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ON THE ALGEBRA OF MULTIPLIERS

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A commutative Banach algebra is called symmetric if, regarded as a function algebra on its maximal ideal space, it is closed under complex conjugation. Varopoulos, [**5**], proved the asymmetry of the tensor algebra $C(T) \hat{\otimes} C(T)$, where T is the unit circle. It is the object of this paper to prove the asymmetry of the Schur multipliers of the space $L^2(T, m) \hat{\otimes} L^2(T, m)$, where m is the Lebesgue measure. In the second part of the paper we study the Hankel multipliers of the space $l^2(Z) \hat{\otimes} l^2(Z)$ and give an application to it.

1. The asymmetry of $M(L^2(T) \hat{\otimes} L^2(T))$**.** Let $C(T)$ denote the space of continuous functions on T and $A(T)$ be the space of those functions in $C(T)$ that have absolutely convergent Fourier series. Consider the mapping F: $C(T) \to C(T \times T)$ defined by $F(f)(x, y) = f(x + y)$. If $\| \|_m$ denotes the multiplier norm in $M(L^2(T) \hat{\otimes} L^2(T))$, then we have

THEOREM 1.1. *The following are equivalent:*

(*i*) $f \in A(T)$
(*ii*) $F(f) \in C(T) \hat{\otimes} C(T)$.

Further $\|f\|_{A(T)} = \|F(f)\|_m$.

Proof. For the equivalence of (*i*) and (*ii*) one can consult [**7**]. To prove the isometric property of F on $A(T)$, let $f \in A(T)$, so

$$f(t) = \sum_{r=-\infty}^{\infty} a_r e^{irt} \quad \text{and} \quad \sum_{r=-\infty}^{\infty} |a_r| < \infty .$$

Hence

$$F(f)(x, y) = \sum_{r=-\infty}^{\infty} a_r e^{irx} \,.\, e^{iry}.$$

Since $\|e^{irx}\|_\infty = 1$ for all r, it follows that $\|F(f)\|_m \leqq \|f\|_{A(T)}$.

To show the other inequality define a mapping

$$P: C(T \times T) \to C(T)$$

such that $P(\varphi)(x) = \int_T \varphi(x - y, y)dy$. Clearly $P \circ F$: $C(T) \to C(T)$ is just the identity mapping. Let $F(f) \in C(T) \hat{\otimes} C(T)$ and $\Sigma_{i=1}^{\infty} \mathcal{U}_i \otimes \mathcal{V}_i$

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be any representation of $F(f)$. Then

$$P(F(f)) = \sum_{i=1}^{\infty} \mathscr{U}_i * \mathscr{V}_i.$$

It follows that

$$\|P(F(f))\|_{A(T)} \leqq \|F(f))\|_{\tau r}.$$

However the function $1 \otimes 1 \in L^2(T) \hat{\otimes} L^2(T)$, so we have

$$\begin{aligned} \|F(f)\|_{\tau r} &= \|F(f) \cdot 1 \otimes 1\|_{\tau r} \\ &\leqq \|F(f)\|_{\mathscr{M}} \cdot \|1 \otimes 1\|_{\tau r} \\ &= \|F(f)\|_{\mathscr{M}} \cdot \end{aligned}$$

Hence $\|P(F(f))\|_{A(T)} \leqq \|F(f)\|_{\mathscr{M}}$. This completes the proof.

Now, we need the following technical lemma.

LEMMA 1.2. *Let ϕ_1 and ϕ_2 be any two elements in the unit ball of $\mathscr{M}(L^2 \hat{\otimes} L^2)$. Assume, further, that* $\sup\phi_1 \subseteq \Omega_1 = X_1 \times Y_1$, $\sup\phi_2 \subseteq \Omega_2 = X_2 \times Y_2$, *where $X_1 \cap X_2 = Y_1 \cap Y_2 = \emptyset$, the empty set. Then there exists a function $\phi \in \mathscr{M}(L^2 \hat{\otimes} L^2)$ such that*

$$\phi|_{\Omega_i} = \phi_i, i = 1, 2 \quad \textit{and} \quad \|\phi\|_{\mathscr{M}} = \max_{i=1,2}\|\phi_i\|_{\mathscr{M}}.$$

