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ON SOME PROPERTIES OF QUASI-DISTANCE-BALANCED GRAPHS

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Abstract

For an edge uv in a graph G, $W^G_{u,v}$ denotes the set of all vertices of G that are closer to u than to v. A graph G is said to be *quasi-distance-balanced* if there exists a constant $\lambda > 1$ such that $|W^G_{u,v}| = \lambda^{\pm 1}|W^G_{v,u}|$ for every pair of adjacent vertices u and v. The existence of nonbipartite quasi-distance-balanced graphs is an open problem. In this paper we investigate the possible structure of cycles in quasi-distance-balanced graphs and generalise the previously known result that every quasi-distance-balanced graph is triangle-free. We also prove that a connected quasi-distance-balanced graph admitting a bridge is isomorphic to a star. Several open problems are posed.

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1. Introduction

Let G be a finite, undirected, connected graph with diameter d and let $V(G)$ and $E(G)$ be the vertex set and the edge set of G, respectively. For $u \in V(G)$, let $N_G(u)$ denote the set of neighbours of u in G. For $u, v \in V(G)$, let $d_G(u, v)$ denote the minimal path-length distance between u and v. When the graph G is clear from the context, we will simply write $d(u, v)$. For a pair of adjacent vertices u, v of G, define the set $W^G_{u,v}$ by

$$W^G_{u,v} = \{x \in V(G) \mid d(x, u) < d(x, v)\}.$$

If the graph G is clear from the context, we write simply $W_{u,v}$. Observe that for a connected bipartite graph G, the sets $W_{u,v}$ and $W_{v,u}$ form a partition of its vertex set. The sets $W_{u,v}$ and $W_{v,u}$ are important in metric graph theory.

A subgraph G of a hypercube H that preserves distances, that is, the distance between any two vertices in G is the same as the distance between those vertices in H, is called a *partial cube*. A set of vertices $S \subseteq V(G)$ of a graph G is *convex* if, for any two points $a, b \in S$, all the points on any shortest path between a and b are contained

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in S. Djoković [4] proved that a connected bipartite graph is a partial cube if and only if the sets $W_{u,v}$ and $W_{v,u}$ are convex for every edge uv of G. These sets also occur in chemical graph theory, where the Szeged index of a graph G, introduced by Gutman in [5], is defined as $Sz(G) = \sum_{uv \in E(G)} |W_{u,v}| \cdot |W_{v,u}|$.

A graph G is called *distance-balanced* (in short, DB) if the sets $W_{u,v}$ and $W_{v,u}$ are of the same size for arbitrary pairs of adjacent vertices u and v. Graphs having this property were first studied by Handa [6], who considered distance-balanced partial cubes. The term itself was introduced by Jerebic *et al.* [8], who gave some basic properties and characterised Cartesian and lexicographic products of distance-balanced graphs. Kutnar *et al.* [9] investigated this property for graphs having a certain type of symmetry and, among other results, proved that every vertex-transitive graph is distance-balanced. The problem of characterising distance-balanced graphs in the family of generalised Petersen graphs was studied in [10, 13]. For more results on this and related concepts, see [2, 3, 7, 11, 12].

Quasi-distance-balanced graphs, introduced by Abedi *et al.* in [1], generalise the concept of distance-balanced graphs. A graph G is *quasi-distance-balanced* (in short, quasi-DB) if there exists a positive rational number $\lambda > 1$ such that, for any edge uv of G, either $|W_{u,v}| = \lambda |W_{v,u}|$ or $|W_{v,u}| = \lambda |W_{u,v}|$. In this case, we set $QDB(G) = \lambda$. From [1], every quasi-distance-balanced graph is triangle-free and the only quasi-distance-balanced graphs with diameter two are the complete bipartite graphs. Quasi-distance-balanced lexicographic and Cartesian products are characterised in [1, Theorems 1.4 and 1.5]. All known examples of quasi-distance-balanced graphs are bipartite, so the following question, which was posed in [1], is still open.

Problem 1.1 [1, Problem 1.1]. Does there exist a nonbipartite quasi-distance-balanced graph?

