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# DERIVATIONS TANGENTIAL TO COMPACT GROUP ACTIONS: SPECTRAL CONDITIONS IN THE WEAK CLOSURE

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**1. Introduction.** Let $G$ be a compact Lie group and $\alpha$ an action of $G$ on a $C^*$-algebra $\mathfrak{A}$ as *-automorphisms. Let $\mathfrak{A}_F^\alpha$ denote the set of $G$-finite elements for this action, i.e., the set of those $x \in \mathfrak{A}$ such that the orbit $\{\alpha_g(x):g \in G\}$ spans a finite dimensional space. $\mathfrak{A}_F^\alpha$ is a common core for all the *-derivations generating one-parameter subgroups of the action $\alpha$. Now let $\delta$ be a *-derivation with domain $D(\delta) = \mathfrak{A}_F^\alpha$ such that $\delta(\mathfrak{A}_F^\alpha) \subset \mathfrak{A}_F^\alpha$. Let us pose the following two problems:

1. Is $\delta$ closable, and is the closure of $\delta$ the generator of a strongly continuous one-parameter group of *-automorphisms?

2. If $\mathfrak{A}$ is simple or prime, under what conditions does $\delta$ have a decomposition

$$\delta = \delta_0 + \tilde{\delta},$$

where $\overline{\delta}_0$ is the generator of a one-parameter subgroup of $\alpha(G)$ and $\tilde{\delta}$ is a bounded, or approximately bounded derivation?

To our knowledge, there are no counterexamples to 1 and there are a number of positive results, [**4**], [**6**], [**7**], [**11**], [**16**], [**21**], [**27**], [**30**]. It is even possible that if $G$ is an arbitrary Lie group, then any *-derivation mapping the algebra $C^\infty(\mathfrak{A}, \alpha)$ of smooth elements with respect to the action $\alpha$ into itself is a pregenerator. (This is more likely if the derivation commutes with the action $\alpha$, see [**16**].)

There are examples with $G = \mathbf{T}$ where the decomposition in 2 is not possible, see [**7**], Theorem 4.7 and Remark 4.10, but there also positive results in this direction; see [**5**], [**6**], [**7**], [**8**], [**10**], [**21**], [**22**], [**25**], [**27**] for situations where such a decomposition is valid.

Our main result for non-abelian groups $G$ (Theorem 2.5) does not require $G$ to be a Lie group, but two interesting applications of the theorem are to product actions of closed subgroups of $U(n)$, the group of $n \times n$ complex unitary matrices, on the UHF algebra $\bigotimes_{\mathbf{N}} \mathbf{M}_n(\mathbf{C})$ and to closed subgroups of $U(n)$ acting canonically on Cuntz's algebra. For these $C^*$-dynamical systems we prove that any *-derivation $\delta$ mapping $\mathfrak{A}_F^\alpha$ into $\mathfrak{A}$ has a unique decomposition $\delta = \delta_0 + \tilde{\delta}$, with $\overline{\delta}_0$ the generator of a one-parameter subgroup of $\alpha(G)$, and $\tilde{\delta}$ an inner derivation. In the case of

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$\mathfrak{D}_n$, this is an improvement of results from [**4**]. In particular, $\bar{\delta}$ is a generator in these two cases since it is a bounded perturbation of the generator $\bar{\delta}_0$. This result is slightly surprising in that no conditions are imposed on the range $\delta(\mathfrak{A}_F^\alpha)$ of $\delta$, while in the classical case, i.e., in the case of abelian $C^*$-algebras, one always has to impose some conditions on the range to ensure that $\delta$ is a generator, for example that $\delta$ maps into once differentiable elements, [**2**].

Our second main result (Theorem 3.4) concerns actions of compact abelian Lie groups. Assuming a spectral condition, that there exists a faithful $G$-covariant representation of $\mathfrak{A}$ such that in this representation the range projections of the ideals $\mathfrak{A}^\alpha(\nu)\mathfrak{A}^\alpha(\nu)^*$ are equal to $\mathbf{1}$ for all $\nu \in \hat{G}$, we prove that $\mathfrak{A}_F^\alpha$ consists of analytic elements for any $*$-derivation $\delta$ mapping $\mathfrak{A}_F^\alpha$ into $\mathfrak{A}_F^\alpha$, and consequently $\delta$ has a generator closure. This generalizes the main theorems of [**7**] and [**30**], and the proof is partly based on ideas from these papers as well as [**4**].

In the case that $G$ is the circle group $\mathbf{T}$, this result (Theorem 3.5) can be strengthened somewhat: It is enough to assume that for each $n \in \mathbf{Z} = \hat{\mathbf{T}}$ the range projections in $(\mathfrak{A}^\alpha)''$ of $\mathfrak{A}^\alpha(n)\mathfrak{A}^\alpha(n)^*$ and $\mathfrak{A}^\alpha(n)^*\mathfrak{A}^\alpha(n)$ are equal, Proposition 3.7. Also, in case that $\mathfrak{A}$ is a separable, simple $C^*$-algebra and $G = \mathbf{T}$, we can use a theorem of Kishimoto to show that all $*$-derivations mapping $\mathfrak{A}_F^\alpha$ into $\mathfrak{A}_F^\alpha$ are pregenerators, Theorem 3.9. After finishing this paper, we received a preprint from Kishimoto, [**20**], where he proves a theorem which contains our Theorem 3.9 as a very special case: His results imply that if $G$ is any compact abelian group, and the action $\alpha$ on $\mathfrak{A}$ admits a faithful family of $\alpha$-covariant irreducible representations, and $\delta$ is a $*$-derivation of $\mathfrak{A}_F^\alpha$ into $\mathfrak{A}_F^\alpha$ such that $\delta_{|\mathfrak{A}^\alpha(\gamma)}$ is bounded for each $\gamma \in \hat{G}$, then $\delta$ is a pregenerator. He also proves in this situation that if

$$\delta_0 = \int_G \alpha_g \circ \delta \circ \alpha_g^{-1} dg$$

denotes the invariant part of $\delta$, then $\delta - \delta_0$ is approximately bounded.

**2. Non-abelian compact group actions.** In this section we consider a $C^*$-dynamical system $(\mathfrak{A}, G, \alpha)$, where $G$ is a (generally non-abelian) compact group, and a $*$-derivation $\delta$ defined on the algebra $\mathfrak{A}_F^\alpha$ of $G$-finite elements in $\mathfrak{A}$, $\delta:\mathfrak{A}_F^\alpha \to \mathfrak{A}$. We give conditions which force $\delta$ to be a bounded perturbation of the generator of a one-parameter subgroup of $\alpha(G)$. Although this conclusion is $C^*$-algebraic, our methods are partly $W^*$-algebraic, and our sufficient condition is the existence of a faithful $\alpha$-covariant representation $\pi$ such that

$$(*) \quad \pi(\mathfrak{A})'' \cap \pi(\mathfrak{A}^\alpha)' = \mathbf{C} \cdot \mathbf{1},$$

see Theorem 2.5. Examples of $C^*$-dynamical systems satisfying this condition are product type actions on UHF algebras and "quasi-product" actions on the Cuntz algebras $\mathfrak{O}_m$. We study these examples in Theorems 2.1 and 2.4; although these two theorems are actually corollaries of Theorem 2.5, we find it instructive to sketch special proofs appropriate to these cases. In Theorem 2.5, we also introduce an auxiliary action $\tau$ of a discrete group $H$ on $\mathfrak{A}$, commuting with both $\alpha(G)$ and $\delta$; the form of our condition which takes into account the automorphism group $(H, \tau)$ is the existence of an $\alpha \times \tau$-covariant representation $\pi$ such that

$$\pi(\mathfrak{A})'' \cap \pi(\mathfrak{A}^\alpha)' \cap [\pi(\mathfrak{A})'']^\tau = \mathbf{C} \cdot \mathbf{1}.$$

This leads to an infinitesimal version of the duality theorem of Araki, Haag, Kastler, and Takesaki, [**29**, p. 207]. We do not, however, recover from our methods the purely $C^*$-algebraic infinitesimal versions of this theorem in [**6**], [**21**], and [**27**].

Before continuing we recall some notions and establish some notation relating to a continuous representation $\alpha$ of a compact group $G$ on a Banach space $\mathfrak{A}$. Associated to a $d(\gamma)$-dimensional irreducible unitary representation $\gamma$ of $G$ are continuous operators on $\mathfrak{A}$,

$$\mathscr{P}^\alpha(\gamma):x \mapsto \int_G d(\gamma)\overline{\operatorname{tr}(\gamma(g))}\alpha_g(x)dg$$

and

$$\mathscr{P}^\alpha_{ji}(\gamma):x \mapsto \int_G d(\gamma)\overline{(\gamma_{ji}(g))}\alpha_g(x)dg \quad (1 \leqq i, j \leqq d(\gamma)).$$

These satisfy the following relations:

(i) $\mathscr{P}^\alpha_{ij}(\gamma)\,\mathscr{P}^\alpha_{kl}(\gamma) = \delta_{li}\mathscr{P}^\alpha_{kj}(\gamma)$.

(ii) $\mathscr{P}^\alpha(\gamma)\,\mathscr{P}^\alpha_{ij}(\gamma) = \mathscr{P}^\alpha_{ij}(\gamma)\mathscr{P}^\alpha(\gamma) = \mathscr{P}^\alpha_{ij}(\gamma)$,
$(\mathscr{P}^\alpha(\gamma))^2 = \mathscr{P}^\alpha(\gamma)$.

(iii) $\alpha_g(\mathscr{P}^\alpha_{ij}(\gamma)(x)) = \sum_k \mathscr{P}^\alpha_{ik}(\gamma)(x)\gamma_{kj}(g)$.

Here (i) and (ii) follow from the orthogonality relations for $\{\gamma_{ij}\}$ and (iii) simply from the fact that $\gamma$ is a unitary representation. Another way to write (iii) is

(iv) $\alpha_g([\mathscr{P}^\alpha_{ij}(\gamma)(x)]) = [\mathscr{P}^\alpha_{ij}(\gamma)(x)] \cdot (\mathbf{1}_{\mathfrak{A}} \otimes \gamma(g))$

(where $[\mathscr{P}^\alpha_{ij}(\gamma)(x)]$ is a $d(\gamma) \times d(\gamma)$ matrix over $\mathfrak{A}$). We denote by $\mathfrak{A}^\alpha(\gamma)$ the range of the projection $\mathscr{P}^\alpha(\gamma)$ and by $\mathfrak{A}^\alpha_F$ the linear span of $\{\mathfrak{A}^\alpha(\gamma):\gamma \in \hat{G}\}$. If $x \in \mathfrak{A}^\alpha(\gamma)$, then

$$x = \sum_{i=1}^{d(\gamma)} \mathscr{P}^\alpha_{ii}(\gamma)(x), \quad \text{and}$$

$$\operatorname{span}\{\alpha_g(x):g \in G\} = \operatorname{span}\{\mathscr{P}^\alpha_{ij}(\gamma)(x):1 \leqq i, j \leqq d(\gamma)\}.$$

Note that $x \in \mathfrak{A}_F^\alpha$ if and only if span $\{\alpha_g(x):g \in G\}$ is finite dimensional. Following [**4**], we say an $n \times d(\gamma)$ matrix $[x^{ij}]$ over $\mathfrak{A}$ is in $\mathfrak{A}_n^\alpha(\gamma)$ if

$$\alpha_g([x^{ij}]) = [x^{ij}] \cdot (\mathbf{1}_{\mathfrak{A}} \otimes \gamma(g)).$$

Thus (iv) says that

$$[\mathscr{P}_{ij}^\alpha(\gamma)(x)] \in \mathfrak{A}_{d(\gamma)}^\alpha(\gamma) \quad \text{for } x \in \mathfrak{A}.$$

In preparation for Theorem 2.1, we review some facts about product type actions on UHF algebras. Let $\mathfrak{A} = \bigotimes_{\mathbf{N}} \mathbf{M}_n(\mathbf{C})$ be the UHF algebra of type $n^\infty$. An automorphism group $\alpha:G \to \mathrm{Aut}(\mathfrak{A})$ is said to be of product type if there is an automorphism group $\theta:G \to \mathrm{Aut}(\mathbf{M}_n(\mathbf{C}))$ such that

$$\alpha(g) = \bigotimes_{\mathbf{N}} \theta(g).$$

Define

$$\mathfrak{A}_0 = \mathbf{C} \cdot \mathbf{1}, \mathfrak{A}_m = \bigotimes_{k=1}^m \mathbf{M}_n(\mathbf{C}), \text{ and } \mathfrak{A}^0 = \bigcup_{m \in \mathbf{N}} \mathfrak{A}_m$$

(without closure). There is an embedding $\sigma \to u(\sigma)$ of the infinite symmetric group $S(\infty)$ in $\mathfrak{A}^0$ with the following properties:

(i) for $\sigma \in S(n) \subset S(\infty)$, and $a = a_1 \otimes \ldots \otimes a_n \in \mathfrak{A}_n$,

$$\mathrm{Ad}(u(\sigma))(a) = a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)},$$

and

(ii) $u(S(\infty))$ is contained in the fixed-point algebra for any product type automorphism group of $\mathfrak{A}$.