Proof. Define the following function ϕ on $T \times T$

$$\phi(x, y) = \begin{cases} \phi_1 & \text{if } (x, y) \in \Omega_1 \\ \phi_2 & \text{if } (x, y) \in \Omega_2 \end{cases}$$

and $\phi \equiv 0$ on the complement of $\Omega_1 \cup \Omega_2$. We claim that the function ϕ is the required function. First, since $\phi = \phi_1 + \phi_2$, it follows that $\phi \in \mathscr{M}(L^2 \hat{\otimes} L^2)$. It remains to estimate the multiplier-norm of ϕ. To do so, let $f \otimes g$ be any atom in the unit ball of $L^2 \hat{\otimes} L^2$. Since

$$f \otimes g = \frac{f}{||f||_2} (||f||_2 \cdot ||g||_2)^{1/2} \cdot \frac{g}{||g||_2} (||f||_2 \cdot ||g||_2)^{1/2},$$

we can assume that $\|f\|_2 = \|g\|_2 \leqq 1$. Further since the support of ϕ is contained in $\Omega_1 \cup \Omega_2$, we let $\text{supp}(f) \subset X_1 \cup X_2$ and $\text{supp}(g) \subset Y_1 \cup Y_2$. Set $f_i = f|_{X_i}$ and $g_i = g|_{Y_i}$, $i = 1, 2$. Then $f = f_1 + f_2$ and $g = g_1 + g_2$.

Further $\|f\|_2{}^2 = \|f_1\|_2{}^2 + \|f_2\|_2{}^2$ and $\|g\|_2{}^2 = \|g_1\|_2{}^2 + \|g_2\|_2{}^2$, since

$$\bigcap_{i=1}^{2} X_i = \bigcap_{i=1}^{2} Y_i = \emptyset.$$

Now, consider

$$\phi \cdot f \otimes g = \phi_1 \cdot f_1 \otimes g_1 + \phi_2 \cdot f_2 \otimes g_2.$$

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Since $\|\phi_i\|_{\mathcal{M}} \leqq 1$, $i = 1, 2$, we deduce

$$\phi_i \cdot f_i \otimes g_i = \sum_{j=1}^{\infty} u_j^{(i)} \otimes v_j^{(i)}$$

$$\sum_{j=1}^{\infty} ||u_j^{(i)}||_2 \cdot ||v_j^{(i)}||_2 \leqq ||f_i||_2 \cdot ||g_i||_2.$$

Again, as above, we can assume that $\|f_i\|_2 = \|g_i\|_2$ and $\|u_j^{(i)}\|_2 = \|v_j^{(i)}\|_2$ for $i = 1, 2$ and $j \geqq 1$. It follows that

$$\sum_{j=1}^{\infty} ||u_j^{(i)}||_2^2 \leqq ||f_i||_2^2$$

$$\sum_{j=1}^{\infty} ||v_j^{(i)}||_2^2 \leqq ||g_i||_2^2, \qquad i = 1, 2.$$

Now define the following functions

$$z_j = u_j^{(1)} + u_j^{(2)}$$
$$w_j = v_j^{(1)} + v_j^{(2)}$$

for all $j \geqq 1$. Then

$$\phi \cdot f \otimes g = \sum_{j=1}^{\infty} (z_j \otimes w_j) \cdot 1_{(X_1 \times Y_1) \cup (X_2 \cup Y_2)},$$

where 1_E denotes the characteristic function of the set E. But since

$$\|z_j\|_2^2 = \|u_j^{(1)}\|_2^2 + \|u_j^{(2)}\|_2^2$$
$$\|w_j\|_2^2 = \|v_j^{(1)}\|_2^2 + \|v_j^{(2)}\|_2^2,$$

it follows that

$$\begin{aligned} ||\phi \cdot f \otimes g||_{\tau\tau} &\leqq \sum_{j=1}^{\infty} ||z_j||_2 ||w_j||_2 \\ &\leqq \sum_{j=1}^{\infty} (||u_j^{(1)}||_2^2 + ||u_j^{(2)}||_2^2)^{1/2} \cdot (||v_j^{(1)}||_2^2 \\ &\qquad + ||v_j^{(2)}||_2^2)^{1/2} \\ &\leqq \left(\sum_{j=1}^{\infty} (||u_j^{(1)}||_2^2 + ||u_j^{(2)}||_2^2)\right)^{1/2} \\ &\qquad \cdot \left(\sum_{j=1}^{\infty} (||v_j^{(1)}||_2^2 + ||v_j^{(2)}||_2^2)\right)^{1/2} \\ &\leqq (||f_1||_2^2 + ||f_2||_2^2)^{1/2} \cdot (||g_1||_2^2 + ||g_2||_2^2)^{1/2} \\ &\leqq ||f||_2 \cdot ||g||_2 \leqq 1. \end{aligned}$$

