beta(y)=\sup_n\phi_n(y)$, and β is thus a measurable function. In order to prove that $h\in L_1(Y)$, use Lemma 2.4 (part c) to obtain

$$\int_X |f(x,y)-h(y)|d\mu\leqq\int_X |f(x,y)|d\mu$$

Integrating over Y, we obtain $\|f-h\|_1\leqq\|f\|_1$, whence $\|h\|_1\leqq 2\|f\|_1$. It is worth noting that A_x and A_y are metric projections of $L_1(X\times Y)$ onto, respectively, $L_1(Y)$ and $L_1(X)$.

Lemma 2.6. *Let $f\in L_\infty(Z)$. To each $m\in M$ satisfying $\|f-m\|_1\leqq\|f\|_1$, there corresponds an $m^*\in M$ such that*

(i) $\|f-m^*\|_1\leqq\|f-m\|_1$ and

(ii) $\|m^*\|_\infty\leqq 32\|f\|_\infty$.

Proof. Firstly, since $m\in M$, we may argue as in the proof of Lemma 2.1 to obtain $\|m\|_1\geqq\|h\|_1$, where $m=g+h$, $g\in L_1(X)$, $h\in L_1(Y)$, and $\int g=0$. Furthermore, since $f\in L_\infty(Z)$ the following pair of inequalities holds:

(a) $\|h\|_1\leqq\|m\|_1\leqq\|m-f\|_1+\|f\|_1\leqq 2\|f\|_1\leqq 2\|f\|_\infty$.

(b) $\|g\|_1\leqq\|g+h\|_1+\|h\|_1\leqq 2\|m\|_1\leqq 4\|f\|_\infty$.

Now let $h^*=A_x(f-g)$ and set $m'=g+h^*$. Since A_x is a metric projection, we have for each y

$$\int_X |f-g-h^*|d\mu\leqq\int_X |f-g-h|d\mu$$

and integrating with respect to y gives us

$$\|f-m'\|_1 \leqq \|f-m\|_1 \leqq \|f\|_1.$$

We can repeat the arguments used at the outset of the proof to obtain $\|h^*\|_1 \leqq 2\|f\|_\infty$. For almost all $y \in Y$ we have

$$\|f(\cdot, y)-g(\cdot)-h^*(y)\|_1 \leqq \|f(\cdot, y)-g(\cdot)\|_1$$

We may write (once again for almost all y)

$$|h^*(y)| = \int_X |h^*(y)| d\mu \leqq \int_X |h^* + g - f| d\mu + \int_X |g - \;| d\mu$$

$$\leqq 2 \int_X |f-g| d\mu \leqq 2\|f\|_\infty + 2\|g\|_1 \leqq 10\|f\|_\infty.$$

We conclude that $\|h^*\|_\infty \leqq 10\|f\|_\infty$. Then defining $g^* = A_y(f-h^*)$ and $m^* = g^* + h^*$, we may repeat the preceding argument, to obtain

$$\|f-m^*\|_1 \leqq \|f-m'\|_1 \leqq \|f-m\|_1$$

and

$$\|g^*\|_\infty \leqq 2\|f\|_\infty + 2\|h^*\|_\infty \leqq 22\|f\|_\infty$$

or

$$\|m^*\|_\infty \leqq \|g^*\|_\infty + \|h^*\|_\infty \leqq 32\|f\|_\infty.$$

Lemma 2.7. *The set $K = \{m \in M : \|m\|_\infty \leqq N\}$ is weakly closed in $L_1(Z, \Phi, \sigma)$.*