These matters are discussed in detail by Powers and Price [**25**]. Let $(\pi, \mathscr{H}, \Omega)$ be the GNS representation of $\mathfrak{A}$ associated with the trace tr. Identify $\mathfrak{A}$ with its image $\pi(\mathfrak{A})$. For integers $r > m \geqq 0$, let $S_{m,r} \subset S(\infty)$ be the subgroup fixing $\mathbf{N}\setminus\{m + 1, \ldots, r\}$ and define

$$\varphi_{m,r}(a) = \frac{1}{|S_{m,r}|} \sum_{\sigma \in S_{m,r}} \mathrm{Ad}(u(\sigma))(a)$$

for $a \in \mathfrak{A}$. Powers and Price show that in the trace representation,

$$\text{strong} \lim_{r \to \infty} \varphi_{m,r}(a) = \varphi_m(a),$$

for all $a \in \mathfrak{A}$, where $\varphi_m$ is the conditional expectation of $\mathfrak{A}$ onto $\mathfrak{A}_m$ with respect to the trace. In particular,

$$\mathrm{tr}(a)\mathbf{1} = \underset{r \to \infty}{\mathrm{st.lim}}\ \varphi_{0,r}(a).$$

Since strong convergence in the trace representation is equivalent to convergence with respect to the trace norm $\|x\|_2 = \mathrm{tr}(x^*x)^{1/2}$ (on bounded sets), $\mathfrak{A}$ is trace-norm dense in $\mathfrak{M} = \mathfrak{A}''$, and the limits above are also valid for $a \in \mathfrak{M}$. In particular,

$$u(S(\infty))' \cap \mathfrak{M} = \mathbf{C} \cdot \mathbf{1};$$

i.e., $S(\infty)$ acts ergodically on $\mathfrak{M}$. This conclusion appears as Lemma 4.2 in [**9**].

If $D \subset \mathfrak{A}$ are *-algebras, we use the notation $\mathrm{Der}(D, \mathfrak{A})$ for the set of *-derivations from $D$ into $\mathfrak{A}$.

THEOREM 2.1. *Let* $\alpha:G \to \mathrm{Aut}(\mathfrak{A})$ *be a product type action of a compact Lie group* $G$ *on the UHF* $C^*$*-algebra* $\mathfrak{A}$ *of type* $n^\infty$.

*Every *-derivation* $\delta:\mathfrak{A}_F^\alpha \to \mathfrak{A}$ *has closure generating a strongly continuous one-parameter group of *-automorphisms of* $\mathfrak{A}$. *Furthermore,* $\delta$ *has a unique decomposition*

$$\delta = \delta_0 + \tilde{\delta},$$

*where* $\bar{\delta}_0$ *generates a one-parameter subgroup of* $\alpha(G)$, *and* $\tilde{\delta}$ *is inner. The inner part* $\tilde{\delta}$ *is zero if and only if* $\delta_{\mathfrak{A}^\alpha} = 0$.

*Remarks*. 1. For each $n \in \mathbf{N} \cup \{\infty\}$, the restriction map $\delta \to \delta|_{\mathfrak{A}_F^\alpha}$ is a bijection of $\mathrm{Der}(C^n(\mathfrak{A}, \alpha), \mathfrak{A})$ onto $\mathrm{Der}(\mathfrak{A}_F^\alpha, \mathfrak{A})$, the inverse map being extension by closure; in particular, the decomposition $\delta = \delta_0 + \tilde{\delta}$ is also valid for $\delta \in \mathrm{Der}(C^n(\mathfrak{A}, \alpha), \mathfrak{A})$. These statements follow from the theorem because each $\delta \in \mathrm{Der}(C^n(\mathfrak{A}, \alpha), \mathfrak{A})$ is automatically continuous with respect to the $C^n$-topology on $C^n(\mathfrak{A}, \alpha)$, and $\mathfrak{A}_F^\alpha$ is $C^n$-dense in $C^n(\mathfrak{A}, \alpha)$. (See [ [**23**] and [**5**], Theorem 3.1].)

2. We repeat for emphasis that by a product type action we mean a restricted product type action

$$G \ni g \mapsto \bigotimes_n \theta(g).$$

The theorem is definitely not valid for unrestricted product actions

$$\prod G_n \ni (g_n) \mapsto \bigotimes_n \theta_n(g_n),$$

see [**7**, Example 5.1.4].

*Proof.* Consider $\mathfrak{A}$ acting in the trace representation with weak closure $\mathfrak{M}$. In this representation, $\alpha$ is implemented by a unitary representation of $G$ and so extends to a $\sigma$-weakly continuous action of $G$ on $\mathfrak{M}$. Let

$$\delta \in \mathrm{Der}(\mathfrak{A}_F^\alpha, \mathfrak{A}).$$

Since $\mathfrak{A}^\alpha$ is $AF$, it follows that there is a skew-adjoint $h \in \mathfrak{M}$ such that $\delta(a) = [h, a]$ for all $a \in \mathfrak{A}^\alpha$; use, for instance, Christensen's theorem [**14**, Theorem 2.3]. We noted above that $(\mathfrak{A}^\alpha)' \cap \mathfrak{M} = \mathbf{C} \cdot \mathbf{1}$ and therefore $h$ is unique up to addition of a scalar. Define

$$\tilde{\delta} = [h, a] \quad \text{for } a \in \mathfrak{A} \quad \text{and}$$

$$\delta_0(a) = \delta(a) - \tilde{\delta}(a) \quad \text{for } a \in \mathfrak{A}_F^\alpha.$$

Observe that

$$\tilde{\delta} = 0 \Leftrightarrow h \text{ is a scalar} \Leftrightarrow \delta_{|\mathfrak{A}^\alpha} = 0.$$

We now have a *-derivation $\delta_0:\mathfrak{A}_F^\alpha \to \mathfrak{M}$ such that $\delta_{0|\mathfrak{A}^\alpha} = 0$ and a bounded derivation $\tilde{\delta}:\mathfrak{A} \to \mathfrak{M}$ such that $\delta = \delta_0 + \tilde{\delta}$ on $\mathfrak{A}_F^\alpha$. We will prove that in fact $\delta_0(\mathfrak{A}_F^\alpha) \subset \mathfrak{A}_F^\alpha$ and $\delta_0$ has a closure generating a one-parameter subgroup of $\alpha(G)$.

The first major step in the proof of Theorem 2.5 below is to show that the restriction of $\delta_0$ to each space $\mathfrak{A}^\alpha(\gamma)$ is $\sigma$-weakly continuous and hence bounded. The proof in this special case is not much different so we will take a short-cut and consider it done, remarking only that it uses the fact recorded above that $(\mathfrak{M}^\alpha)' \cap \mathfrak{M} = \mathbf{C} \cdot \mathbf{1}$. Since $\alpha$ is of product type,

$$\mathfrak{A}^0 \subset \mathfrak{A}_F^\alpha \subset D(\delta_0).$$

Since $\delta_{0|\mathfrak{A}_m}$ is implemented by an element of $\mathfrak{M}$ for each $m \in \mathbf{N}$, we have

$$\text{tr} \circ \delta_{0|\mathfrak{A}^0} = 0;$$

but since for each $\gamma \in \hat{G}$, $\mathfrak{A}^0 \cap \mathfrak{A}^\alpha(\gamma)$ is dense in $\mathfrak{A}^\alpha(\gamma)$, and $\delta_0$ is bounded on $\mathfrak{A}^\alpha(\gamma)$, it follows that $\text{tr} \circ \delta_0 = 0$ on all of $\mathfrak{A}_F^\alpha$. Hence $\delta_0$ is weakly closable.

We can now employ the argument of Powers and Price [**25**] to show that $\delta_0$ leaves each matrix algebra $\mathfrak{A}_m$ invariant: If $a \in \mathfrak{A}_m$, then

$$a = \varphi_{m,r}(a) \quad \text{for all } r > m$$

and $\delta_0(\varphi_{m,r}(a)) = \varphi_{m,r}(\delta_0(a))$ converges to $\varphi_m(\delta_0(a))$ strongly. By weak closability of $\delta_0$,

$$\delta_0(a) = \varphi_m(\delta_0(a)).$$

It follows that $\delta_0(\mathfrak{A}^0) \subset \mathfrak{A}^0$ and that $\mathfrak{A}^0$ is a norm dense space of analytic elements for $\delta_0$. Since both $\delta$ and $\delta_0$ map $\mathfrak{A}^0$ into $\mathfrak{A}$, so does $\tilde{\delta}$, and since $\tilde{\delta}$ is bounded, this implies that $\tilde{\delta}(\mathfrak{A}) \subset \mathfrak{A}$. But then, because $\delta(\mathfrak{A}_F^\alpha) \subset \mathfrak{A}$, we have $\delta_0(\mathfrak{A}_F^\alpha) \subset \mathfrak{A}$. Because $\text{tr} \circ \delta_0 = 0$ and $\delta_0$ has a dense family of analytic vectors, we can conclude that $\bar{\delta}_0$ generates a strongly continuous one-parameter group of *-automorphisms of $\mathfrak{A}$, say $\{\beta_t\}$ [**12**, Theorem 3.2.57]. The group $\{\beta_t\}$ also extends to a $\sigma$-weakly continuous group of automorphisms of $\mathfrak{M}$. Since $\delta_{0|\mathfrak{A}^\alpha} = 0$, each $\beta_t$ is the identity on $\mathfrak{A}^\alpha$, and hence on $\mathfrak{M}^\alpha$. Each $\beta_t$ commutes with the ergodic group $\{\text{Ad}(u(\sigma)):\sigma \in S(\infty)\}$ of automorphisms of $\mathfrak{M}$, as do the automorphisms $\{\alpha(g):g \in G\}$, and therefore the duality theorem of Araki, Haag, Kastler, and Takesaki [**1**], [**29**, p. 207] implies that $\beta_t \in \alpha(G)$.

Finally, $\bar{\delta} = \bar{\delta}_0 + \tilde{\delta}$ is a generator by perturbation theory, and $\tilde{\delta}$ is inner by Sakai's theorem [**28**, Theorem 4.1.11]. The decomposition is unique because if $\delta = \delta_0 + \tilde{\delta}$ and $\delta = \delta_0' + \tilde{\delta}'$ were two such decompositions, then $\delta_0 - \delta_0'$ would be a bounded, hence inner, *-derivation killing $\mathfrak{A}^\alpha$. But any element implementing $\delta_0 - \delta_0'$ would be contained in $(\mathfrak{A}^\alpha)' \cap \mathfrak{A} = \mathbf{C} \cdot \mathbf{1}$, so $\delta_0 = \delta_0'$.

Our next theorem is an analogue of Theorem 2.1 for the Cuntz $C^*$-algebra $\mathfrak{O}_m$ $(m = 2, 3, \dots)$ and extends Theorem 3.3 of [**4**]. The $C^*$-algebra $\mathfrak{O}_m$ is generated by $m$ partial isometries $s_1, \dots, s_m$ satisfying

$$\sum_{i=1}^{m} s_i s_i^* = \mathbf{1}, \quad \text{and} \quad s_i^* s_j = \delta_{i,j} \cdot \mathbf{1}.$$

Since $m$ is fixed, we will denote this $C^*$-algebra by $\mathfrak{O}$ in the remainder of this discussion. The full unitary group $U(m)$ has a canonical action on $\mathfrak{O}$ which on the generators is given by

$$\tau(g)(s_i) = \sum_{j=1}^{m} g_{ji} s_j \quad (g \in U(m)).$$

The action of the circle $\mathbf{T} \cdot \mathbf{1} = \mathbf{Z}(U(m))$ is given simply by

$$\tau(t)(s_i) = t s_i.$$

The fixed point algebra $\mathcal{F}$ for the circle action, which is isomorphic to the UHF algebra of type $m^\infty$ [**15**], is evidently globally invariant under $U(m)$, and the induced action of $U(m)$ on $\mathcal{F}$ may be identified with the canonical product type action of $U(m)$ on $\bigotimes_{\mathbf{N}} \mathbf{M}_m(\mathbf{C})$, namely

$$g \to \bigotimes_{\mathbf{N}} \mathrm{Ad}(g).$$

Let

$$\omega = \mathrm{tr} \circ \int_{\mathbf{T}} \tau(t) dt,$$

where tr is the unique normalized trace on $\mathcal{F}$. Denote the GNS representation corresponding to $\omega$ by $(\pi, \mathcal{H}, \Omega)$; since $\omega$ is $\tau$-invariant, $\tau$ is implemented by a unitary representation of $U(m)$ on $\mathcal{H}$, and therefore extends to a $\sigma$-weakly continuous action on $\pi(\mathfrak{O})'' = \mathfrak{M}$. We henceforth consider $\mathfrak{O}$ as acting in this representation and suppress the notation $\pi$. The following is an extension of Theorem 3.2 in [**4**].

LEMMA 2.2. *Let $\mathfrak{O}$ be the Cuntz $C^*$-algebra generated by $m$ isometries, acting in the cyclic representation arising from the canonical state $\omega$. Let $\mathcal{F}$ be the gauge invariant subalgebra, $\mathcal{F} = \mathfrak{O}^{\mathbf{T}}$, and let $u:S(\infty) \to \mathcal{F}$ be the standard embedding of $S(\infty)$ in the* UHF *algebra $\mathcal{F}$.*

*Then $u(S(\infty))' \cap \mathfrak{O}'' = \mathbf{C} \cdot \mathbf{1}$.*

*Proof.* Let $\mathfrak{M}$ denote $\mathfrak{D}''$. First we observe that

$$\mathfrak{M}^{\mathbf{T}} \cap u(S(\infty))' = \mathbf{C} \cdot \mathbf{1}:$$

The representation of $\mathscr{F}$ on $[\mathscr{F}\Omega]$ is evidently equivalent to the trace representation, so if $x \in \mathfrak{M}^{\mathbf{T}} \cap u(S(\infty))'$, then the restriction of $x$ to $[\mathscr{F}\Omega]$ is a scalar, by the ergodicity of $S(\infty)$ in the trace representation (see above). Since $x \to x_{|[\mathscr{F}\Omega]}$ is faithful, $x \in \mathbf{C} \cdot \mathbf{1}$.