Since $f \otimes g$ was an arbitrary atom in the unit ball of $L^2 \hat{\otimes} L^2$, it follows that $\|\phi\|_{\mathcal{M}} \leqq 1$. This completes the proof of the lemma.

Now we prove

THEOREM 1.2. *The space* $\mathcal{M}(L^2 \hat{\otimes} L^2)$ *is not symmetric.*

Proof. To prove the asymmetry of a space it is enough to produce an element in such a space which has independent powers, [**7**].

Let P be a Cantor independent set which is not Helson in T. The existence of P is illustrated in [**4**]. Take ν to be a non-negative measure concentrated on $P \cup (-P)$. Then ν has mutually singular convolution powers, and if we choose $\|\nu\|_{M(T)} = 1$, we obtain

$$\left\|\sum_{r=1}^{n} \lambda_r \nu^r\right\|_{M(T)} = \sum_{r=1}^{n} |\lambda_r|,$$

for all $\lambda_r \in C$ and $n \in \mathbf{N}$. Since discrete measures on T are dense in $M(T)$ in the weak-* topology [**1**], then we can find a sequence $(\nu_n)_{n=1}^{\infty}$ of finitely supported discrete measures (the support of each ν_n is a finite subgroup of T) such that

$$\hat{\nu}_n(j) \rightarrow \hat{\nu}(j)$$

for all $j \in Z$. That P is not Helson enables us to choose ν such that $\|\hat{\nu}\|_{\infty}$ is as small as we like and $\hat{\nu}$ to be real. If E_n denotes the support of ν_n, then we can find a sequence $(f_n)_{n=1}^{\infty}$ of real functions on T such that

$$\|f_n\|_{A(E_n)} \leqq 1 \quad (n \geqq 1),$$

$$\|f_n\|_{\infty} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$\sup_n \left\|\sum_{r=1}^{s} \lambda_r f_n^r\right\|_{A(E_n)} = \sum_{r=1}^{s} |\lambda_r|,$$

for all $s \in \mathbf{N}$ and $\lambda_r \in C$.

Now, let $(X_n^{(i)})_{n=1}^{\infty}$ $i = 1, 2$, be two sequences of sets in T such that $X_n^{(i)} \cap X_m^{(i)} = \emptyset$ for $n \neq m$, $i = 1, 2$ and $X_n^{(i)}$ has the same cardinality as E_n. Identify, then, $X_n^{(i)}$ with E_n for every $n \geqq 1$, and $i = 1, 2$. If $F\colon C(T) \rightarrow C(T \times T)$ is the function defined in Theorem 1.1, then set $\phi_n = F(f_n)$, $n \geqq 1$. A simple application of Lemma 1.1 implies that $\phi_n \in \mathcal{M}(L^2 \hat{\otimes} L^2)$ and

$$\|\phi_n\|_{\mathcal{M}} \leqq 1 \quad (n \geqq 1);$$

$$\|\phi\|_{\infty} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\sup_n \left\|\sum_{r=0}^{s} \lambda_r \phi_n^r\right\|_{\mathcal{M}} = \sum_{r=0}^{s} |\lambda_r|;$$

for all $s \in N$ and $\lambda_r \in C$. Using Lemma 1.2 repeatedly we construct a sequence of real functions $(\psi_n)_{n=1}^{\infty}$ in $\mathcal{M}(L^2 \hat{\otimes} L^2)$ such that