Proof. Assume to the contrary, that there exists a generalised sequence $\{m_\alpha\}$ in K which converges weakly to an element m which is not in K. Then there exists a set $A \subset Z$ such that $\sigma(A) > 0$ and $|m(z)| > N$ for all $z \in A$. Take $v \in L_\infty(Z)$ such that $v = 0$ on $Z \backslash A$ and $v = \operatorname{sgn} m(z)$ on A. Since $L_1^*(Z)$ can be identified with $L_\infty(Z)$ we can define a functional $\delta \in L_1^*(Z)$ by $\delta(f) = \int_Z fv\, d\sigma = \int_A fv\, d\sigma$. Since $\{m_\alpha\}$ is weakly convergent we must have

$$\int_A m_\alpha v\, d\sigma \to \int_A mv\, d\sigma = \int_A |m| d\sigma > N\sigma(A)$$

and this implies that eventually $\int_A m_\alpha v\, d\sigma > N\sigma(A)$. But this is not possible, because

$$\int_A m_\alpha v\, d\sigma \leqq \|m_\alpha\|_\infty \|v\|_\infty \cdot \sigma(A) \leqq N\sigma(A).$$

Theorem 2.2. *Let $f \in L_\infty(Z)$. Then in the metric of $L_1(Z)$, f possesses a best approximation from the subspace $M = L_1(X) + L_1(Y)$.*

Proof. By Lemma 2.6 we may confine our search for a best approximation to the set

$$B = \{m \in M : \|m\|_\infty \leqq 32\|f\|_\infty\}$$

which is weakly relatively compact by the Dunford–Pettis Theorem ([**4**], p. 294, [**9**], p. 274). It is weakly closed by Lemma 2.7. Since the norm is weakly lower-semicontinuous, the expression $\|f - m\|_1$ *attains* its infimum on B.

The arguments of Theorem 2.2 also establish the following result.

Theorem 2.3. *Each $f \in L_1(Z)$ has a best approximation in each of the sets $B_n = \{m \in M : \|m\|_\infty \leqq n\}$ for $n = 1, 2, 3, \ldots$.*

We do not know, however, whether each $f \in L_1(Z)$ has a best approximation in M, or alternatively, whether the numbers $d_n = \operatorname{dist}(f, B_n)$ are eventually constant as $n \to \infty$.

Lemma 2.8. *The operators A_x and A_y previously defined have the properties*

$$A_x(f + h) = A_x f + h \qquad f \in L_1(X \times Y), \quad h \in L_1(Y)$$

$$A_y(f + g) = A_y f + g \qquad f \in L_1(X \times Y), \quad g \in L_1(X)$$

Proof This follows from Lemma 2.4.

3. Existence of best L_1-approximations for continuous functions

In this section we consider spaces of continuous functions furnished with L_1-norms. The spaces X and Y are now compact Hausdorff and we work with the spaces $C(X)$, $C(Y)$ and $C(X \times Y)$. Let μ and v be Borel measures on X and Y, respectively. Is it true that for each $f \in C(X \times Y)$ there exist $g \in C(X)$ and $h \in C(Y)$ which minimise the expression

$$\iint |f(x, y) - g(x) - h(y)| \, d\mu \, dv? \tag{1}$$

Theorem 3.1. *Each element of $C(X \times Y)$ has a best L_1-approximation in the subspace $C(X) + C(Y)$.*

Proof. Let δ denote the infimum of the expression in (1) as g and h range over $C(X)$ and $C(Y)$, respectively. Select $g_n \in C(X)$ and $h_n \in C(Y)$ such that $\|f - g_n - h_n\|_1 \to \delta$. Define $h_n^* = A_x(f - g_n)$, where A_x is the operator defined in the preceding section. We shall show that the sequence $\{h_n^*\}$ is equicontinuous. Let y_0 be a point of Y at which equicontinuity is to be established. Let $\varepsilon > 0$. By the lemma following, $\{f(x, \cdot) : x \in X\}$ is equicontinuous. Hence there is a neighborhood V of y_0 such that

$$|f(x, y) - f(x, y_0)| < \varepsilon \quad (x \in X, \ y \in V).$$

If $y \in V$, then by Lemma 2.4, part (d),

$$\begin{aligned}|h_n^*(y) - h_n^*(y_0)| &= |A_x(f - g_n)(y) - A_x(f - g_n)(y_0)| \\ &\leqq \sup_x |(f - g_n)(x, y) - (f - g_n)(x, y_0)| \\ &= \sup_x |f(x, y) - f(x, y_0)| \leqq \varepsilon \end{aligned} \tag{2}$$

This proves the equicontinuity of the sequence $\{h_n^*\}$.

