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# RECOGNISING REE GROUPS $^2G_2(q)$ USING THE CODEGREE SET

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**Abstract**

The codegree of an irreducible character $\chi$ of a finite group $G$ is $|G : \ker\chi|/\chi(1)$. We show that the Ree group $^2G_2(q)$, where $q = 3^{2f+1}$, is determined up to isomorphism by its set of codegrees.

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## 1. Introduction

Let $G$ be a finite group and $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of $G$. The concept of codegrees was introduced by Chillag and Herzog in [6], where the codegree of $\chi$ was defined as $|G|/\chi(1)$ for a character $\chi \in \mathrm{Irr}(G)$. The definition was modified to $\mathrm{cod}(\chi) = |G : \ker(\chi)|/\chi(1)$ by Qian *et al.* in [18] so that there is no different meaning for $\mathrm{cod}(\chi)$ when $\chi$ is considered as a character in some quotient group of $G$. Because the relationship between codegrees and degrees is very close, we may expect to characterise the structure of groups by codegrees. During the past few years, the study of character codegrees has been very active and many results have been obtained, including the relationship between codegrees and element orders, codegrees of $p$-groups and groups with few codegrees (see, for example, [2, 8, 11, 14, 15, 17]).

Let $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. Huppert made the following conjecture (which has been verified for sporadic simple groups, alternating groups and some simple groups of Lie type with low rank).

HUPPERT'S CONJECTURE: Let $H$ be any finite nonabelian simple group and $G$ a finite group such that $\mathrm{cd}(G) = \mathrm{cd}(H)$. Then $G \cong H \times A$, where $A$ is abelian.

We denote $\mathrm{cod}(G) = \{\mathrm{cod}(\chi) \mid \chi \in \mathrm{Irr}(G)\}$. Qian made the following conjecture (Question 20.79 in the Kourovka Notebook [12]).

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Codegree version of Huppert's conjecture: Let $H$ be any finite nonabelian simple group and $G$ a finite group such that $\mathrm{cod}(G) = \mathrm{cod}(H)$. Then $G \cong H$.

This conjecture was shown to hold for $\mathrm{PSL}(2, q)$ in [4]. In [1], the conjecture was proven for ${}^2B_2(2^{2f+1})$, where $f \geq 1$, $\mathrm{PSL}(3, 4)$, $\mathrm{Alt}_7$ and $J_1$. The conjecture also holds in the cases where $H$ is $M_{11}, M_{12}, M_{22}, M_{23}$ or $\mathrm{PSL}(3, 3)$ by [9]. For $\mathrm{PSL}(3, q)$ and $\mathrm{PSU}(3, q)$, the conjecture was confirmed in [16]. In this paper, we continue the study of the conjecture and establish the following result.

Theorem 1.1. *If $H$ is isomorphic to a simple Ree group ${}^2G_2(3^{2f+1})$, $f \geq 1$ and $G$ is a finite group such that* $\mathrm{cod}(G) = \mathrm{cod}(H)$*, then $G \cong H$.*

## 2. Preliminary results

First, we give a list of simple groups with few codegrees. The cases for finite groups $G$ with $|\mathrm{cod}(G)| \leq 3$ are studied in [2]. In particular, those groups are all solvable.

Lemma 2.1. *Let $G$ be a nonabelian finite simple group. If $|\mathrm{cod}(G)| \leq 11$, then one of the following holds:*

(a) $|\mathrm{cod}(G)| = 4$ *and* $G = \mathrm{PSL}(2, 2^f)$ *for* $f \geq 2$;
(b) $|\mathrm{cod}(G)| = 5$ *and* $G = \mathrm{PSL}(2, p^f)$, $p \neq 2$, $p^f > 5$;
(c) $|\mathrm{cod}(G)| = 6$ *and* $G = {}^2B_2(2^{2f+1})$, $f \geq 1$, *or* $G = \mathrm{PSL}(3, 4)$;
(d) $|\mathrm{cod}(G)| = 7$ *and* $G = \mathrm{PSL}(3, 3), \mathrm{Alt}_7, M_{11}$ *or* $J_1$;
(e) $|\mathrm{cod}(G)| = 8$ *and* $G = \mathrm{PSL}(3, q)$, *where* $4 < q \not\equiv 1 \pmod 3$, *or* $G = \mathrm{PSU}(3, q)$, *where* $4 < q \not\equiv -1 \pmod 3$, *or* $G = G_2(2)'$;
(f) $|\mathrm{cod}(G)| = 9$ *and* $G = \mathrm{PSL}(3, q)$, *where* $4 < q \equiv 1 \pmod 3$, *or* $G = \mathrm{PSU}(3, q)$, *where* $4 < q \equiv -1 \pmod 3$;
(g) $|\mathrm{cod}(G)| = 10$ *and* $G = M_{22}$;
(h) $|\mathrm{cod}(G)| = 11$ *and* $G = \mathrm{PSL}(4, 2), M_{12}, M_{23}$ *or* ${}^2G_2(3^{2f+1})$, $f \geq 1$.

