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# REMARKABLE HYPERPLANES IN LOCALLY CONVEX SPACES OF DIMENSION AT MOST $c$

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ABSTRACT. Every locally convex space $E$ of dimension at most $c$ contains a hyperplane $G$ with the following property: the linear hull of each bounded Banach disk in $G$ is finite-dimensional.

Let $I$ be an infinite index set. Denote by $m_0(I)$ the linear span of the indicator functions of the subsets of $I$, equipped with the weak topology $\sigma(m_0(I), K^{(I)})$. Using the technique of summable sequences, J. Batt, P. Dierolf, and J. Voigt proved in [1] that $m_0(I)$ has a curious property: the linear hull of each bounded Banach disk in $m_0(I)$ is finite-dimensional. Later M. Valdivia proved, [4], that every ultrabornological space whose topology is different from the finest locally convex topology, contains at least one non-ultrabornological hyperplane. In proving this, Valdivia implicitly obtained the following result: if $E$ is a locally convex space with a separable weak dual, then $E$ contains a hyperplane $H$ such that the linear hull of each bounded Banach disk in $H$ is finite-dimensional. In this article we shall show that using the method of Valdivia's proof, the same conclusion can also be obtained, if the requirement for $(E', \sigma(E', E))$ to be separable, is replaced by a weaker one: that $\dim E \leq c$.

Let $E$ be a locally convex Hausdorff topological vector space over the field $K$ of the real or complex numbers, or briefly, a locally convex space. We shall denote by $E'$ and $E^*$ its topological and algebraic duals, respectively. An absolutely convex set will be called a *disk*. Let $B$ be a bounded disk in $E$. We denote by $E_B$ the linear hull of $B$, equipped with the norm, associated with $B$. We say that $B$ is finite- or infinite-dimensional, if $E_B$ is finite- or infinite-dimensional. We say that $B$ is a Banach disk, if $E_B$ is a Banach space. The dimension of $E$ is the cardinality of its Hamel basis. We denote by $c$ the cardinality of continuum. Given a dual pair $\langle E, F\rangle$, we denote by $\mu(E, F)$ and $\sigma(E, F)$ the Mackey and the weak topology on $E$, respectively.

THEOREM. *Let $E$ be a locally convex space of dimension at most $c$. There exists a hyperplane $G$ in $E$ such that no infinite-dimensional bounded Banach disk is contained in $G$.*

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**Proof.** If $E$ does not admit an infinite-dimensional bounded Banach disk, then every hyperplane in $E$ has the required property.

Suppose $E$ has an infinite-dimensional bounded Banach disk $B$. Since by ([2], Theorem I-1), $\dim E_B = c$, we conclude that $\dim E = c$.

In [4] M. Valdivia considered the family of *all* weakly compact disks of $E$. Here we shall apply the arguments of M. Valdivia to another family. Since $B$ is infinite-dimensional, it contains a sequence of linearly independent elements, converging to zero in $E_B$, hence it contains a disk $A$, which is compact and metrizable when equipped with the topology induced by $E$. Now we shall consider the family $\mathcal{A} = \{A_i : i \in I\}$ of *all* infinite-dimensional compact and metrizable disks of $E$. Since each $A_i$ is separable, and therefore defined by a countable subset of $E$, we conclude that Card $I \leq c$. Clearly Card $I$ is infinite. Let $\Omega$ be the set of all ordinal numbers whose cardinality is less than Card $I$. Since Card $\Omega$ = Card $I$, we may suppose that $\mathcal{A} = \{A_\alpha : \alpha \in \Omega\}$.

Now we shall use transfinite induction. Take $x_0 \neq 0$ in $E$. There is $y_0 \in x_0 + A_0$, such that $x_0$ and $y_0$ are linearly independent. Let $\alpha \in \Omega$. Suppose for each $\beta$, satisfying: $0 \leq \beta < \alpha$, there is $y_\beta \in x_0 + A_\beta$ such that $\{y_\beta : 0 \leq \beta < \alpha\} \cup \{x_0\}$ are linearly independent. Then, since Card $\alpha <$ Card $I \leq c$ and $\dim E_{A_\alpha} = c$, there is $y_\alpha \in x_0 + A_\alpha$ such that $\{y_\beta : 0 \leq \beta \leq \alpha\} \cup \{x_0\}$ are linearly independent. Having constructed the elements $\{y_\alpha : \alpha \in \Omega\}$, take $f \in E^*$ such that $f(y_\alpha) = 0$ for every $\alpha \in \Omega$ and $f(x_0) = 1$. Denote by $G = \{x \in E : f(x) = 0\}$.

The hyperplane $G$ satisfies the required condition. Indeed, if $B$ is an infinite-dimensional bounded Banach disk, contained in $G$, then $G$ contains a Banach disk $A_\alpha$ from the family $\mathcal{A}$, which implies: $x_0 \in y_\alpha + A_\alpha \subset G$. This is a contradiction, hence we conclude, that every Banach disk of $G$ is finite-dimensional.

Corollary. *There exists a topology $\tau$ on $E$, such that $E'$ has codimension one in the dual of $(E, \tau)$, and such that every bounded Banach disk of $(E, \tau)$ is finite-dimensional.*

Indeed, let $f \neq 0$ be an element of $E^*$, such that $f(G) = 0$. Denote by $F$ the linear hull of $E' \cup \{f\}$. Then $\mu(E, F)$ is the required topology.

We notice that the topology $t = \mu(E, F)$ preserves any given barrelledness property of $E$, (see Part II of [3]).

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