The inequality (2) shows also that if h_n^* is replaced by $h_n^* - h_n^*(y_0)$ for some fixed y_0, then the resulting functions are bounded above by $2\|f\|_\infty$. We suppose, therefore, that the sequence $\{h_n^*\}$ is bounded in the supremum norm.

Now define $g_n^* = A_y(f - h_n^*)$. By a repetition of the previous argument we infer that $\{g_n^*\}$ is an equicontinuous sequence in $C(X)$. By Lemma 2.4, part (d), $\|g_n^*\|_\infty \leqq \|f - h_n^*\|_\infty$, and the g_n^*-sequence is bounded. By the Ascoli Theorem, we select a sequence of integers $\{n_k\}$ such that

$$g_{n_k}^* \to g \in C(X), \quad h_{n_k}^* \to h \in C(Y) \quad \text{(uniformly)}.$$

By the continuity of the L_1-norm in $C(X \times Y)$, we have

$$\|f - g - h\| = \delta.$$

Lemma 3.1. *If X is compact and $f \in C(X \times Y)$ then the family $\{f(x, \cdot) : x \in X\}$ is equicontinuous in $C(Y)$.*

Proof. Let $\varepsilon > 0$. For each point (x, y) in $X \times Y$ there are open neighborhoods $U(x, y)$ of x and $V(x, y)$ of y such that

$$|f(x, y) - f(s, t)| < \varepsilon, \quad s \in U(x, y),\ t \in V(x, y). \tag{3}$$

Fix y. Since $x \in U(x, y)$ for all x, the family $\{U(x, y) : x \in X\}$ covers X. Since X is compact, there exist $x_1, \ldots, x_n$ such that X is covered by $U(x_1, y), \ldots, U(x_n, y)$. Define $V = V(x_1, y) \cap \cdots \cap V(x_n, y)$. Then V is a neighborhood of y. If x is arbitrary in X and $t \in V$, then for some i, $x \in U(x_i, y)$. Of course, $t \in V(x_i, y)$. Hence by (3) we have $|f(x, y) - f(x, t)| < \varepsilon$. This establishes equicontinuity at y.

The analogue of the Diliberto–Straus algorithm in the present setting is defined by the formulas

$$f_1 = f, \quad f_{2n} = f_{2n-1} - A_y f_{2n-1}, \quad f_{2n+1} = f_{2n} - A_x f_{2n}.$$

Theorem 3.2. *There exist functions $f \in C(X \times Y)$ for which the generalised Diliberto–Straus algorithm does not work, i.e., $\lim \|f_n\|_1 > \operatorname{dist}(f, M)$, where $M = C(X) + C(Y)$.*

Proof. Let $X = Y = [-1, 1]$ and define $f \in C(X \times Y)$ by

$$f(x,y) = \begin{cases} xy : (x,y) \in [0,1] \times [0,1] \\ -xy : (x,y) \in [-1,0] \times [-1,0] \\ 0 : \text{elsewhere.} \end{cases}$$

From the characterisation theorem for best approximation by constants we obtain $A_x f = A_y f = 0$. Hence, in the algorithm all iterates are identical. However, 0 is not a best approximation to f. If it were, then by Theorem 2.1 the inequality $\iint m \operatorname{sgn} f \leqq \iint_{A(f)} |m|$ would be valid for all $m \in M$. The function $x+y$ does not satisfy this, as an elementary calculation shows. More careful calculation reveals that $\iint |f(x,y) - c(x+y)|$ attains a minimum of 0·354 near $c = 0{\cdot}19$.

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