$$\|\psi_n\|_{\mathcal{M}} \leqq 1 \quad (n \geqq 1);$$

$$\text{supp } \psi_n = \bigcup_{j=1}^{n} X_j^{(1)} \times X_j^{(2)};$$

$$\psi_n|_{X_n^{(1)} \times X_n^{(2)}} = \phi_n,$$

$$\|\psi_n\|_{\infty} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Clearly, the sequence $(\psi_n)_{n=1}^{\infty}$ converges uniformly to a function $\psi \in \mathcal{M}(L^2 \hat{\otimes} L^2)$. Furthermore

$$\|\psi\|_{\mathcal{M}} = \sup_n \|\psi_n\|_{\mathcal{M}}.$$

Hence

$$\left\|\sum_{r=0}^{s} \lambda_r \psi^r\right\|_{\mathcal{M}} = \sum_{r=0}^{s} |\lambda_r|.$$

This completes the proof of the theorem.

As a corollary of the previous theorem we have

THEOREM 1.3. *The space $\mathcal{M}(L^2 \hat{\otimes} L^2)$ is not separable.*

Proof. The functions $(\psi_n)_{n=1}^{\infty}$ in Theorem 1.2 have the property that

$$\|\psi_n - \psi_m\|_{\mathcal{M}} \geqq \alpha > 0 \quad \text{for } n \neq m.$$

This proves the claim.

2. The Hankel multipliers of $l^2(Z) \hat{\otimes} l^2(Z)$. Let $f \in l^\infty(Z)$ and ϕ be a function on $Z \times Z$ defined by $\phi(r, s) = f(r + s)$. If $\phi \in \mathcal{M}(l^2(Z) \hat{\otimes} l^2(Z))$, then ϕ will be called a *Hankel multiplier* of $l^2(Z) \hat{\otimes} l^2(Z)$. It is the purpose of this section to characterize the Hankel multipliers of $l^2(Z) \hat{\otimes} l^2(Z)$.

Let $M(T)$ denote the space of all complex valued regular bounded Borel measures on T. Set $B(Z)$ to be the set of functions $f \in l^\infty(Z)$ such that $f = \hat{\nu}$ for some $\nu \in M(T)$.

THEOREM 2.1. *Let $\phi \in l^\infty(Z \times Z)$ be defined by: $\phi(r, s) = f(r + s)$ for some $f \in l^\infty(Z)$ then the following are equivalent:*

(*i*) $\phi \in \mathcal{M}(l^2(Z) \hat{\otimes} l^2(Z))$.

(*ii*) $f \in B(Z)$.

Furthermore, $\|f\|_{B(Z)} = \|\phi\|_{\mathcal{M}}$.

Proof. (*ii*) $\Rightarrow$ (*i*). Let ν be any element in $M(T)$. It is well known, [**1**], that there exists a sequence of discrete measures in $M(T)$ such that:

$$\hat{\nu}_n(j) \rightarrow \hat{\nu}(j) \text{ for all } j, \text{ and } \|\nu_n\|_{M(T)} \leqq \|\nu\|_{M(T)}.$$

For any discrete measure ν, we have

$$\nu = \sum_{j=1}^{\infty} \alpha_j \delta_{t_j}, \quad \hat{\nu}(r) = \sum_{j=1}^{\infty} \alpha_j e^{-irt_j}, \quad \text{and}$$

$$\|\hat{\nu}\|_{B(Z)} = \sum_{j=1}^{\infty} |\alpha_j| < \infty,$$

where δ_{t_j} is the unit mass at the point t_j. Now, let

$$\phi(r, s) = \hat{\nu}(r + s) = f(r + s).$$

Then

$$\phi(r, s) = \sum_{j=1}^{\infty} \alpha_j e^{-i(r+s)t_j}$$

$$= \sum_{j=1}^{\infty} \alpha_j e^{-irt_j} e^{-ist_j}.$$

Setting $f_j(r) = \alpha_j e^{-irt_j}$ and $g_j(s) = e^{-ist_j}$, we see that $\phi \in l^\infty(Z) \hat{\otimes} l^\infty(Z)$. Further

$$||\phi||_{\mathscr{M}} \leqq ||\phi||_{\tilde{V}(Z)} \leqq \sum_{j=1}^{\infty} |\alpha_j| = ||f||_{B(Z)}.$$