The remainder of the proof is like that of Theorem 3.2 in [**4**].

We need one more lemma before proceeding to Theorem 2.4. This cohomology result is also a useful complement to Theorem 2.5. We already used the same technique in the case of a compact abelian group $G$ in [**7**, Lemma 5.1].

LEMMA 2.3. *Let $\mathfrak{A}$ be a $C^*$ algebra acting on a Hilbert space $\mathscr{H}$, and let $\alpha:G \to \mathrm{Aut}(\mathfrak{A})$ be an action of a compact group $G$ on $\mathfrak{A}$. Let $\delta:\mathfrak{A}^\alpha \to \mathfrak{A}^\alpha_F$ be a derivation. Then*:

(i) *There is a finite subset $\Lambda \subset \hat{G}$ such that $\delta(\mathfrak{A}^\alpha) \subset \mathfrak{A}^\alpha(\Lambda)$.*

(ii) *There is an element $h$ in the $\sigma$-weak closure of $\mathfrak{A}^\alpha(\Lambda)$ such that*

$$\delta(a) = [h, a] \text{ for all } a \in \mathfrak{A}^\alpha.$$

*If $\delta$ is a *derivation, then $h$ can be chosen skew-adjoint.*

*Remark.* The same proof also shows the following: Suppose $\alpha:G \to \mathrm{Aut}(\mathfrak{A})$ extends to a $\sigma$-weakly continuous automorphism group of $\mathfrak{M} = \mathfrak{A}''$; if $\delta:\mathfrak{A}^\alpha \to \mathfrak{M}^\alpha_F$ is a derivation, then $\delta$ is implemented by an element of $\mathfrak{M}^\alpha_F$.

*Proof.* First note that for $f \in C(G)$,

$$a \mapsto \int_G f(g)\alpha_g(\delta(a))dg$$

is a derivation from $\mathfrak{A}^\alpha$ into $\mathfrak{A}^\alpha_F$. In particular for $\gamma \in \hat{G}$, $\mathscr{P}^\alpha(\gamma) \circ \delta$ and $\mathscr{P}^\alpha_{ij}(\gamma) \circ \delta$ are derivations. If (i) were false, then there would be a sequence $\gamma_i (i \in \mathbf{N})$ of distinct elements in $\hat{G}$ such that

$$\mathscr{P}^\alpha(\gamma_i) \circ \delta(\mathfrak{A}^\alpha) \neq (0).$$

The operator

$$\mathscr{P} = \sum_{i \in \mathbf{N}} \mathscr{P}^\alpha(\gamma_i)$$

makes sense on $\mathfrak{A}^\alpha_F$ and $\mathscr{P} \circ \delta$ is a derivation from $\mathfrak{A}^\alpha$ into span $\{\mathfrak{A}^\alpha(\gamma_i): i \in \mathbf{N}\}$. By Ringrose's theorem [**26**], $\mathscr{P} \circ \delta$ is continuous, so

$$X_n = \{a \in \mathfrak{A}^\alpha: \mathscr{P} \circ \delta(a) \in \text{span } \{\mathfrak{A}^\alpha(\gamma_i): 1 \leqq i \leqq n\}\}$$

is an increasing sequence of proper closed subspace of $\mathfrak{A}^\alpha$ with union equal to $\mathfrak{A}^\alpha$. This contradicts the Baire category theorem, and the contradiction proves (i).

Fix $\gamma \in \hat{G}$ of dimension $d$ and let

$$\pi(a) = a \otimes \mathbf{1}_d \quad \text{for } a \in \mathfrak{A}^\alpha.$$

For $1 \leqq i \leqq d$, the map

$$V_i(a) = \begin{bmatrix} \mathscr{P}^\alpha_{i1}(\gamma)(\delta(a)) \\ \cdot \\ \cdot \\ \cdot \\ \mathscr{P}^\alpha_{id}(\gamma)(\delta(a)) \end{bmatrix}$$

satisfies the derivation identity

$$V_i(ab) = \pi(a)V_i(b) + V_i(a)b,$$

and $V_i(a)^*V_i(b) \in \mathfrak{A}^\alpha$ for all $a, b \in \mathfrak{A}^\alpha$, since $V_i(a)^* \in \mathfrak{A}^\alpha_1(\bar{\gamma})$. Therefore a theorem of Christensen and Evans [**13**, Theorem 2.1] gives an element

$$h_i = \begin{bmatrix} h_{i1} \\ \cdot \\ \cdot \\ \cdot \\ h_{id} \end{bmatrix}$$

in the $\sigma$-weak closure of the linear span of $\{V_i(a)b:a, b \in \mathfrak{A}^\alpha\}$ such that

$$V_i(a) = h_i a - \pi(a)h_i \quad \text{for } a \in \mathfrak{A}^\alpha.$$

In particular,

$$\mathscr{P}^\alpha_{ii}(\gamma)(\delta(a)) = h_{ii}a - ah_{ii}, \quad \text{and}$$

$$\mathscr{P}^\alpha(\gamma) \circ \delta(a) = \sum_i \mathscr{P}^\alpha_{i\,i}(\gamma) \circ \delta(a) = \left[\sum_i h_{ii}, a\right].$$

Define

$$h(\gamma) = \sum_i h_{ii}.$$

It follows that $\delta$ is implemented by the element

$$h = \sum_{\gamma \in \Lambda} h(\gamma).$$

If $\delta$ is a *-derivation, it is also implemented by $\dfrac{h - h^*}{2}$.

The proof of the next theorem has a lot in common with that of Theorem 3.3 in [**4**]; the reader is referred to that paper for further details.

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THEOREM 2.4. *Let $\mathfrak{O}$ be the Cuntz $C^*$-algebra generated by m isometries* $\{s_1, \ldots, s_m\}$. *Let G be a compact subgroup of $U(m)$ and $\alpha:G \to \mathrm{Aut}(\mathfrak{O})$ the restriction to G of the canonical action $\tau:U(m) \to \mathrm{Aut}(\mathfrak{O})$.*

*Every $*$-derivation $\delta:\mathfrak{O}_F^\alpha \to \mathfrak{O}$ has a generator closure. Furthermore $\delta$ has a unique decomposition*

$$\delta = \delta_0 + \tilde{\delta},$$

*where $\bar{\delta}_0$ generates a one-parameter subgroup of $\alpha(G)$ and $\tilde{\delta}$ is inner. The inner part $\tilde{\delta}$ is zero if and only if $\delta_{|_{\mathfrak{O}^\alpha}} = 0$.*

*Remarks.* 1. As in Theorem 2.1, it follows that the decomposition $\delta = \delta_0 + \tilde{\delta}$ is also valid for

$$\delta \in \mathrm{Der}(C^n(\mathfrak{O}, \alpha), \mathfrak{O}), \quad \text{for } n \in \mathbf{N} \cup \{\infty\}.$$

2. (due to George Elliott, David E. Evans, and Palle E. T. Jorgensen). This decomposition is also unique with respect to the property that $\tilde{\delta}$ is approximately inner. This is because the generator $\delta_0$ of a one-parameter subgroup of the $U(m)$ action on $\mathfrak{O}$ is not approximately inner unless $\delta_0 = 0$. This can be seen as follows:

Let $\xi \in \mathbf{C}^m$ with $\|\xi\| = 1$, and let $\omega_\xi = \bigotimes_{\mathbf{N}} \langle \xi \mid \cdot\, \xi\rangle$ be the corresponding product state on $\mathfrak{O}^{\mathbf{T}} = \mathscr{F}$. Then if $h \in \mathscr{F}$,

$$s_i[h, s_i^*] \in \mathscr{F} \quad \text{for } i = 1, \ldots, m, \text{ and}$$

$$\sum_{i=1}^m \omega_\xi(s_i[h, s_i^*])$$

$$= \omega_\xi\left(\sum_{i=1}^m s_i h s_i^* - \sum_{i=1}^m s_i s_i^* h\right)$$

$$= \omega_\xi(h) - \omega_\xi(h) = 0,$$

where we used that

$$h \mapsto \sum_{i=1}^m s_i h s_i^*$$

is the one sided shift on $\mathscr{F} = \bigotimes_{\mathbf{N}} \mathbf{M}_m(\mathbf{C})$, $\omega_\xi$ is shift invariant, and

$$\sum_{i=1}^m s_i s_i^* = \mathbf{1}, \quad [4].$$

Next let $\delta_0$ be the generator of a one-parameter subgroup of the $U(m)$ action and let $X = [x_{ij}]$ be the corresponding (skew-adjoint) element of the Lie algebra of $U(m)$. Then

$$\sum_i \omega_\xi(s_i \delta_0(s_i^*))$$
$$= \sum_i \omega_\xi\left(s_i \sum_j s_j^* \cdot \bar{x}_{ji}\right)$$
$$= \sum_i \sum_j \bar{x}_{ji} \omega_\xi(s_i s_j^*)$$
$$= \sum_i \sum_j \bar{x}_{ji} \langle \xi | \, s_i s_j^* \xi \rangle$$
$$= \sum_i \sum_j \bar{x}_{ji} \bar{\xi}_i \xi_j$$
$$= \langle \xi | X^* \xi \rangle.$$

Now suppose that $\delta_0$ is approximately inner on the polynomial algebra generated by $\{s_i\}$; i.e., there exists a sequence $k_n$ in $\mathfrak{O}$ such that

$$\delta_0(x) = \lim_{n \to \infty} [k_n, x]$$

for $x$ in this algebra. Since $\mathbf{T}$ is the center of $U(m)$, $\delta_0$ commutes with $\tau_t$ for $t \in \mathbf{T}$, and thus

$$\delta_0(x) = \tau_t \delta_0 \tau_t^{-1}(x) = \lim_{n \to \infty} [\tau_t(k_n), x].$$

Integrating this over $\mathbf{T}$, we get

$$\delta_0(x) = \lim_{n \to \infty} [h_n, x]$$

where $h_n = \int_{\mathbf{T}} \tau_t(k_n) dt$ lies in $\mathscr{F}$.

But then for all $\xi \in \mathbf{C}^m$,

$$\langle \xi | X^* \xi \rangle$$
$$= \sum_i \omega_\xi(s_i \delta_0(s_i^*))$$
$$= \lim_{n \to \infty} \sum_i \omega_\xi(s_i [h_n, s_i^*]) = 0,$$

and hence $X = 0$; i.e., $\delta_0 = 0$.

In particular, this shows that any approximately inner derivation $\delta: \mathfrak{O}_F^\alpha \to \mathfrak{O}$ is actually inner. This situation is quite different from that of Theorem 2.1, where $\delta_0$ is always approximately inner.

*Proof of Theorem* 2.4. Consider $\mathfrak{O}$ acting in the GNS representation arising from canonical state $\omega$, with weak closure $\mathfrak{M}$. The fixed point

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algebra $\mathfrak{D}^\tau$ of the $U(m)$ action is an $AF$ algebra, so proceeding as in the proof of Theorem 2.2, we get a decomposition $\delta = \delta_0 + \tilde{\delta}$, where $\tilde{\delta}:\mathfrak{D} \to \mathfrak{M}$ is bounded and $\delta_0:\mathfrak{D}_F^\alpha \to \mathfrak{M}$ satisfies $\delta_{0|_{\mathfrak{D}^\tau}} = 0$. We will prove that in fact $\delta_{0|_{\mathfrak{D}^\alpha}} = 0$, $\delta_0(\mathfrak{D}_F^\alpha) \subset \mathfrak{D}_F^\alpha$, and $\bar{\delta}_0$ generates a one-parameter subgroup of $\alpha(G)$.

Define a matrix $L = [L_{ji}] \in \mathfrak{M} \otimes \mathbf{M}_m(\mathbf{C})$ by

$$L_{ji} = s_j^*\delta_0(s_i).$$

The discussion surrounding [**4**, Lemma 3.1] shows that

(i) $[\delta_0(s_1), \dots, \delta_0(s_m)] = [s_1, \dots, s_m] \cdot L$.

(ii) $L$ is skew-adjoint.

(iii) $L \in [\mathfrak{M} \cap (\mathfrak{D}^\tau)'] \otimes \mathbf{M}_m(\mathbf{C})$.

But since $\mathfrak{M} \cap (\mathfrak{D}^\tau) = \mathbf{C} \cdot \mathbf{1}$ (Lemma 2.2), $L$ is a skew adjoint matrix of scalars and

$$\{\exp(tL)\} \subset \mathbf{1} \otimes U(m).$$

Therefore $\delta_0$ agrees on the *-algebra $\mathscr{A}_0$ generated by $\{s_1, \dots, s_m\}$ with the generator $\delta_1$ of the one-parameter group $\{\tau(\exp(tL))\}$.