Since a graph is bipartite if and only if it contains no odd cycle, Problem 1.1 naturally leads to the investigation of the possible structure of cycles in quasi-distance-balanced graphs. Edges of a quasi-distance-balanced graph G can be naturally oriented in the following way: for two adjacent vertices $u, v \in V(G)$, we define $u \to v$ if and only if $|W_{u,v}| = \lambda |W_{v,u}|$. (See Figure 1 for an example.) Let $Q(G)$ be the directed graph obtained in this way.

Let $C = v_1, \ldots, v_n$ be a cycle of length n in a quasi-DB graph G. Let C^+ be the set of indices $i \in \{1, \ldots, n\}$ for which $v_i \to v_{i+1}$, that is, $C^+ = \{i \in \{1, \ldots, n\} \mid v_i \to v_{i+1}\}$ (here we identify v_{n+1} with v_1 and v_0 with v_n). Similarly, let $C^- = \{i \in \{1, \ldots, n\} \mid v_{i+1} \to v_i\}$. (See Figure 2 for an example.)

The main focus of this paper is the investigation of the cycle structure in quasi-distance-balanced graphs. The main results are as follows.

Theorem 1.2. *Let G be a quasi-DB graph and let $C = v_1, \ldots, v_n$ be a cycle in G. Then*

$$\sum_{i \in C^+} |W_{v_i, v_{i+1}}| = \sum_{i \in C^-} |W_{v_{i+1}, v_i}|.$$

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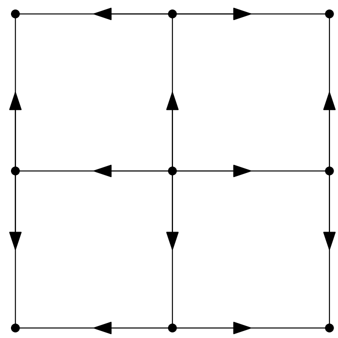

FIGURE 1. Orientation of edges in $P_3 \square P_3$.

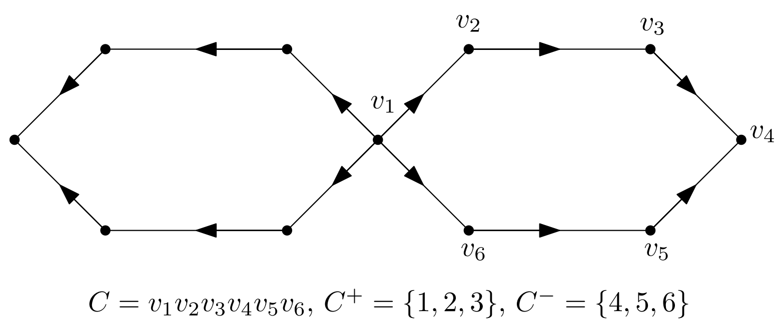

FIGURE 2. Example of C^+ and C^- in a quasi-DB graph.

The following result generalises [1, Theorem 1.2], where it was proved that every quasi-DB graph is triangle-free.

THEOREM 1.3. *Let G be a quasi-DB graph and let $C = v_1, \ldots, v_n$ be a cycle of length n in G. Then $2 \leq |C^+| \leq n - 2$ and $2 \leq |C^-| \leq n - 2$.*

For bipartite quasi-DB graphs, we have equality between the sizes of C^+ and C^-.

THEOREM 1.4. *Let G be a bipartite quasi-DB graph and let $C = v_1, \ldots, v_{2n}$ be a cycle of length $2n$ in G. Then $|C^+| = |C^-| = n$.*

Recall that a *bridge* in a graph is an edge whose removal increases the number of connected components of the graph. Quasi-DB graphs admitting a bridge are characterised in the following theorem.

THEOREM 1.5. *The only connected quasi-DB graphs having a bridge are stars.*

This paper is organised as follows. In Section 2, we develop some important tools and obtain results about partitions of the vertex set defined by distance from a given closed walk in a graph (see Theorem 2.5). In Section 3, we prove Theorems 1.2, 1.3 and 1.4. In Section 4, we prove Theorem 1.5.

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2. Preliminaries

In this section, we develop some important tools that will be useful later on. We first need to introduce some terminology.