For $\phi(r, s) = f(r + s)$, where f is any function in $B(Z)$, we have

$$\phi(r, s) = \lim_n f_n(r + s),$$

where $f_n(r + s) = \hat{\nu}_n(r + s)$ for some discrete measure ν_n and $\|f_n\|_{B(Z)} \leqq \|f\|_{B(Z)}$. Hence the function ϕ is the pointwise limit of a uniformly bounded sequence of elements in $l^\infty \hat{\otimes} l^\infty$. It follows, [**5**],

$$\varphi \in \tilde{V}(Z) = l^1(Z) \check{\otimes} l^1(Z)^* \quad \text{and} \quad \|\phi\|_{\tilde{V}(Z)} \leqq \|f\|_{B(Z)}.$$

Hence, [**3**], $\phi \in \mathscr{M}(l^2(Z) \hat{\otimes} l^2(Z))$. Further

$$\|\phi\|_{\mathscr{M}} \leqq \|\phi\|_{\tilde{V}(Z)} \leqq \|f\|_{B(Z)}.$$

Conversely $(i) \Rightarrow (ii)$. Let $F: l^\infty(Z) \rightarrow l^\infty(Z \times Z)$ be the mapping $f(u)(r, s) = u(r + s)$, and E be the set of functions ϕ in $\mathscr{M}(l^2(Z) \hat{\otimes} l^2(Z))$ such that $\phi = F(u)$ for some u in $l^\infty(Z)$. It follows, [**3**], that $E \subseteq \tilde{V}(Z)$. Hence if $\phi_n = \phi|_{Z_n \times Z_n}$, then

$$\phi_n \in l^\infty(Z_n) \hat{\otimes} l^\infty(Z_n).$$

Let $\sum_{i=1}^k f_i \otimes g_i$ be a representation of ϕ_n in $l^\infty(Z_n) \hat{\otimes} l^\infty(Z_n)$. Then

$$\phi_n(r, s) = (F(u))_n(r, s)$$

$$= \sum_{i=1}^{k} f_i(r) \cdot g_i(s)$$

$$= \sum_{i=1}^{k} f_i(\alpha) \cdot g_i(\beta) \tag{*}$$

for all α and β in Z such that $\alpha + \beta = r + s$. For each $n \in \mathbf{N}$, define a mapping P_n on E as follows:

$$P_n : E \rightarrow l^\infty(Z),$$

$$P_n(\phi) = \frac{1}{2n + 1} \sum_{i=1}^{k} f_i * g_i.$$

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The function $P_n(\phi)$ is independent of the representation of ϕ_n, for

$$\begin{aligned} P_n(\phi)(k) &= \frac{1}{2n+1} \sum_{i=1}^{k} (f_i * g_i)(k) \\ &= \frac{1}{2n+1} \sum_{j=-n}^{n} \left(\sum_{i=1}^{k} f_i(k-j) v_i(j) \right) \\ &= \frac{1}{2n+1} \sum_{j=-n}^{n} \phi_n(k-j, j). \end{aligned}$$

Let $A(Z)$ be the space $l^2(Z)*l^2(Z)$ which is, by the Plancherel theorem, the same space as $FL^1(T)$, the Fourier transforms of $L^1(T)$. Then $P_n(\phi) \in A(Z) \subseteq (BZ)$. Further, if $\| \|_{\tau r}$ denotes the norm in $l^2(Z_n) \hat{\otimes} l^2(Z_n)$ and 1_{Z_n} is the characteristic function of Z_n, then

$$\begin{aligned} \|P_n(\phi)\|_{A(Z)} &\leqq (2n+1)^{-1} \cdot \|\phi_n\|_{\tau r} \\ &\leqq (2n+1)^{-1} \cdot \|\phi_n \cdot 1_{Z_n} \otimes 1_{Z_n}\|_{\tau r} \\ &\leqq (2n+1)^{-1} \cdot \|\phi_n\|_{\mathscr{M}} \cdot \|1_{Z_n} \otimes 1_{Z_n}\|_{\tau r} \\ &\leqq \|\phi_n\|_{\mathscr{M}} \\ &\leqq \|\phi\|_{\mathscr{M}} \ (**). \end{aligned}$$

On the other hand, since $\phi = F(u)$,

$$\begin{aligned} P_n(\ )(k) &= P_n(F(u))(k) \\ &= \frac{1}{2n+1} \sum_{j=-n}^{n} \phi_n(k-j, j) \\ &= \frac{1}{2n+1} \sum_{j=-n}^{n} u(k) \\ &= u(k). \end{aligned}$$

Hence $P_n(F(u)) \to u$ pointwise. Since $(P_n(F(u))_{n=1}^{\infty}$ is a uniformly bounded sequence in $A(Z)$ which converges pointwise to u, we obtain that $u \in B(Z)$. Furthermore, relation $(**)$ implies that

$$\|u\|_{B(Z)} \leqq \|\phi\|_{\mathscr{M}}.$$

This completes the proof of the theorem.