The first step in the proof of Theorem 2.5 shows that $\delta_0$ is bounded on each spectral subspace $\mathfrak{D}^\alpha(\gamma)(\gamma \in \hat{G})$. The key to this is that $\delta_{0|_{\mathfrak{D}^\alpha}}$ is bounded and that

$$\mathfrak{M} \cap (\mathfrak{D}^\alpha)' = \mathbf{C} \cdot \mathbf{1}.$$

Now for each $\gamma \in \hat{G}$ we have that $\delta_0 = \delta_1$ on $\mathscr{A}_0 \cap \mathfrak{D}^\alpha(\gamma)$, $\mathscr{A}_0 \cap \mathfrak{D}^\alpha(\gamma)$ is dense in $\mathfrak{D}^\alpha(\gamma)$, $\delta_0$ is bounded on $\mathfrak{D}^\alpha(\gamma)$ and $\delta_1$ is closed; it follows that $D(\delta_1) \supset \mathfrak{D}_F^\alpha$ and $\delta_1$ extends $\delta_0$. In particular

$$\delta_0(\mathfrak{D}_F^\alpha) \subset \mathfrak{D} \text{ and } \tilde{\delta}(\mathfrak{D}) \subset \mathfrak{D}.$$

Now we want to show that

$$\delta_{0|_{\mathfrak{D}^\alpha}} = 0.$$

The linear span of $\{\tau_g\delta_1\tau_g^{-1}:g \in U(m)\}$ is finite dimensional (say as operators on $C^\infty(\mathfrak{D}, \tau)$), and therefore span $\{\alpha_g\delta_0\alpha_g^{-1}:g \in G\}$ is finite dimensional (say as operators on $\mathscr{A}_0$). It follows by continuity that for $a \in \mathfrak{D}^\alpha$, $\{\alpha_g\delta_0(a):g \in G\}$ has finite dimensional span; that is,

$$\delta_0(\mathfrak{D}^\alpha) \subset \mathfrak{D}_F^\alpha.$$

Hence by Lemma 2.3, there is a skew adjoint $k \in \mathfrak{M}$ such that

$$\delta_0(a) = [k, a] \quad \text{for all } a \in \mathfrak{D}^\alpha.$$

But $\delta_0$ already satisfies $\delta_{0|_{\mathfrak{D}^\tau}} = 0$, and therefore

$$k \in \mathfrak{M} \cap (\mathfrak{D}^\tau)' = \mathbf{C} \cdot \mathbf{1} \quad \text{and} \quad \delta_{0|_{\mathfrak{D}^\alpha}} = 0$$

as desired.

The last paragraph implies that the automorphism group $\{\tau(\exp(tL))\}$ is the identity on $\mathfrak{D}^\alpha$ and hence also on $\mathfrak{M}^\alpha$. Since both $\alpha(G)$ and $\{\tau(\exp(tL))\}$ commute with the ergodic automorphism group $\{\mathrm{Ad}(u(\sigma)):\sigma \in S(\infty)\}$, the duality theorem of Araki et al., [**29**], shows that

$$\exp(t\delta_1) = \tau(\exp(tL)) \subset \alpha(G).$$

But then $\mathfrak{D}_F^\alpha$ is a dense invariant space of analytic vectors for $\delta_1$ and hence $\delta_1 = \bar{\delta}_0$.

The remaining points are taken care of exactly as in the last paragraphs of the proof of Theorem 2.2.

We now proceed to the main abstract result of this section.

THEOREM 2.5. *Let $\mathfrak{A}$ be a $C^*$-algebra, $\alpha:G \to \mathrm{Aut}(\mathfrak{A})$ a strongly continuous action of a compact group $G$ on $\mathfrak{A}$, and $\tau:H \to \mathrm{Aut}(\mathfrak{A})$ an action of a discrete group $H$ such that $[\alpha_g, \tau_h] = 0$ for all $g \in G$ and $h \in H$. Suppose that $\delta:\mathfrak{A}_F^\alpha \to \mathfrak{A}$* is a *-derivation such that $[\delta, \tau_h] = 0$ for all $h \in H$.

*Assume that $\{\mathfrak{A}, G \times H, \alpha \times \tau\}$ admits a faithful covariant representation $\{\pi, U \times V, \mathscr{H}\}$ such that*

$$(*) \quad \pi(\mathfrak{A})'' \cap \pi(\mathfrak{A}^\alpha)' \cap \{V(H)\}' = \mathbf{C} \cdot \mathbf{1}.$$

(i) *If $\delta_{|\mathfrak{A}^\alpha} = 0$, then $\delta$ has a closure generating a one-parameter subgroup of $\alpha(G)$.*

(ii) *If $\delta_{|\mathfrak{A}^\alpha}$ is implemented by an element of $\pi(\mathfrak{A})''$ and $H$ is amenable, then $\delta$ has a generator closure. Furthermore, $\delta$ has a unique decomposition*

$$\delta = \delta_0 + \tilde{\delta},$$

*where $\delta_0$ has a closure generating a one-parameter subgroup of $\alpha(G)$, and $\tilde{\delta}$ is bounded.*

*Remarks*. 1. The condition in (ii) that $\delta_{|\mathfrak{A}^\alpha}$ is implemented by an element of $\pi(\mathfrak{A})''$ means that there exists a skew adjoint element $h \in \pi(\mathfrak{A})''$ such that

$$\pi(\delta(a)) = [h, \pi(a)] \quad \text{for all } a \in \mathfrak{A}^\alpha.$$

It may be that this always holds; in any case, the condition is valid under the following circumstances:

a. $\delta(\mathfrak{A}^\alpha) \subset \mathfrak{A}_F^\alpha$ (Lemma 2.3).
b. $\pi(\mathfrak{A}^\alpha)''$ is injective or $\pi(\mathfrak{A})''$ is finite ([**14**, Theorem 2.3]).

2. Two special cases of the theorem are of importance, the case that $H$ is trivial and

$$\pi(\mathfrak{A})'' \cap \pi(\mathfrak{A}^\alpha)' = \mathbf{C} \cdot \mathbf{1},$$

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and the case that $H$ acts ergodically on $\pi(\mathfrak{A})''$. The $C^*$-dynamical systems considered in Theorems 2.1 and 2.4 can be regarded as instances of either special case; take the trace representation for the UHF algebra and the $\omega$-GNS representation for the Cuntz algebra, and in both cases take $H = S(\infty)$.

3. The case that $H$ acts ergodically on $\pi(\mathfrak{A})''$ gives an infinitesimal version of the duality theorem of Araki et. al., [**29**, p. 207].

4. In case $G$ is a compact Lie group, the conclusions are also valid for derivations defined on $C^n(\mathfrak{A}, \alpha)$ $(n \in \mathbf{N} \cup \{\infty\})$. See the remarks following Theorem 2.1.

5. (due to George A. Elliott). Part (i) of the theorem is rather similar to Theorem 3.1 of [**6**], which says that if $\tau$ is a strongly topologically transitive action of a group $H$ on a $C^*$-algebra $\mathfrak{A}$, $\alpha$ an action of a compact group $G$ on $\mathfrak{A}$ such that $[\alpha, \tau] = 0$ and $\delta$ is a *-derivation on $\mathfrak{A}$ with $D(\delta) = \mathfrak{A}_F^\alpha$, $\delta_{|\mathfrak{A}^\alpha} = 0$, and $[\delta, \tau] = 0$, then $\delta$ is closable and $\bar{\delta}$ generates a one-parameter subgroup of $\alpha(G)$. It may be that one of these theorems implies the other, but there are technical difficulties in establishing either implication. If there were an $\alpha \times \tau$-covariant representation $\pi$ such that $\tau$ remained strongly topologically transitive on $\pi(\mathfrak{A})''$, then $\tau$ would be ergodic on $\pi(\mathfrak{A})''$, and Theorem 2.5 would apply to give an alternate proof of Theorem 3.1 of [**6**]. On the other hand, if the hypothesis $(*)$ is satisfied in some $\alpha \times \tau$-covariant representation $\pi$, then $H$ together with the unitary group of $\pi(\mathfrak{A}^\alpha)$ define an ergodic action of a group $\tilde{H}$ on $\pi(\mathfrak{A})''$, and both $\alpha$ and $\delta$ commute with this action if $\delta_{|\mathfrak{A}^\alpha} = 0$. If ergodicity implied strong topological transitivity for a von Neumann algebra, then Theorem 2.5(i) would follow from Theorem 3.1 of [**6**] together with Observation 2 below.

*Proof of Theorem* 2.5. We drop the notation $\pi$ and consider $\mathfrak{A}$ as acting on $\mathscr{H}$; we put $\mathfrak{M} = \mathfrak{A}''$. To begin the proof let us take any $\delta \in \mathrm{Der}(\mathfrak{A}_F^\alpha, \mathfrak{M})$. Take $\gamma \in \hat{G}$ of dimension $d = d(\gamma)$, and define $\delta$ on $\mathfrak{A}_n^\alpha(\gamma)$ by

$$\delta([x^{ij}]) = [\delta(x^{ij})].$$

*Observation* 1. $\delta:\mathfrak{A}_1^\alpha(\gamma) \to \mathfrak{M}^d$ is $\sigma$-weakly closable.

*Proof of Observation* 1. The restriction of $\delta$ to $\mathfrak{A}^\alpha$ is bounded and in fact $\sigma$-weakly continuous, by an argument of Kadison [**12**, 3.2.24], or by [**26**]. Let $x_n$ be a net in $\mathfrak{A}_1^\alpha(\gamma)$ such that $x_n \to 0$ and $\delta(x_n) \to z$ $\sigma$-weakly. Then for all $y \in \mathfrak{A}_1^\alpha(\gamma)$, $x_n y^*$ is an element of $\mathfrak{A}^\alpha$ and

$$\delta(x_n y^*) = \delta(x_n)y^* + x_n\delta(y^*).$$

Taking $\sigma$-weak limits we get $zy^* = 0$, and therefore $zp = 0$, where $p \in \mathfrak{M} \otimes \mathbf{M}_d(\mathbf{C})$ is the range projection of $y^*y$. Hence $zE = 0$, where $E$ is the

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supremum in $\mathfrak{M} \otimes \mathbf{M}_d(\mathbf{C})$ of all such range projections. We claim that in fact $E = \mathbf{1}$, whence $z = 0$.

In this paragraph $y$ denotes an arbitrary element of $\mathfrak{A}_1^\alpha(\gamma)$ and $p$ is the range projection of $y^*y$. If $a \in \mathfrak{A}^\alpha$, then $y(a^* \otimes \mathbf{1}_d)$ is also contained in $\mathfrak{A}_1^\alpha(\gamma)$, so

$$[ (a \otimes \mathbf{1}_d)p\mathscr{H}^d] \subset E \cdot \mathscr{H}^d.$$

Hence

$$E \in (\mathfrak{M} \cap (\mathfrak{A}^\alpha)') \otimes \mathbf{M}_d(\mathbf{C}).$$

Secondly, if $h \in H$, then

$$\tau_h(y) = [\tau_h(y^1), \ldots, \tau_h(y^d)]$$

is also in $\mathfrak{A}_1^\alpha(\gamma)$. The range projection of $\tau_h(y)^*$ is $\tau_h(p)$, and it follows that $E$ is invariant under $\tau_h$,

$$E \in (\mathfrak{M}^\tau \cap (\mathfrak{A}^\alpha)') \otimes \mathbf{M}_d(\mathbf{C}),$$

which is $\mathbf{1} \otimes \mathbf{M}_d(\mathbf{C})$, by hypothesis $(*)$. Finally, since

$$\alpha_g(y^*y) = \mathrm{Ad}(\mathbf{1}_{\mathfrak{A}} \otimes \gamma(g)^*)(y^*y),$$

the range projection of $\alpha_g(y^*)\alpha_g(y)$ is

$$\mathrm{Ad}(\mathbf{1}_{\mathfrak{A}} \otimes \gamma(g)^*)(p).$$

But since $E$ is a matrix of scalars and $y = yE$,

$$\alpha_g(y) = \alpha_g(yE) = \alpha_g(y)\alpha_g(E) = \alpha_g(y)E.$$

Therefore $E$ dominates the range projection $\mathrm{Ad}(\mathbf{1}_{\mathfrak{A}} \otimes \gamma(g)^*)(p)$ of $\alpha_g(y^*)$, and hence $E$ also dominates the sup of all such projections as $y$ varies, namely $\mathrm{Ad}(\mathbf{1}_{\mathfrak{A}} \otimes \gamma(g)^*)(E)$. Thus $E$ is invariant under the group $\{\mathrm{Ad}(\mathbf{1}_{\mathfrak{A}} \otimes \gamma(g))\}$, and so $E = \mathbf{1}$, as $\gamma$ is irreducible. This completes the proof of Observation 1.

*Observation* 2. There is an extension of $\delta$ to a *-derivation $\delta:\mathfrak{M}_F^\alpha \rightarrow \mathfrak{M}$ which is $\sigma$-weakly continuous on each spectral subspace $\mathfrak{M}^\alpha(\gamma)$.

*Proof of Observation* 2. Since $\mathfrak{A}_1^\alpha(\gamma)$ is norm closed and $\delta$ is $\sigma$-weakly closable on $\mathfrak{A}_1^\alpha(\gamma)$, $\delta$ is bounded and $\sigma$-weakly continuous on bounded subsets of $\mathfrak{A}_1^\alpha(\gamma)$. It follows that $\delta$ extends by $\sigma$-weak continuity to $\mathfrak{M}_1^\alpha(\gamma)$; we have only to show that any $x \in \mathfrak{M}_1^\alpha(\gamma)$ is the $\sigma$-weak limit of a bounded net in $\mathfrak{A}_1^\alpha(\gamma)$. Write $x = [x^1, \ldots, x^d]$. By Kaplansky's density theorem, there is a bounded net $y_n$ in $\mathfrak{A}$ converging to $x^1$ $\sigma$-weakly. For each $n$ and for $1 \leqq i \leqq d$, define

$$z_n^i = \mathscr{P}_{1i}^\alpha(\gamma)(y_n).$$

Then

$$z_n = [z_n^1, \dots, z_n^d] \in \mathfrak{A}_1^\alpha(\gamma) \quad \text{and}$$
$$z_n^i \to \mathscr{P}_{1i}^\alpha(\gamma)(x^1) = x^1 \ \sigma\text{-weakly},$$

by $\sigma$-weak continuity of $\mathscr{P}_{1i}^\alpha$.