Definition 2.1. Let G be a graph and let $C = v_1, v_2, \ldots, v_n, v_{n+1}$ be a walk of length n in G. Define a mapping $\varphi_C : V(G) \to \mathbb{Z}^n$ by

$$\varphi_C(v) = (x_1, \ldots, x_n), \quad \text{where } x_i = d_G(v, v_{i+1}) - d_G(v, v_i) \text{ for } i \in \{1, \ldots, n\}.$$

For $i \in \{1, \ldots, n\}$, let $A_i^{\pm} = \{(x_1, \ldots, x_n) \in \{-1, 0, 1\}^n \mid x_i = \pm 1\}$.

For any walk C in a graph G and any vertex $v \in V(G)$, we have $\varphi_C(v) \in \{-1, 0, 1\}^n$ and hence the following observation.

Observation 2.2. Let G be a graph and let $C = v_1, v_2, \ldots, v_n, v_{n+1}$ be a walk of length n in G. The set $\{\varphi_C^{-1}((x_1, \ldots, x_n)) \mid (x_1, \ldots, x_n) \in \{-1, 0, 1\}^n\}$ is a partition of $V(G)$.

The next lemma gives a connection between the sets W_{uv} and the mapping φ_C.

Lemma 2.3. *Let G be a graph and let $C = v_1, v_2, \ldots, v_n, v_{n+1}$ be a walk of length n in G. Then $W_{v_i,v_{i+1}} = \varphi_C^{-1}(A_i^+)$ and $W_{v_{i+1},v_i} = \varphi_C^{-1}(A_i^-)$ for every $i \in \{1, \ldots, n\}$.*

Proof. Let $v \in V(G)$. Suppose first that $v \in W_{v_i,v_{i+1}}$. Then $d(v, v_i) < d(v, v_{i+1})$. Moreover, since $v_i v_{i+1} \in E(G)$, it follows that $d(v, v_{i+1}) = d(v, v_i) + 1$. Therefore, $d(v, v_{i+1}) - d(v, v_i) = 1$ and hence $\varphi_C(v) \in A_i^+$. Conversely, let v be a vertex such that $\varphi_C(v) \in A_i^+$. By the definition of the mapping φ_C, it follows that $d(v, v_{i+1}) - d(v, v_i) = 1$ and hence $v \in W_{v_i,v_{i+1}}$. This proves that $W_{v_i,v_{i+1}} = \varphi_C^{-1}(A_i^+)$. Similarly, $W_{v_{i+1},v_i} = \varphi_C^{-1}(A_i^-)$. □

Lemma 2.4. *Let G be a graph and let $C = v_1, v_2, \ldots, v_n, v_{n+1}$ be a walk of length n in G. Let $v \in V(G)$ and let $\varphi_C(v) = (x_1, \ldots, x_n)$. Then $\sum_{i=1}^n x_i = d(v, v_{n+1}) - d(v, v_1)$.*

Proof. By the definition of the mapping φ, it follows that $x_i = d(v, v_{i+1}) - d(v, v_i)$. Hence, $\sum_{i=1}^n x_i = \sum_{i=1}^n (d(v, v_{i+1}) - d(v, v_i)) = d(v, v_{n+1}) - d(v, v_1)$. □

Theorem 2.5. *Let $v_1, \ldots, v_n, v_1$ be a closed walk in a graph G. Then*

$$\sum_{i=1}^n |W_{v_i,v_{i+1}}| = \sum_{i=1}^n |W_{v_{i+1},v_i}|.$$

Proof. Let $C = v_1, \ldots, v_n, v_1$ be a closed walk in G and let v be an arbitrary vertex of G. Let $\varphi_C(v) = (x_1, \ldots, x_n)$. Suppose that v contributes k to the sum $\sum_{i=1}^n |W_{v_i,v_{i+1}}|$, that is, there exist k indices $i \in \{1, \ldots, n\}$ such that $v \in W_{v_i,v_{i+1}}$. Lemma 2.3 implies that there are exactly k coordinates of $\varphi_C(v)$ equal to 1. By Lemma 2.4, it follows that $\sum_{i=1}^n x_i = 0$. Since $x_i \in \{-1, 0, 1\}$, there are also exactly k coordinates of $\varphi_C(v)$ equal to -1. Therefore, v contributes k to the sum $\sum_{i=1}^n |W_{v_{i+1},v_i}|$. Since this is true for every $v \in V(G)$, the result follows. □