A similar result was proved by Varopoulus [**5**], where he proved the isometry of $B(Z)$ and its image under F in the tensor algebra norm.

As an application of Theorem 2.1, we estimate the multiplier norm of the matrix ψ, as an element in $\mathscr{M}(l^2(Z) \hat{\otimes} l^2(Z))$, where

$$\psi(i,j) = \begin{cases} 1 & \text{if } 0 < i + j \leqq n \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.3.1. *$\|\psi\|_{\mathscr{M}} \sim C \cdot \log n$, where C is a constant independent of n.*

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Proof. Let f be a function defined on Z as follows:

$$f(i) = \begin{cases} 1 & \text{if } 0 < i \leqq n \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\psi(i, j) = f(i + j)$. Since f has a finite support in Z, then $f \in B(Z)$. Let $f = \hat{\nu}$ for some $\nu \in M(T)$. By the Riesz-representation theorem, there exists a continuous linear functional $S : C(T) \to \mathbf{C}$ such that $S(h) = \int_T h d\nu$ and $\|S\| = \|\nu\|_{M(T)}$, where

$$\|S\| = \sup_h \frac{|S(h)|}{\|h\|}, \quad h \in C(T).$$

It follows from Theorem 2.1 that

$$\|\psi\|_m = \|f\|_{B(Z)} = \|\nu\|_{M(T)} = \|S\|.$$

Hence it is enough to estimate the norm of S. Further, since the trigonometric polynomials are dense in $C(T)$ under the supremum norm, it is enough to take h, in the definition of $\|S\|$, to be a trigonometric polynomial. Setting

$$\hat{\nu}(r) = \int_T e^{irt} d\nu(t) = f(r),$$

we see that

$$S(e^{irt}) = \begin{cases} 1 & \text{if } 0 < r \leqq n \\ 0 & \text{otherwise.} \end{cases}$$

Thus if $h(t) = \sum_{j=-k}^{k} \alpha_j e^{ijt}$, then

$$S(h) = \begin{cases} \sum_1^n \alpha_j & \text{if } k > n \\ \sum_1^k \alpha_j & \text{if } k < n. \end{cases}$$

Consider the following function in $C(T)$:

$$\begin{aligned} \tilde{D}_n(t) &= \sum_{r=1}^{n} e^{irt} \\ &= \sum_{r=1}^{n} \cos rt + i \sum_{r=1}^{n} \sin rt \\ &= (D_n - \tfrac{1}{2})(t) + \bar{D}_n(t), \end{aligned}$$

where D_n is the Dirichlet kernel and $\bar{D}_n$ is the conjugate kernel to D_n. A classical result in harmonic analysis, [**2**], asserts that $\|D_n\|_1 \approx \alpha \log n$ and $\|\bar{D}_n\|_1 \approx \log n$, where $\| \ \|_1$ denotes the norm in $L^1(T)$. Hence

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$\|\tilde{D}_n\|_1 \approx c \log n$ for some constant c independent of n. Next we observe that

$$\sum_{j=1}^{n} \alpha_j = (\tilde{D}_n * h)(0),$$

from which we conclude

$$\begin{aligned} |S(h)| &= \left|\sum_{j=1}^{n} \alpha_j\right| \\ &= |(\tilde{D}_n * h)(0)| \\ &\leqq \|\tilde{D}_n\|_1 \cdot \|h\|_\infty \\ &\leqq c \log n \cdot \|h\|_\infty. \end{aligned}$$

Hence

$$\|S\| = \sup_h \frac{|S(h)|}{\|h\|_\infty} \leqq c \log n.$$

This completes the proof of the lemma.

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