It also follows that $\delta$ is $\sigma$-weakly continuous on $\mathfrak{A}^\alpha(\gamma)$ and so extends by continuity to $\mathfrak{M}^\alpha(\gamma)$; in fact, the map

$$\mathfrak{A}^\alpha(\gamma) \ni x \mapsto [\mathscr{P}_{ij}^\alpha(\gamma)(x)] = [x^{ij}] \in \mathfrak{A}_d^\alpha(\gamma)$$

is $\sigma$-weakly continuous, as is the map

$$\delta:[x^{ij}] \mapsto [\delta(x^{ij})].$$

But $\delta(x) = \sum_i \delta(x^{ii})$ is then also $\sigma$-weakly continuous. Letting $\gamma$ vary in $\hat{G}$, we extend $\delta$ to $\cup_\gamma \mathfrak{M}^\alpha(\gamma)$, and then by linearity to $\mathfrak{M}_F^\alpha$. The extended $\delta$ is $\sigma$-weakly continuous on each set

$$\mathfrak{M}^\alpha(\Lambda) = \sum_{\gamma\in\Lambda}^{\oplus} \mathfrak{M}^\alpha(\gamma),$$

for $\Lambda$ a finite subset of $\hat{G}$. Given $\gamma_1, \gamma_2 \in \hat{G}$, there is a finite subset $\Lambda$ such that

$$\mathfrak{M}^\alpha(\gamma_1)\mathfrak{M}^\alpha(\gamma_2) \subset \mathfrak{M}^\alpha(\Lambda);$$

we can therefore conclude that $\delta$ satisfies the derivation identity on $\mathfrak{M}_F^\alpha$, by a standard argument: Given $x \in \mathfrak{A}^\alpha(\gamma_1)$ and $y \in \mathfrak{M}^\alpha(\gamma_2)$, one checks that

$$\delta(xy) = \delta(x)y + x\delta(y)$$

by approximating $y$ by a net in $\mathfrak{A}^\alpha(\gamma_2)$ and using the continuity of $\delta$ on $\mathfrak{M}^\alpha(\Lambda)$. A second, similar step establishes the identity for $x \in \mathfrak{M}^\alpha(\gamma_1)$ and $y \in \mathfrak{M}^\alpha(\gamma_2)$. It is evident that the extended $\delta$ also preserves adjoints.

*Observation* 3. Let $x = [x^{ij}]$ be an element of $\mathfrak{M}_n^\alpha(\gamma)$, and let $x = (xx^*)^{1/2} u$ be its polar decomposition. Then $u \in \mathfrak{M}_n^\alpha(\gamma)$.

*Proof of Observation* 3. On the one hand

$$\alpha_g(x) = x(\mathbf{1}_{\mathfrak{A}} \otimes \gamma(g)) = (xx^*)^{1/2}u(\mathbf{1}_{\mathfrak{A}} \otimes \gamma(g)),$$

and on the other hand,

$$\alpha_g(x) = \alpha_g(xx^*)^{1/2}\alpha_g(u) = (xx^*)^{1/2}\alpha_g(u).$$

Comparing these two expressions and using the uniqueness of the polar decomposition, we get

$$\alpha_g(u) = u(\mathbf{1}_{\mathfrak{A}} \otimes \gamma(g)),$$

and this proves the observation.

We now begin the proof of statement (ii), and take

$$\delta \in \operatorname{Der}(\mathfrak{A}_F^\alpha, \mathfrak{A})$$

such that $\delta_{|\mathfrak{A}^\alpha}$ is implemented by an element of $\mathfrak{M}$. Then there is a skew-adjoint $r_0 \in \mathfrak{M}$ such that

$$\delta(x) = [r_0, x] \quad \text{for all } x \in \mathfrak{M}^\alpha.$$

Every element in the $\sigma$-weak closure of the convex hull of $\{\tau_h(r_0):h \in H\}$ also implements $\delta$ on $\mathfrak{M}^\alpha$, and by amenability of $H$, this set contains an $H$-invariant element $r$. Define $\tilde{\delta}(x) = [r, x]$ for $x \in \mathfrak{M}$, and $\delta_0 = \delta - \tilde{\delta}$ on $\mathfrak{M}^\alpha_F$. Then $\delta_0:\mathfrak{M}^\alpha_F \to \mathfrak{M}$ is $\tau$-invariant and is zero on $\mathfrak{M}^\alpha$. We show that in fact $\delta_0(\mathfrak{A}^\alpha_F) \subset \mathfrak{A}^\alpha_F$, and $\overline{\delta}_0$ generates a one-parameter subgroup of $\alpha(G)$. The proof of this also proves statement (i).

*Observation* 4. For each $\gamma \in \hat{G}$, there is a skew-symmetric matrix of scalars

$$L(\gamma) \in \mathbf{1}_{\mathfrak{A}} \otimes \mathbf{M}_{d(\gamma)}(\mathbf{C})$$

such that $\delta_0(x) = xL$ for $x \in \mathfrak{M}^\alpha_1(\gamma)$.

*Proof of Observation* 4. Fix $\gamma \in \hat{G}$ of dimension $d$. Let $\mathbf{F}$ be the union over $n \in \mathbf{N}$ of the partial isometries in $\mathfrak{M}^\alpha_n(\gamma)$, and give $\mathbf{F}$ the following partial order:

$$u \leqq_{\mathbf{F}} v \quad \text{if } u^*u \leqq v^*v \text{ in } \mathfrak{M} \otimes \mathbf{M}_d(\mathbf{C}).$$

Observe that $\mathbf{F}$ is directed since if $u, v \in \mathbf{F}$ and $w$ is the partial isometry part of $\begin{bmatrix} u \\ v \end{bmatrix}$, then $w^*w$ dominates both $u^*u$ and $v^*v$. For each $u \in \mathbf{F}$, define

$$L(u) = u^*\delta_0(u) \in \mathfrak{M} \otimes \mathbf{M}_d(\mathbf{C}).$$

If $x \in \mathfrak{M}^\alpha_1(\gamma)$ has the polar decomposition $x = (xx^*)^{1/2}u$ and if $v \geqq_{\mathbf{F}} u$, then $x = xv^*v$ and

$$\delta_0(x) = xv^*\delta_0(v) = xL(v),$$

since $xv^*$ is a matrix over $\mathfrak{M}^\alpha$. Hence if $v \geqq_{\mathbf{F}} u$, then

$$u^*uL(u) = u^*\delta_0(u) = u^*uL(v). \tag{2.1}$$

We define an operator $L(\gamma)$ on $\cup_{u \in \mathbf{F}} u^*u \cdot \mathscr{H}^d \equiv W$ by putting

$$L(\gamma)\xi = -L(u)^*\xi \quad \text{for } \xi \in u^*u \cdot \mathscr{H}^d.$$

Note that $W$ is a vector space since $\mathbf{F}$ is directed and (2.1) implies that $L(\gamma)$ is well defined on $W$. We claim that $L(\gamma)$ is skew-symmetric on $W$. Given $\xi, \eta \in W$, choose $u \in \mathbf{F}$ such that

$$u^*u\xi = \xi \quad \text{and} \quad u^*u\eta = \eta.$$

Then

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$$\begin{aligned}&\langle \eta | L(\gamma)\xi \rangle \\ &= -\langle \eta | L(u)^* u^* u\xi \rangle \\ &= -\langle u^* u\eta | \delta_0(u)^* u\xi \rangle \\ &= -\langle u\eta | u\delta_0(u^*) u\xi \rangle \\ &= -\langle u\eta | [\delta_0(uu^*) - \delta_0(u)u^*] u\xi \rangle \\ &= \langle \delta_0(u)^* u\eta | u^* u\xi \rangle \\ &= \langle L(u)^* \eta | \xi \rangle \\ &= \langle -L(\gamma)\eta | \xi \rangle.\end{aligned}$$

Next we show that $L(\gamma)$ is affiliated with $(\mathfrak{M}^\alpha)' \otimes \mathbf{M}_d(\mathbf{C})$. If $v \in \mathfrak{M}^\alpha$ is unitary and $u \in \mathbf{F}$, then $u(v \otimes \mathbf{1}_d) \in \mathbf{F}$, so

$$(v^* \otimes \mathbf{1}_d)W = W.$$

Take $\xi \in W$ and choose $u \in \mathbf{F}$ such that $u^* u\xi = \xi$. Then

$$\begin{aligned}&(v \otimes \mathbf{1}_d)L(\gamma)(v^* \otimes \mathbf{1}_d)\xi \\ &= (v \otimes \mathbf{1}_d)L(\gamma)(v^* \otimes \mathbf{1}_d)u^* u\xi \\ &= -(v \otimes \mathbf{1}_d)\delta_0[(v^* \otimes \mathbf{1}_d)u^*]u\xi \\ &= -\delta_0(u)^* u\xi = -L(u)^*\xi = L(\gamma)\xi.\end{aligned}$$

The next step is to show that $L(\gamma)$ is affiliated with

$$\{V(h):h \in H\}' \otimes \mathbf{M}_d(\mathbf{C}).$$

(Recall that $h \mapsto V(h)$ is the unitary representation of $H$ on $\mathscr{H}$ implementing $\tau$ on $\mathfrak{M}$.) For $h \in H$ and $u \in \mathbf{F}$,

$$\begin{aligned}&(V(h) \otimes \mathbf{1}_d)u^* u\mathscr{H}^d \\ &= (V(h) \otimes \mathbf{1}_d)u^* u(V(h)^* \otimes \mathbf{1}_d)\mathscr{H}^d \\ &= \tau_h(u^*)\tau_h(u)\mathscr{H}^d.\end{aligned}$$

Since $\tau_h(u) \in \mathbf{F}$, $W$ is thus invariant under $\{(V(h) \otimes \mathbf{1}_d)\}$. Take $\xi \in W$ and choose $u$ so that $u^* u\xi = \xi$. Then for all $h \in H$,

$$\begin{aligned}&(V(h) \otimes \mathbf{1}_d)L(\gamma)(V(h)^* \otimes \mathbf{1}_d)\xi \\ &= (V(h) \otimes \mathbf{1}_d)L(\gamma)(V(h)^* \otimes \mathbf{1}_d)u^* u\xi \\ &= (V(h) \otimes \mathbf{1}_d)L(\gamma)\tau_h^{-1}(u^*)\tau_h^{-1}(u)(V(h)^* \otimes \mathbf{1}_d)\xi \\ &= -(V(h) \otimes \mathbf{1}_d)\delta_0(\tau_h^{-1}(u))^*\tau_h^{-1}(u)(V(h)^* \otimes \mathbf{1}_d)\xi \\ &= -(V(h) \otimes \mathbf{1}_d)\tau_h^{-1}(\delta_0(u)^* u)(V(h)^* \otimes \mathbf{1}_d)\xi \\ &= -\delta_0(u)^* u\xi = L(\gamma)\xi.\end{aligned}$$

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The last paragraph together with the hypothesis (*) imply that $L(\gamma)$ is affiliated with the finite matrix algebra $\mathbf{1}_{\mathfrak{A}} \otimes \mathbf{M}_d(\mathbf{C})$ and therefore $L(\gamma)$ is itself a matrix of scalars. Finally for $u \in \mathbf{F}$,

$$\begin{aligned} &u^*uL(\gamma) \\ &= -u^*uL(\gamma)^* \\ &= -(L(\gamma)u^*u)^* \\ &= (L(u)^*u^*u)^* \\ &= u^*uL(u). \end{aligned}$$

Hence if $x \in \mathfrak{M}_1^\alpha(\gamma)$ and $u \in \mathbf{F}$ is its partial isometry part, then

$$xL(\gamma) = xu^*uL(\gamma) = xu^*uL(u) = xL(u) = \delta_0(x).$$

This completes the proof of Observation 4.

Given $x \in \mathfrak{M}^\alpha(\gamma)$, let

$$[x^{ij}] = [\mathscr{P}_{ij}^\alpha(\gamma)(x)] \in \mathfrak{M}_d^\alpha(\gamma).$$

Then

$$[\delta_0(x^{ij})] = [x^{ij}] \cdot L(\gamma),$$

and hence

$$\delta_0(x) = \sum_{i=1}^d \delta_0(x^{ii})$$

is in the linear span of $\{x^{ij}\}$ and therefore in the (at most $d^2$) dimensional linear span of $\{\alpha_g(x):g \in G\}$. In particular, $\delta_0$ maps $\mathfrak{A}^\alpha(\gamma)$ into itself and $\mathfrak{M}^\alpha(\gamma)$ into itself, and these spaces consist of analytic elements for $\delta_0$. Furthermore, the faithful conditional expectation $\mathscr{P}^\alpha(0)$ onto the fixed point algebra $\mathfrak{M}^\alpha$ satisfies

$$\mathscr{P}^\alpha(0) \circ \delta_0 = 0,$$

so it follows from [**18**, Lemma 2.2] that $\delta_{0|_{\mathfrak{A}_F^\alpha}}$ has norm closure generating a strongly continuous one-parameter group $\beta_t$ of *-automorphisms of $\mathfrak{A}$. Similarly $(\delta_{0|_{\mathfrak{M}_F^\alpha}})^{-\sigma\text{-weak}}$ generates a $\sigma$-weakly continuous one-parameter automorphism group of $\mathfrak{M}$, which we also call $\beta_t$. Each $\beta_t$ is the identity on $\mathfrak{M}^\alpha$, since $\delta_{0|_{\mathfrak{M}^\alpha}} = 0$, and $\beta_t$ also commutes with $\tau(H)$. Thus $\beta_t$ commutes with the ergodic group of automorphisms of $\mathfrak{M}$ generated by

$$\{\mathrm{Ad}(u):u \in \mathscr{U}(\mathfrak{M}^\alpha)\} \cup \{\mathrm{Ad}(V(h)):h \in H\},$$

as does $\alpha(G)$, and by the duality theorem of Araki et al., [**29**], $\beta_t$ is a one-parameter subgroup of $\alpha(G)$.