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3. Cycles in quasi-distance-balanced graphs

Let G be a quasi-DB graph with $QDB(G) = \lambda > 1$. As explained in the introduction, there is a natural orientation of the edges of G defined by $u \to v$ if and only if $|W_{u,v}| = \lambda |W_{v,u}|$. Again, from the introduction, for a cycle $C = v_1, \ldots, v_n$, we defined $C^+ = \{i \in \{1, \ldots, n\} \mid v_i \to v_{i+1}\}$ and $C^- = \{i \in \{1, \ldots, n\} \mid v_{i+1} \to v_i\}$, where $v_{n+1} = v_1$ and $v_0 = v_n$.

Proof of Theorem 1.2. It is clear that $C^+ \cap C^- = \emptyset$ and $C^+ \cup C^- = \{1, \ldots, n\}$. By Theorem 2.5,

$$\sum_{i \in C^+} |W_{v_i,v_{i+1}}| + \sum_{i \in C^-} |W_{v_i,v_{i+1}}| = \sum_{i \in C^+} |W_{v_{i+1},v_i}| + \sum_{i \in C^-} |W_{v_{i+1},v_i}|$$

and consequently

$$\sum_{i \in C^+} |W_{v_i,v_{i+1}}| - \sum_{i \in C^-} |W_{v_{i+1},v_i}| = \sum_{i \in C^+} |W_{v_{i+1},v_i}| - \sum_{i \in C^-} |W_{v_i,v_{i+1}}|. \tag{3.1}$$

By the definition of the sets C^+ and C^-,

$$|W_{v_i,v_{i+1}}| = \lambda |W_{v_{i+1},v_i}| \quad (\text{for all } i \in C^+), \tag{3.2}$$

$$|W_{v_{i+1},v_i}| = \lambda |W_{v_i,v_{i+1}}| \quad (\text{for all } i \in C^-). \tag{3.3}$$

By combining (3.2) and (3.3), it is easy to see that

$$\sum_{i \in C^+} |W_{v_i,v_{i+1}}| - \sum_{i \in C^-} |W_{v_{i+1},v_i}| = \lambda\Big(\sum_{i \in C^+} |W_{v_{i+1},v_i}| - \sum_{i \in C^-} |W_{v_i,v_{i+1}}|\Big). \tag{3.4}$$

The result now follows from (3.1), (3.4) and the fact that $\lambda > 1$. □

Proof of Theorem 1.3. Suppose first that $|C^-| = 0$. Then it is clear that $C^+ = \{1, \ldots, n\}$ and that $\sum_{i \in C^-} |W_{v_{i+1},v_i}| = 0$. By Theorem 1.2, $\sum_{i \in C^+} |W_{v_i,v_{i+1}}| = 0$. However, since $v_i \in W_{v_i,v_{i+1}}$, it follows that $|W_{v_i,v_{i+1}}| \geq 1$ for every $i \in \{1, \ldots, n\}$, which contradicts the fact that $\sum_{i \in C^+} |W_{v_i,v_{i+1}}| = 0$. Thus, $|C^-| \geq 1$.

Suppose now that $|C^-| = 1$. Without loss of generality, we may assume that $C^- = \{n\}$ and $C^+ = \{1, \ldots, n-1\}$. We claim that