This completes the proof except for the uniqueness statement in (ii). Suppose $\delta = \delta_0 + \tilde{\delta} = \delta_0' + \tilde{\delta}'$ are two decompositions. Then the

$\sigma$-weakly continuous extension of $\Delta = \tilde{\delta} - \tilde{\delta}'$ to $\mathfrak{M}$ commutes with $\tau(H)$ and kills $\mathfrak{M}^\alpha$. By the derivation theorem of Kadison and Sakai,

$$\Delta = \text{ad}(r) \quad \text{for } r \in \mathfrak{M} \cap (\mathfrak{M}^\alpha)',$$

and, averaging over the amenable group $H$, $r$ can be chosen in $\mathfrak{M}^\tau \cap (\mathfrak{M}^\alpha)' = \mathbf{C} \cdot \mathbf{1}$. Thus $\Delta = 0$.

**3. Abelian compact group actions.** Let $\alpha$ be an action of a compact abelian group $G$ on a $C^*$-algebra $\mathfrak{A}$, and let $\delta$ be a *-derivation defined on the $G$-finite elements $\mathfrak{A}_F^\alpha$ such that $\delta(\mathfrak{A}_F^\alpha) \subset \mathfrak{A}_F^\alpha$. We will show that under some circumstances $\delta$ is a generator, and the core of the proof is to show that $\mathfrak{A}_F^\alpha$ consists of analytic elements for $\delta$. To prove this we will consider a decomposition $\delta = \delta_0 + \tilde{\delta}$ similar to the one considered in Section 2, but where $\delta_0$ is now characterized only by $\delta_{0|_{\mathfrak{A}^\alpha}} = 0$. We thus have to study the structure of *-derivations $\delta$ with the properties

$$D(\delta) = \mathfrak{A}_F^\alpha \quad \text{and} \quad \delta_{|\mathfrak{A}^\alpha} = 0.$$

In case that the dynamics satisfy the condition $\Gamma$ of [7], i.e.,

$$\mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^* = \mathfrak{A}^\alpha \quad \text{for all } \gamma \in \hat{G},$$

it was proved in [7] that these derivations are characterized by a certain cocycle $\hat{G} \ni \gamma \mapsto L(\gamma)$ with values in the relative commutant of $\mathfrak{A}^\alpha$ in the multiplier algebra $M(\mathfrak{A})$ of $\mathfrak{A}$. The cocycle relation is

$$\text{(c)} \qquad L(\gamma_1 + \gamma_2) = L(\gamma_1) + \beta_{\gamma_1}(L(\gamma_2))$$

where $\beta_\gamma$ is the unique automorphism of $(\mathfrak{A}^\alpha)' \cap M(\mathfrak{A})$ determined by

$$\beta_\gamma(a)x = xa$$

for all $x \in \mathfrak{A}^\alpha(\gamma)$ and $a \in (\mathfrak{A}^\alpha)' \cap M(\mathfrak{A})$. The relation between $\delta$ and $L(\gamma)$ is given by

$$\delta(x) = L(\gamma)x$$

for all $x \in \mathfrak{A}^\alpha(\gamma)$. Since $\delta$ preserves adjoints, $L$ satisfies the additional relation

$$L(-\gamma) = \beta_{-\gamma}(L(\gamma)^*),$$

which implies that $L$ is skew-adjoint, $L(\gamma)^* = -L(\gamma)$.

We will now extend this description of $\delta$ to the case where condition $\Gamma$ is not fulfilled. To this end, we suppose that $\mathfrak{A}$ is faithfully represented on a Hilbert space $\mathscr{H}$ in such a fashion that $\alpha$ extends to a $\sigma$-weakly continuous action (also called $\alpha$) of $G$ on $\mathfrak{M} = \mathfrak{A}''$. (We say that the representation is $\alpha$-covariant.) For each $\gamma \in \hat{G}$, let $E(\gamma)$ be the projection onto

$$[\mathfrak{M}^\alpha(\gamma)\mathscr{H}] = [\mathfrak{A}^\alpha(\gamma)\mathscr{H}].$$

Under a very weak (and possibly unnecessary) regularity condition on $\delta$, we obtain bounded operators $L(\gamma)(\gamma \in \hat{G})$ on $\mathscr{H}$ such that

$$\delta(x) = L(\gamma)x \quad \text{for all } x \in \mathfrak{A}^{\alpha}(\gamma).$$

In general, these operators satisfy only a certain partial cocycle relation. However, in case all of the projections $E(\gamma)$ are equal to **1**, the regularity condition mentioned before is automatic and the full cocycle relation (c) is satisfied. This will be true for any $\alpha$-covariant representation in case a weakened form of $\Gamma$ is satisfied:

$$(\Gamma_n) \quad \overline{\mathfrak{A}^{\alpha}(\gamma)\mathfrak{A}^{\alpha}(\gamma)^*}^{\|\cdot\|} = \mathfrak{A}^{\alpha} \quad \text{for all } \gamma \in \hat{G}.$$

All of the conclusions of [7] and [**30**] which depended on condition $\Gamma$ and the existence of a cocycle $L$ determining $\delta$ can be recovered when the weaker condition $\Gamma_n$ is satisfied. See, for example Theorem 3.4 of this section, which generalizes the theorem in [**30**]; the proof we give here is shorter and depends in part on ideas from [**4**].

Let us now proceed with the construction of $L$. Note that

$$E(\gamma) = [\mathfrak{M}^{\alpha}(\gamma)\mathscr{H}]$$

is a central projection in $\mathfrak{M}^{\alpha} = (\mathfrak{A}^{\alpha})''$, and we have that

$$\overline{\mathfrak{M}^{\alpha}(\gamma)\mathfrak{M}^{\alpha}(\gamma)^*}^{\sigma\text{-weak}} = \mathfrak{M}^{\alpha} \cdot E(\gamma) = E(\gamma) \cdot \mathfrak{M}^{\alpha},$$

since the left hand side is a closed ideal in $\mathfrak{M}^{\alpha}$.

Let $\mathscr{C}(\gamma)$ be the relative commutant of $\mathfrak{M}^{\alpha} \cdot E(\gamma)$ in $E(\gamma) \cdot \mathfrak{M} \cdot E(\gamma)$. Then, for each $\gamma$ there exists a *-isomorphism

$$\beta_{\gamma}: \mathscr{C}(-\gamma) \to \mathscr{C}(\gamma)$$

uniquely defined by

$$\beta_{\gamma}(a)x = xa$$

for $a \in \mathscr{C}(-\gamma)$, and $x \in \mathfrak{M}^{\alpha}(\gamma)$. The proof of this is exactly as the proof of Lemma 1.5 of [**10**]; the assumption there that $a$ is contained in the center of $\mathfrak{M}^{\alpha} \cdot E(\gamma)$ is not essential for the proof and can be replaced by the assumption that $a \in \mathscr{C}(-\gamma)$.

LEMMA 3.1. *Adopt the assumptions and notation of the previous paragraphs, and let $\delta$ be a *-derivation with the properties*

$$\delta: \mathfrak{A}^{\alpha}_F \to \mathfrak{M},$$

$$\delta_{|\mathfrak{A}^{\alpha}} = 0.$$

*Assume in addition that for all $\gamma \in \hat{G}$ and all $x \in \mathfrak{A}^{\alpha}(\gamma)$,*

$$E(\gamma)\delta(x) = \delta(x).$$

*Then*:

(i) $\delta_{|\mathfrak{A}^\alpha(\gamma)}$ *is bounded and σ-weakly continuous for each* $\gamma$, *and therefore* $\delta$ *extends to a *-derivation on* $\mathfrak{M}_F^\alpha$ *which is σ-weakly continuous on each spectral subspace* $\mathfrak{M}^\alpha(\gamma)$.

(ii) *For each* $\gamma \in \hat{G}$, *there is a skew-adjoint operator* $L(\gamma) \in \mathscr{C}(\gamma)$ *such that*

$$\delta(x) = L(\gamma)x \quad \textit{for all } x \in \mathfrak{M}^\alpha(\gamma).$$

*If* $E(\gamma_1, \gamma_2)$ *denotes the projection*

$$E(\gamma_1, \gamma_2) = E(\gamma_1)\beta_{\gamma_1}(E(-\gamma_1)E(\gamma_2))$$

*then* $L$ *satisfies the partial cocycle relation*

$$L(\gamma_1 + \gamma_2)E(\gamma_1, \gamma_2) = L(\gamma)E(\gamma_1, \gamma_2) + \beta_{\gamma_1}(E(-\gamma_1)L(\gamma_2))E(\gamma_1, \gamma_2)$$

*and the skew-adjointness relation*:

$$L(-\gamma) = \beta_{-\gamma}(L(\gamma)^*).$$

*Remark* 3.2. We do not know whether the condition

$$E(\gamma)\delta(x) = \delta(x) \quad \text{for } x \in \mathfrak{A}^\alpha(\gamma)$$

is automatically satisfied, but the condition is fulfilled in the following cases:

1. $E(\gamma) = \mathbf{1}$ for all $\gamma \in \hat{G}$ (trivially).

2. The derivation $\delta$ is closable, or only: $\delta_{|\mathfrak{A}^\alpha(\gamma)}$ is bounded for each $\gamma \in \hat{G}$.

The ideal $\mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^*$ in $\mathfrak{A}^\alpha$ contains an approximate identity of the form

$$e_\tau = \sum_i a_i^\tau a_i^{\tau *},$$

with $a_i^\tau \in \mathfrak{A}^\alpha(\gamma)$; see the proof of Lemma 4.4 in [**3**]. Hence for $x \in \mathfrak{A}^\alpha(\gamma)$,

$$x = \lim_{\tau \to \infty} e_\tau x,$$

and if $\delta$ is bounded on $\mathfrak{A}^\alpha(\gamma)$, then

$$\delta(x) = \lim_{\tau \to \infty} \delta(e_\tau x) = \lim_{\tau \to \infty} e_\tau \delta(x).$$

As $e_\tau \leqq E(\gamma)$ for all $\tau$, the condition $\delta(x) = E(\gamma)\delta(x)$ follows.

3. $\mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^*\mathfrak{A}^\alpha(\gamma) = \mathfrak{A}^\alpha(\gamma)$ for all $\gamma \in \hat{G}$.

It follows from the existence of an approximate identity as in (2) that the left hand side is always dense in $\mathfrak{A}^\alpha(\gamma)$, and we do not know any

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example where equality does not hold.

If the condition is fulfilled, then each $x$ in $\mathfrak{A}^\alpha(\gamma)$ has the form

$$x = \sum_i y_i z_i,$$

where $y_i \in \mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^*$, $z_i \in \mathfrak{A}^\alpha(\gamma)$, and the sum is finite. Then

$$\delta(x) = \sum_i y_i \delta(z_i) = E(\gamma)\delta(x).$$

*Remark* 3.3. If one assumes from the outset that $\delta(\mathfrak{A}_F^\alpha) \subset \mathfrak{A}$, then it follows that $L(\gamma)$ lies in the relative commutant $\mathscr{D}(\gamma)$ of

$$\overline{\mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^*}^{\|\|}$$

in the multiplier algebra of

$$\overline{\mathfrak{A}^\alpha(\gamma) \cdot \mathfrak{A} \cdot \mathfrak{A}^\alpha(\gamma)^*}^{\|\|}.$$

In this case $\beta_\gamma$ defines a *-isomorphism between $\mathscr{D}(-\gamma)$ and $\mathscr{D}(\gamma)$. Conversely, a mapping

$$\hat{G} \ni \gamma \mapsto L(\gamma) \in \mathscr{D}(\gamma)$$

with the cocycle and skew-adjointness properties defines a *-derivation $\delta$ with

$$D(\delta) = \mathfrak{A}_F^\alpha \quad \text{and} \quad \delta_{|\mathfrak{A}^\alpha} = 0.$$

This follows from the proof; see also [**3**], [**10**], [7] and [**24**].

*Proof of Lemma* 3.1. Let $e_\tau$ be an approximate identity in $\mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^*$ of the sort described in Remark 3.2. Then

$$\lim_{\tau\to\infty} e_\tau = E(\gamma) \quad \text{strongly on } \mathscr{H}.$$

Define

$$L_\tau = \sum_i \delta(a_i^\tau)a_i^{\tau*}x.$$

If $x \in \mathfrak{A}^\alpha(\gamma)$, then

$$\begin{aligned} L_\tau x &= \sum_i \delta(a_i^\tau)a_i^{\tau}{}^*x \\ &= \sum_i \delta(a_i^\tau a_i^{\tau*}x) \\ &= \sum_i a_i^\tau a_i^{\tau*}\delta(x) = e_\tau\delta(x). \end{aligned}$$

Thus if $x$ has the particular form $x = ay$, where $a \in \mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^*$ and $y \in \mathfrak{A}^\alpha(\gamma)$, then

$$\lim_{\tau\to\infty} L_\tau ay = \lim_{\tau\to\infty} e_\tau a\delta(y) = a\delta(y) = \delta(ay),$$

where the limit exists in norm. Hence we may define an operator $L_0(\gamma)$ from $E(\gamma)\mathscr{H}$ into $E(\gamma)\mathscr{H}$ with domain

$$D(L_0(\gamma)) = \mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^*\mathfrak{A}^\alpha(\gamma)\mathscr{H}$$

by

$$L_0(\gamma)\left(\sum_i x_i\xi_i\right) = \lim_{\tau\to\infty} L_\tau\left(\sum_i x_i\xi_i\right) = \sum_i \delta(x_i)\xi_i$$

for $x_i \in \mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^*\mathfrak{A}^\alpha(\gamma)$ and $\xi_i \in \mathscr{H}$. As

$$0 = \sum_i \delta(a_i^\tau a_i^{\tau *}) = \sum_i (\delta(a_i^\tau)a_i^{\tau *} + a_i^\tau\delta(a_i^{\tau *})) = L_\tau + L_\tau^*,$$

each $L_\tau$ is skew-adjoint; hence $L_0(\gamma)$ is skew symmetric and therefore closable. Denote the closure of $L_0(\gamma)$ by $L(\gamma)$.