$$\sum_{i=1}^{n-1} |W_{v_i,v_{i+1}}| - |W_{v_1,v_n}| > 0. \tag{3.5}$$

We are first going to prove that

$$W_{v_1,v_n} \subseteq \bigcup_{i=1}^{n-1} W_{v_i,v_{i+1}}.$$

Let $v \in W_{v_1,v_n}$ and $\varphi_C(v) = (x_1, \ldots, x_n)$. By Lemma 2.3, $x_n = -1$ and, by Lemma 2.4, $\sum_{i=1}^{n} x_i = 0$. Since $x_i \in \{-1, 0, 1\}$ and $x_n = -1$, there exists $j \in \{1, \ldots, n-1\}$ such that $x_j = 1$. By Lemma 2.3, $v_j \in W_{v_j,v_{j+1}}$. Therefore, $W_{v_1,v_n} \subseteq \bigcup_{i=1}^{n-1} W_{v_i,v_{i+1}}$. Now observe that $v_{n-1} \in W_{v_{n-1},v_n}$ and $v_{n-1} \notin W_{v_1,v_n}$. This proves that W_{v_1,v_n} is a proper subset of $\bigcup_{i=1}^{n-1} W_{v_i,v_{i+1}}$, which gives (3.5). However, (3.5) contradicts Theorem 1.2. Thus, $|C^-| \geq 2$ and similarly $|C^+| \geq 2$. □

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COROLLARY 3.1. *Let G be a quasi-DB graph and let $C = v_1, v_2, v_3, v_4$ be a 4-cycle in G. Then $|C^+| = |C^-| = 2$.*

We now show that in a bipartite quasi-DB graph, $|C^+| = |C^-|$ for every cycle C.

PROOF OF THEOREM 1.4. Let G be a bipartite quasi-DB graph. Recall that for any edge uv in G, the sets $W_{u,v}$ and $W_{v,u}$ form a partition of $V(G)$. Hence, there exists some constant M with $|V(G)|/2 < M < |V(G)|$ such that

$$|W_{v_i,v_{i+1}}| = M, \quad |W_{v_{i+1},v_i}| = |V(G)| - M \quad (\text{for all } i \in C^+)$$

and

$$|W_{v_i,v_{i+1}}| = |V(G)| - M, \quad |W_{v_{i+1},v_i}| = M \quad (\text{for all } i \in C^-).$$

By Theorem 2.5,

$$|C^+| \cdot M + |C^-| \cdot (|V(G)| - M) = |C^+| \cdot (|V(G)| - M) + |C^-| \cdot M,$$

which implies that

$$2(|C^+| - |C^-|) \cdot M = (|C^+| - |C^-|) \cdot |V(G)|.$$

If $|C^+| - |C^-| \neq 0$, then $M = |V(G)|/2$, which is a contradiction. Thus, $|C^+| = |C^-|$. □

REMARK 3.2. Since all known examples of quasi-DB graphs are bipartite, it follows that in all known quasi-DB graphs, $|C^+| = |C^-|$ for every cycle C.

We now consider the existence of cycles of length 5 in quasi-DB graphs. A 5-cycle v_1, v_2, v_3, v_4, v_5 in a graph G is said to be *central* if every vertex in G is at distance at most 2 from every vertex on the 5-cycle, that is, $d(v, v_i) \leq 2$ for all $i \in \{1, 2, 3, 4, 5\}$ and for all $v \in V(G)$. The following result shows that there is no central 5-cycle in a quasi-DB graph.

PROPOSITION 3.3. *Let G be a graph having a central 5-cycle. Then G is not quasi-DB.*

PROOF. Suppose on the contrary that G is quasi-DB and that $C = v_1, v_2, v_3, v_4, v_5$ induces a central 5-cycle in G. We claim that $W_{v_{i+1},v_i} \backslash \{v_{i+2}\} = W_{v_{i+1},v_{i+2}} \backslash \{v_i\}$ for every $i \in \{1, 2, 3, 4, 5\}$. Let $v \in W_{v_{i+1},v_i} \backslash \{v_{i+2}\}$. If $v = v_{i+1}$, then clearly $v \in W_{v_{i+1},v_{i+2}} \backslash \{v_i\}$. If $v \neq v_{i+1}$, then since C is a central 5-cycle in G, it follows that $d(v, v_{i+1}) = 1$. Since G is triangle-free by [1, Theorem 1.2], it follows that $d(v, v_i) = d(v, v_{i+2}) = 2$. It is now clear that $v \in W_{v_{i+1},v_{i+2}} \backslash \{v_i\}$, which proves that $W_{v_{i+1},v_i} \backslash \{v_{i+2}\} \subseteq W_{v_{i+1},v_{i+2}} \backslash \{v_i\}$. It is easy to see that the reverse inclusion also holds. It is now clear that

$$|W_{v_{i+1},v_i}| = |W_{v_{i+1},v_{i+2}}| \quad (\text{for all } i \in \{1, 2, 3, 4, 5\}). \tag{3.6}$$

Since G is quasi-DB, $|W_{v_i,v_{i+1}}| = \lambda^{e_i} |W_{v_{i+1},v_i}|$, where $e_i = \pm 1$. By multiplying these equalities and using (3.6),

$$\prod_{i=1}^{5} |W_{v_i,v_{i+1}}| = \lambda^{e_1+e_2+e_3+e_4+e_5} \cdot \prod_{i=1}^{5} |W_{v_i,v_{i+1}}|.$$

Since $|W_{v_i,v_{i+1}}| \geq 1$ for each $i \in \{1, 2, 3, 4, 5\}$, it follows that $\lambda^{e_1+e_2+e_3+e_4+e_5} = 1$. But this is impossible, since $\lambda > 1$ and $e_1 + e_2 + e_3 + e_4 + e_5 \neq 0$. □

PROBLEM 3.4. Does there exist a quasi-DB graph admitting a 5-cycle?

4. Bridges in quasi-DB graphs

For a graph G, the *minimum degree of* G, denoted by $\delta(G)$, is the minimum degree of vertices in G. The following lemma characterises quasi-DB graphs with $\delta = 1$.

Lemma 4.1. *Let G be a connected quasi-DB graph. If $\delta(G) = 1$, then G is isomorphic to a star.*

Proof. Let G be a connected quasi-DB graph and let u be a vertex of degree 1 in G. Let v be the unique neighbour of u. It is easy to see that $|W_{u,v}| = 1$ and $|W_{v,u}| = |V(G)| - 1$, which implies that $QDB(G) = |V(G)| - 1$. Let w be a neighbour of v different from u. Since $|W_{v,w}| \geq 2$, it follows that $|W_{v,w}| = |V(G)| - 1$ and $|W_{w,v}| = 1$. This shows that every neighbour of v is a leaf in G and hence G is isomorphic to a star. □

We are now going to characterise quasi-DB graphs admitting a bridge. Recall that a *bridge* (or *cut edge*) in a graph G is an edge whose removal increases the number of connected components of G.

Proof of Theorem 1.5. Let G be a connected quasi-DB graph and let v_1v_2 be a bridge in G. Let $\lambda = QDB(G)$. For $i \in \{1, 2\}$, let G_i be the component containing v_i after removing the bridge v_1v_2. We assume, without loss of generality, that $|V(G_1)| \geq |V(G_2)|$. It is clear that $W^G_{v_1,v_2} = V(G_1)$ and $W^G_{v_2,v_1} = V(G_2)$. It follows that $\lambda = |V(G_1)|/|V(G_2)|$. If $V(G_2) = \{v_2\}$, then $\delta(G) = 1$ and hence by Lemma 4.1 it follows that G is isomorphic to a star. Let $x \in V(G_2)\backslash\{v_2\}$. It is now easy to see that $|W^G_{v_2,x}| \geq |V(G_1)| + 1$ and that $|W^G_{x,v_2}| \leq |V(G_2)| - 1$. It is also clear that $|W^G_{v_2,x}| \geq |W^G_{x,v_2}|$, implying that $|W^G_{v_2,x}| = \lambda|W^G_{x,v_2}|$, that is,

$$\lambda = \frac{|W^G_{v_2,x}|}{|W^G_{x,v_2}|} \geq \frac{|V(G_1)| + 1}{|V(G_2)| - 1} > \frac{|V(G_1)|}{|V(G_2)|} = \lambda,$$

which is a contradiction, showing that a quasi-DB graph with a bridge is isomorphic to a star. □

The next natural question is the characterisation of quasi-DB graphs with a cut vertex. As shown in [1, Proposition 3.4], there are infinitely many examples of such graphs. The examples constructed in [1, Proposition 3.4] are formed from bipartite DB graphs with the same number of vertices, glued along a vertex. All the graphs constructed in this way are bipartite. We conclude with the following problem.

Problem 4.2. Characterise quasi-DB graphs having a cut vertex.

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