Since $\mathfrak{A}^\alpha(\gamma)$ is a module over $\mathfrak{A}^\alpha$, it follows that for $x \in \mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^*\mathfrak{A}^\alpha(\gamma)$, $a \in \mathfrak{A}^\alpha$, and $\xi \in \mathscr{H}$, $ax\xi \in D(L(\gamma))$ and

$$L(\gamma)ax\xi = \delta(ax)\xi = a\delta(x)\xi = aL(\gamma)x\xi.$$

As $L(\gamma)$ is closed the relation

$$L(\gamma)a\xi = aL(\gamma)\xi$$

holds for all $a \in (\mathfrak{A}^\alpha)'' = \mathfrak{M}^\alpha$ and $\xi \in D(L(\gamma))$; i.e., $L(\gamma)$ is affiliated with $(\mathfrak{M}^\alpha)'E(\gamma)$.

Next take any $x \in \mathfrak{A}^\alpha(\gamma)$ and $\xi \in \mathscr{H}$. Then (with $e_\tau$ the approximate identity used above)

$$e_\tau x\xi \in D(L(\gamma)) \text{ for all } \tau, \quad \text{and}$$

$$L(\gamma)e_\tau x\xi = \delta(e_\tau x)\xi = e_\tau\delta(x)\xi.$$

Now

$$\lim_{\tau\to\infty} e_\tau x\xi = x\xi,$$

and

$$\lim_{\tau\to\infty} e_\tau\delta(x)\xi = E(\gamma)\delta(x)\xi,$$

and hence, as $L(\gamma)$ is closed,

$$x\xi \in D(L(\gamma)) \quad \text{and} \quad L(\gamma)x\xi = E(\gamma)\delta(x)\xi.$$

Thus

$$L(\gamma)x = E(\gamma)\delta(x)$$

for all $x \in \mathfrak{A}^{\alpha}(\gamma)$.

Using the (presumably redundant) technical assumption

$$E(\gamma)\delta(x) = \delta(x),$$

we have

$$L(\gamma)x = \delta(x)$$

for all $x \in \mathfrak{A}^{\alpha}(\gamma)$.

We now argue that $\delta_{|_{\mathfrak{A}^{\alpha}(\gamma)}}$ is bounded. If $x_n \in \mathfrak{A}^{\alpha}(\gamma)$ is a sequence such that $x_n \to 0$ and $\delta(x_n) \to y$, then, for all $\xi \in \mathscr{H}$,

$$x_n\xi \to 0$$

while

$$L(\gamma)x_n\xi = \delta(x_n)\xi \to y\xi.$$

As $L(\gamma)$ is closed, it follows that $y\xi = 0$ and so $y = 0$. Thus $\delta_{|_{\mathfrak{A}^{\alpha}(\gamma)}}$ is closed and therefore bounded.

Next we observe that $\mathfrak{M}^{\alpha}(\gamma)\mathscr{H} \subset D(L(\gamma))$ and that $L(\gamma)\mathfrak{M}^{\alpha}(\gamma) \subset \mathfrak{M}$. Let $x \in \mathfrak{M}^{\alpha}(\gamma)$. Using the Kaplansky density theorem and applying the $\sigma$-weakly continuous projection

$$\mathscr{P}^{\alpha}(\gamma) = \int_G \overline{\langle \gamma, g \rangle} \alpha_g dg$$

from $\mathfrak{M}$ onto $\mathfrak{M}^{\alpha}(\gamma)$, we can find a net $x_\tau$ in $\mathfrak{A}^{\alpha}(\gamma)$ such that $\|x_\tau\| \leqq \|x\|$ and $x_\tau$ converges weakly to $x$. Considering the boundedness of $\delta$ on $\mathfrak{A}^{\alpha}(\gamma)$ and the weak compactness of the unit ball of $\mathfrak{M}$, and passing to a subnet of $x_\tau$, we can suppose that $\delta(x_\tau)$ converges to some $y \in \mathfrak{M}$. But then as $L(\gamma)$ is closed with respect to the weak topology on $\mathscr{H}$ and

$$\delta(x_\tau) = L(\gamma)x_\tau \quad \text{for all } \tau,$$

it follows by limiting that

$$x\mathscr{H} \subset D(L(\gamma)) \text{ and } y = L(\gamma)x.$$

We can now define an operator $\delta$ from $\mathfrak{M}_F^{\alpha}$ into $\mathfrak{M}$ by

$$\delta(x) = L(\gamma)x \quad \text{for } x \in \mathfrak{M}^{\alpha}(\gamma).$$

As $L(\gamma)$ is closed it follows as above that $\delta$ is $\sigma$-weakly closed and hence $\sigma$-weakly continuous on each spectral subspace $\mathfrak{M}^{\alpha}(\gamma)$. The operator $\delta$ extends our original $\delta$, and it follows by taking limits that the extended $\delta$ is a *-derivation. This completes the proof of (i).

We already proved that $L(\gamma)$ is affiliated with $(\mathfrak{M}^{\alpha})'E(\gamma)$; but as $L(\gamma)$ is a limit of operators in $E(\gamma) \cdot \mathfrak{M} \cdot E(\gamma)$, it is also affiliated with

$E(\gamma) \cdot \mathfrak{M} \cdot E(\gamma)$, and hence with $\mathscr{C}(\gamma)$. Our goal is now to show that $L(\gamma)$ is bounded.

If $x \in \mathfrak{M}^\alpha(\gamma)$, and $x = u|x|$ is its polar decomposition, then $u \in \mathfrak{M}^\alpha(\gamma)$ and $|x| \in \mathfrak{M}^\alpha$. If $\{u_\tau\}$ is a family of partial isometries in $\mathfrak{M}^\alpha(\gamma)$ with mutually orthogonal initial projections $u_\tau^* u_\tau \leqq E(-\gamma)$ and mutually orthogonal range projections $u_\tau u_\tau^* \leqq E(\gamma)$, then

$$w = \sum_\tau u_\tau$$

is a partial isometry in $\mathfrak{M}^\alpha(\gamma)$, and by Zorn's lemma, there exists a maximal partial isometry in $\mathfrak{M}^\alpha(\gamma)$. Maximality is characterized by the property

$$(E(\gamma) - uu^*)\mathfrak{M}^\alpha(\gamma)(E(-\gamma) - u^*u) = 0,$$

since if this space is not zero, then it contains a partial isometry which can be added to $u$, and, on the other hand, if $w$ is a partial isometry properly containing $u$, then

$$(E(\gamma) - uu^*)w(E(-\gamma) - u^*u) \neq 0.$$

Next we show that if $u$ is a maximal partial isometry in $\mathfrak{M}^\alpha(\gamma)$, then there exists a projection $E$ in the center of $\mathfrak{M}^\alpha E(\gamma)$ with the property

$$uu^*(E(\gamma) - E) = E(\gamma) - E$$

and

$$u^*u\beta_{-\gamma}(E) = \beta_{-\gamma}(E).$$

This can be seen as follows: Let $E$ be the central support of the projection $E(\gamma) - uu^*$ in $\mathfrak{M}^\alpha E(\gamma)$. Then clearly

$$uu^*(E(\gamma) - E) = E(\gamma) - E,$$

while $uu^*P \lneqq P$ whenever $P \leqq E$ is a non-zero projection in the center of $\mathfrak{M}^\alpha$. Suppose now that

$$\beta_{-\gamma}(E)(E(-\gamma) - u^*u) \neq 0;$$

then

$$\begin{aligned} 0 &\neq \mathfrak{M}^\alpha(\gamma)\beta_{-\gamma}(E)(E(-\gamma) - u^*u) \\ &= E\mathfrak{M}^\alpha(\gamma)\beta_{-\gamma}(E)(E(-\gamma) - u^*u). \end{aligned}$$

If $x$ is a non-zero element in the latter space, then

$$\begin{aligned} x &= uu^*x + (E(\gamma) - uu^*)x \\ &= uu^*Ex + (E(\gamma) - uu^*)Ex, \end{aligned}$$

and hence either

1. $(E(\gamma) - uu^*)Ex \neq 0$, or
2. $uu^*Ex \neq 0$.

In case 1, the condition

$$(E(\gamma) - uu^*)\mathfrak{M}^\alpha(\gamma)(E(\gamma) - u^*u) = 0$$

is contradicted. In case 2, as $uu^*P \lneqq P$ for each central projection $P \leqq E$ in $\mathfrak{M}^\alpha$, there is an element $y \in \mathfrak{M}^\alpha$ with

$$\begin{aligned} 0 &\neq (E(\gamma) - uu^*)yuu^*Ex \\ &= (E(\gamma) - uu^*)yuu^*Ex(E(\gamma) - u^*u), \end{aligned}$$

and since $yuu^*Ex \in \mathfrak{M}^\alpha(\gamma)$, this again contradicts the maximality of $u$. This contradiction shows that

$$u^*u\beta_{-\gamma}(E) = \beta_{-\gamma}(E).$$

It now follows from this and the other identity

$$uu^*(E(\gamma) - E) = E(\gamma) - E$$

that any $x \in \mathfrak{M}^\alpha(\gamma)$ has the decomposition

$$\begin{aligned} x &= (E(\gamma) - E)x + Ex \\ &= (E(\gamma) - E)x + x\beta_{-\gamma}(E) \\ &= (E(\gamma) - E)uu^*x + xu^*u\beta_{-\gamma}(E) \end{aligned}$$

and as $u^*x$ and $xu^*$ lie in $\mathfrak{M}^\alpha$, we get

$$\begin{aligned} L(\gamma)x &= \delta(x) \\ &= (E(\gamma) - E)\delta(u)u^*x + xu^*\delta(u)\beta_{-\gamma}(E) \\ &= [\,(E(\gamma) - E)\delta(u)u^* + \beta_\gamma(u^*\delta(u)\,)E] \cdot x. \end{aligned}$$

It follows that

$$L(\gamma) = (E(\gamma) - E)\delta(u)u^* + \beta_\gamma(u^*\delta(u)\,)E,$$

and $L(\gamma)$ is a bounded element in $(\mathfrak{M}^\alpha)' \cap \mathfrak{M}$.

The partial cocycle relation is now a straightforward consequence of the derivation property of $\delta$, and the skew-adjointness relation follows from $\delta(x^*) = \delta(x)^*$; see [**7**], proof of Proposition 2.3.

THEOREM 3.4. *Let $\alpha$ be an action of a compact abelian Lie group $G$ on a $C^*$-algebra $\mathfrak{A}$, and assume that there exists a faithful $G$-covariant representation of $\mathfrak{A}$ such that for each $\gamma \in \hat{G}$ the range projection of the ideal $\mathfrak{A}^\alpha(\gamma)\mathfrak{A}^\alpha(\gamma)^*$ in $\mathfrak{A}^\alpha$ is equal to* $\mathbf{1}$.

*Let $\delta$ be a $*$-derivation with domain $D(\delta)$ equal to the algebra $\mathfrak{A}_F^\alpha$ of $G$-finite elements of $\mathfrak{A}$, and with the range contained in $\mathfrak{A}_F^\alpha$. It follows that $\mathfrak{A}_F^\alpha$*

*consists of analytic elements for $\delta$, that $\delta$ is closable, and that $\bar{\delta}$ generates a one-parameter group of *-automorphisms.*

*Remark.* The dynamical assumption $E(\gamma) = \mathbf{1}$ is fulfilled in the case $\Gamma(\alpha) = \hat{G}$, where $\Gamma(\alpha)$ denotes the gamma-spectrum of the extension of $\alpha$ to $\mathfrak{M} = \mathfrak{A}''$. In fact, the single condition $\Gamma(\alpha) = \hat{G}$ is equivalent to the two conditions

$$E(\gamma) = \mathbf{1}$$

and

$$\mathfrak{M}^\alpha \cap (\mathfrak{M}^\alpha)' \subset \mathfrak{M} \cap \mathfrak{M}';$$

see [**3**], Remark 4.9. The dynamical assumption is in particular fulfilled in the case $(\mathfrak{M}^\alpha)' \cap \mathfrak{M} = \mathbf{C} \cdot \mathbf{1}$, but in this case we already have the stronger Theorem 2.5.

Note that Theorem 3.4 generalizes the main theorem of [**7**] and [**30**].

Example 5.14 in [**7**] shows that the theorem may fail if $G$ is not a Lie group.

*Proof of Theorem* 3.4. We follow partly ideas from [**4**] and [**30**], showing that all the elements of $\mathfrak{A}_F^\alpha$ are analytic for $\delta$ with a uniform radius of analyticity.

By Lemma 2.3, $\delta$ has a decomposition $\delta = \delta_0 + \tilde{\delta}$, where $\delta_0$ and $\tilde{\delta}$ are *-derivations mapping $\mathfrak{A}_F^\alpha$ into $\mathfrak{M}_F^\alpha$, $\delta_{0|_{\mathfrak{A}^\alpha}} = 0$, and $\tilde{\delta} = \mathrm{ad}(h)$, with $h \in \mathfrak{M}_F^\alpha$. Next, Lemma 3.1 implies that $\delta_0$ and $\delta$ extend to $\mathfrak{M}_F^\alpha$, and $\delta_0$ is defined by a cocycle

$$\hat{G} \ni \gamma \mapsto L(\gamma) \in (\mathfrak{M}^\alpha)' \cap \mathfrak{M}_F^\alpha.$$

In particular, $\delta(\mathfrak{M}_F^\alpha) \subset \mathfrak{M}_F^\alpha$.

For each $\gamma \in \hat{G}$, let $u = u(\gamma)$ be a maximal partial isometry in $\mathfrak{M}_F^\alpha$; i.e.,

$$(\mathbf{1} - uu^*)\mathfrak{M}^\alpha(\gamma)(\mathbf{1} - u^*u) = 0.$$

One can see from the proof of Lemma 3.1 that this implies that the central supports of both $uu^*$ and $u^*u$ in $\mathfrak{M}^\alpha$ are equal to $\mathbf{1}$.

Now if all the $u(\gamma)$'s were unitary, the techniques of [**4**] would apply. We can reduce to this case by the following artifice. Let $\mathscr{H}$ be the Hilbert space on which $\mathscr{M}$ is acting, and replace $\mathscr{M}$ by $\mathscr{L}(\mathscr{H}) \bar{\otimes} \mathfrak{M}$, $\alpha$ by $\mathrm{id} \otimes \alpha$, $\delta$ by $\mathrm{id} \otimes \delta$ (and similarly for $\delta_0$ and $\tilde{\delta}$). Then clearly

$$(\mathscr{L}(\mathscr{H}) \bar{\otimes} \mathfrak{M})^{\mathrm{id}\otimes\alpha}(\gamma) = \mathscr{L}(\mathscr{H}) \bar{\otimes} \mathfrak{M}^\alpha(\gamma),$$

where the last space is the weak closure of the algebraic tensor product. As the restriction of $\mathrm{id} \otimes \delta_0$ to the algebraic tensor product $\mathscr{L}(\mathscr{H}) \otimes \mathfrak{M}^\alpha(\gamma)$ is defined by left multiplication by $\mathbf{1} \otimes L(\gamma)$, which is bounded, it follows that $\mathrm{id} \otimes \delta_0$, and hence $\mathrm{id} \otimes \delta$, extend to $(\mathscr{L}(\mathscr{H}) \bar{\otimes} \mathfrak{M})_F$. A simple

Cantor-Bernstein type argument establishes that the initial and final projections of the partial isometry $\mathbf{1} \otimes u(\gamma)$ are equivalent to $\mathbf{1} \otimes \mathbf{1}$ by partial isometries in $\mathscr{L}(\mathscr{H}) \bar{\otimes} \mathfrak{M}^{\alpha}$, and modifying $\mathbf{1} \otimes u(\gamma)$ by these partial isometries we obtain unitaries

$$U(\gamma) \in (\mathscr{L}(\mathscr{H}) \bar{\otimes} \mathfrak{M})^{\mathrm{id}\otimes\alpha}(\gamma).$$

Therefore the condition $\Gamma$ of [**4**] is fulfilled for the action of $G$ on the norm-closure of $(\mathscr{L}(\mathscr{H}) \bar{\otimes} \mathfrak{M})_F$. It now follows from Lemmas 2.4 and 2.5 of [**4**] that $(\mathscr{L}(\mathscr{H}) \bar{\otimes} \mathfrak{M})_F$ consists of analytic elements for $\mathrm{id} \otimes \delta$ and that $\pm(\mathrm{id} \otimes \delta)$ is dissipative on $(\mathscr{L}(\mathscr{H}) \bar{\otimes} \mathfrak{M})_F$; it is crucial here that $G$ is a Lie group. But then $\pm\delta$ is dissipative on $\mathfrak{A}_F^{\alpha}$ and $\mathfrak{A}_F^{\alpha}$ consists of analytic elements for $\delta$. It follows from the Lumer-Phillips theorem that $\delta$ is closable and its closure is a generator, [**12**, Lemma 3.1.14 and Theorem 3.1.16].

We do not have any very general results about the situation where the central projections $E(\gamma)$ vary with $\gamma$. The few results we do have depend on the following lemma, which has some interest in its own right:

LEMMA 3.6. *Let $\delta_0$ be a *-derivation on a $C^*$-algebra or von Neumann algebra $\mathfrak{A}$. Let $h = -h^* \in \mathfrak{A}$ be an analytic element for $\delta_0$, and define another derivation $\delta$ by*

$$D(\delta) = D(\delta_0),$$

$$\delta(x) = \delta_0(x) + [h, x] \quad \text{for } x \in D(\delta_0).$$

*If $x$ is any analytic element for $\delta_0$ and $t_0$ is a positive number such that*

$$\sum_{n=0}^{\infty} \frac{t_0^{\,n}\|\delta_0^{\,n}(x)\|}{n!} < \infty, \quad \textit{and}$$

$$\sum_{n=0}^{\infty} \frac{t_0^{\,n}\|\delta_0^{\,n}(h)\|}{n!} < \infty,$$

*then $x$ is also analytic for $\delta$ and*

$$\sum_{n=0}^{\infty} \frac{t^n\|\delta^n(x)\|}{n!} < \infty$$

*for all $t < t_0$.*

*Proof.* A direct proof of the last estimate seems to lead to dire combinatorial difficulties, so we will give a proof based on the cocycle formalism for perturbations of derivations as presented in [**12**] or [**9**]. This proof was suggested in conversation with Palle Jorgensen.

First define

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$$e^{t\delta_0}(x) = \sum_{n=0}^{\infty} \frac{t^n \delta_0{}^n(x)}{n!}$$

for $|t| < t_0$, and define $e^{t\delta_0}(h)$ analogously. Next define

$$\Gamma_t^h = \mathbf{1} + \sum_{n \geqq 1} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n e^{t_n\delta_0}(h) \ldots e^{t_1\delta_0}(h)$$

for $|t| < t_0$. Then $\Gamma_t^h$ is the unique solution of the differential equation

$$\frac{d}{dt}(\Gamma_t^h) = \Gamma_t^h \, e^{t\delta_0}(h),$$

and the functions $t \to e^{t\delta_0}(h)$ and $t \to \Gamma_t^h$ clearly have analytic extensions to the disk $\{z: |z| < t_0\}$.

Next define

$$e^{t\delta}(x) = \Gamma_t^h e^{t\delta_0}(x) \Gamma_t^{h*}$$

for $|t| < t_0$. Then $t \to e^{t\delta}(x)$ has an analytic extension to the disk $|z| < t_0$, and we have

$$\begin{aligned} &\frac{d}{dt}(e^{t\delta}(x)) \\ &= \Gamma_t^h \, e^{t\delta_0}(\delta_0(x) + hx - xh) \, \Gamma_t^{h*} \\ &= \Gamma_t^h e^{t\delta_0}(\delta(x)) \Gamma_t^{h*}. \end{aligned}$$

Iterating this computation, we get

$$\frac{d^n}{dt^n}(e^{t\delta}(x))\big|_{t=0} = \delta^n(x).$$

Therefore the analytic function $e^{z\delta}(x)$ has Taylor series

$$e^{z\delta}(x) = \sum_{n \geqq 0} \frac{t^n \delta^n(x)}{n!},$$

where the power series converges in norm for $|z| < t_0$, [**12**, Theorem 2.5.21]. The lemma follows.

We can now prove

PROPOSITION 3.7. *Let $\alpha$ be an action of the circle group $\mathbf{T}$ on a $C^*$-algebra $\mathfrak{A}$, and assume that there exists a faithful G-covariant representation of $\mathfrak{A}$ such that $E(n) = E(-n)$ for all $n \in \mathbf{Z}$, where $E(n)$ denotes the range projection of the ideal $\mathfrak{A}^\alpha(n)\mathfrak{A}^\alpha(n)^*$ of $\mathfrak{A}^\alpha$.*

*Let $\delta$ be a *-derivation mapping $\mathfrak{A}_F^\alpha$ into $\mathfrak{A}_F^\alpha$ satisfying the condition*

$$E(n)\delta(x) = \delta(x) \quad \textit{for } x \in \mathfrak{A}^\alpha(n).$$

*It follows that* $\mathfrak{A}_F^\alpha$ *consists of analytic elements for* $\delta$, *and* $\delta$ *has a closure generating a one-parameter group of* *-*automorphisms of* $\mathfrak{A}$.

*Remark* 3.8. Note that the condition $E(n) = E(-n)$ is automatically fulfilled if $\mathfrak{A}$ is abelian.

*Proof.* Let $\mathfrak{M}$ denote the weak closure of $\mathfrak{A}$. Lemma 3.1 allows us to extend $\delta$ to a derivation of $\delta:\mathfrak{M}_F^\alpha \to \mathfrak{M}_F^\alpha$, and Lemma 2.3 gives us a decomposition $\delta = \delta_0 + \tilde{\delta}$, with $\delta_{0|_{\mathfrak{A}^\alpha}} = 0$ and $\tilde{\delta}$ implemented by an element of $\mathfrak{M}_F^\alpha$. Using the technique of tensoring with $\mathscr{L}(\mathscr{H})$ as in the previous proof, we may assume that each spectral subspace $\mathfrak{M}^\alpha(m)$ contains an operator $U(m)$ with the property

$$U(m)U(m)^* = E(m) = E(-m) = U(m)^*U(m).$$

Thus any element $x \in E(m)\mathfrak{M}_F^\alpha E(m)$ has a decomposition

$$x = \sum_{n=-\infty}^{\infty} x_n U(m)^n,$$

where all the terms except a finite number are zero, and

$$x_n \in \mathfrak{M}^\alpha(\{0, 1, \dots, m - 1\}).$$

As $\delta_0(E(m)) = 0$, $\delta_0$ maps the algebra $E(m)\mathfrak{M}_F^\alpha E(m)$ into itself, and also the restriction of $\delta_0$ to

$$E(m)\mathfrak{M}^\alpha(\{0, 1, \dots, m - 1\})E(m)$$

is bounded and maps this space into some finite spectral subspace $\mathfrak{M}^\alpha(\Lambda)$, where $\Lambda$ is a finite subset of $\mathbf{Z}$. Now one deduces as in [**4**], Lemma 2.3, that there exist finitely many bounded maps

$$\delta_n:E(m)\mathfrak{M}^\alpha(\{0, 1, \dots, m - 1\})E(m) \to E(m)\mathfrak{M}^\alpha(\{0, 1, \dots, m - 1\})E(m)$$

such that

$$\delta_0(x) = \sum_{n=-M}^{M} \delta_n(x)U(m)^n$$

for all $x \in E(m)\mathfrak{M}^\alpha(\{0, 1, \dots, m - 1\})E(m)$. Next one concludes, as in [**4**], Lemmas 2.4 and 2.5, that $E(m)\mathfrak{M}_F^\alpha E(m)$ consists of analytic elements for $\delta_0$ with a uniform radius of analyticity, and that $\pm\delta_0$ is dissipative on $E(m)\mathfrak{M}_F^\alpha E(m)$. The same holds for the linear span of a finite number of the spaces $E(m)\mathfrak{M}_F^\alpha E(m)$, and hence $\mathfrak{M}_F^\alpha$ consists of analytic vectors for $\delta_0$ and $\delta_0$ is dissipative on $\mathfrak{M}_F^\alpha$. It follows from Lemma 3.6 that $\mathfrak{M}_F^\alpha$ also consists of analytic elements for $\delta = \delta_0 + \tilde{\delta}$, and $\pm\delta$ is dissipative as $\pm\delta_0$ and $\pm\tilde{\delta}$ are so.

We end by stating a result which is essentially a corollary of a theorem of Kishimoto, [**19**].

THEOREM 3.9. *Let $\alpha$ be an action of the circle group* $\mathbf{T}$ *on a separable simple $C^*$-algebra $\mathfrak{A}$, and let $\delta$ be a closable *-derivation with $D(\delta) = \mathfrak{A}_F^\alpha$ and $\delta(\mathfrak{A}_F^\alpha) \subset \mathfrak{A}_F^\alpha$. It follows that $\mathfrak{A}_F^\alpha$ consists of entire analytic elements for $\delta$, and $\bar{\delta}$ is a generator.*

*Proof.* By [**19**], Theorem 2.1, $\mathfrak{A}$ has an irreducible $\mathbf{T}$-covariant representation, and the action of $\mathbf{T}$ on $\mathfrak{A}'' = \mathscr{L}(\mathscr{H})$ is then implemented by a unitary group $t \mapsto \exp(2\pi itN)$, where $N$ is a self-adjoint operator with spectrum contained in $\mathbf{Z}$. By Lemmas 2.3 and 3.1 (and Remark 3.2; here the closability of $\delta$ is used), $\delta$ has a decomposition $\delta = \delta_0 + \tilde{\delta}$ and $\delta_0$ extends to $\mathscr{L}(\mathscr{H})_F$. In this case, $\delta_0 = \text{ad}(iH)$, where $H$ is affiliated to $(\mathscr{L}(\mathscr{H})^\alpha)'$, the maximal abelian von Neumann algebra generated by the bounded functions of $N$. Thus $H = f(N)$ for a suitable function $f$, and it easily follows that

$$\delta_0(\mathscr{L}(\mathscr{H})^\alpha(n)) \subset \mathscr{L}(\mathscr{H})^\alpha(n) \quad \text{for all } n.$$

Thus $\delta_0$ commutes with $\alpha$, and $\delta_0$ is a generator by [**8**] (or for elementary reasons). Also $\mathscr{L}(\mathscr{H})_F^\alpha$ consists of entire analytic elements for $\delta_0$ and hence for $\delta$, by Lemma 3.6. Furthermore, $\pm\delta = \pm(\delta_0 + \tilde{\delta})$ is dissipative on $\mathscr{L}(\mathscr{H})_F^\alpha$. Hence $\bar{\delta}$ is a generator.

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