Proof. For a simple group $G$, each nontrivial irreducible character is faithful. Then $|\mathrm{cod}(G)| = |\mathrm{cd}(G)|$ and the result follows from [3, Theorem 1.1]. □

The character degree sets for the relevant simple groups have been worked out. Using the definition of codegrees and the fact that the kernel of a nontrivial character is trivial, it is easy to calculate the codegrees. We list the relevant codegree sets in Table 1 for easy reference. Please see [1, 4, 7, 19–21] for the details.

Lemma 2.2. *If a simple group $G$ is isomorphic to* $\mathrm{PSL}(2, k)$, ${}^2B_2(q^2)$, $\mathrm{PSL}(3, 4)$, $\mathrm{Alt}_7$, $J_1$, $M_{11}$, $\mathrm{PSL}(3, 3)$, $G_2(2)'$, $M_{22}$, $\mathrm{PSL}(4, 2)$, $M_{12}$, $M_{23}$, $\mathrm{PSL}(3, q)$, $\mathrm{PSU}(3, q)$ *or* ${}^2G_2(q)$, *then* $\mathrm{cod}(G)$ *can be found in Table 1.*

To end this section, we state a result about the maximal subgroups of ${}^2G_2(3^{2f+1})$, $f \geq 1$. This is obtained from [13, Theorem C].

Lemma 2.3. *Let $K$ be a maximal subgroup of ${}^2G_2(q)$, where $q = 3^{2f+1}$, $f \geq 1$. Then $K$ is isomorphic to one of the groups in Table 2.*

TABLE 1. Codegree sets for some simple groups.

| Group $G$ | $\mathrm{cod}(G)$ |
|---|---|
| $\mathrm{PSL}(2,k)$ $(k=2^f\geq 4)$ | $\{1, k(k-1), k(k+1), k^2-1\}$ |
| $\mathrm{PSL}(2,k)$ $(k>5)$ | $\{1, k(k-1)/2, k(k+1)/2, (k^2-1)/2, k(k-\epsilon(k))\}$, $\epsilon(k)=(-1)^{(k-1)/2}$ |
| ${}^2B_2(q),\ q=2r^2=2^{2f+1}$ | $\{1,\ (q-1)(q^2+1),\ q^2(q-1),\ 2^{3f+2}(q^2+1),\ q^2(q-2r+1),\ q^2(q+2r+1)\}$ |
| $\mathrm{PSL}(3,4)$ | $\{1,\ 2^4{\cdot}3^2{\cdot}7,\ 2^6{\cdot}3^2,\ 2^6{\cdot}5,\ 2^6{\cdot}7,\ 3^2{\cdot}5{\cdot}7\}$ |
| $\mathrm{Alt}_7$ | $\{1,\ 2^2{\cdot}3{\cdot}5{\cdot}7,\ 2^2{\cdot}3^2{\cdot}7,\ 2^2{\cdot}3^2{\cdot}5,\ 2^3{\cdot}3{\cdot}7,\ 2^3{\cdot}3{\cdot}5,\ 2^3{\cdot}3^2\}$ |
| $J_1$ | $\{1,\ 3{\cdot}5{\cdot}11{\cdot}19,\ 2{\cdot}3{\cdot}5{\cdot}7{\cdot}11,\ 2^3{\cdot}3{\cdot}5{\cdot}19,\ 7{\cdot}11{\cdot}19,\ 2^3{\cdot}3{\cdot}5{\cdot}11,\ 2^3{\cdot}3{\cdot}5{\cdot}7\}$ |
| $M_{11}$ | $\{1,\ 2^3{\cdot}3^2{\cdot}11,\ 2^4{\cdot}3^2{\cdot}5,\ 3^2{\cdot}5{\cdot}11,\ 2^2{\cdot}3^2{\cdot}5,\ 2^4{\cdot}11,\ 2^4{\cdot}3^2\}$ |
| $\mathrm{PSL}(3,3)$ | $\{1,\ 2^2{\cdot}3^2{\cdot}13,\ 2^4{\cdot}3^3,\ 3^3{\cdot}13,\ 2^3{\cdot}3^3,\ 2^4{\cdot}13,\ 2^4{\cdot}3^2\}$ |
| $G_2(2)'$ | $\{1,\ 2^4{\cdot}3^2{\cdot}7,\ 2^5{\cdot}3^3,\ 2^4{\cdot}3^3,\ 2^5{\cdot}3^2,\ 2^5{\cdot}7, 2^3{\cdot}3^3,\ 3^3{\cdot}7\}$ |
| $M_{22}$ | $\{1,\ 2^7{\cdot}3{\cdot}5{\cdot}11,\ 2^7{\cdot}7{\cdot}11,\ 2^7{\cdot}3^2{\cdot}7,\ 2^7{\cdot}5{\cdot}7,\ 2^6{\cdot}3^2{\cdot}5,\ 2^6{\cdot}3{\cdot}11,\ 2^7{\cdot}3{\cdot}5, 2^4{\cdot}3^2{\cdot}11,\ 2^7{\cdot}3^2\}$ |
| $\mathrm{PSL}(4,2)$ | $\{1,\ 2^6{\cdot}3^2{\cdot}5,\ 2^5{\cdot}3^2{\cdot}5,\ 2^4{\cdot}3^2{\cdot}7,\ 2^6{\cdot}3{\cdot}5,\ 2^4{\cdot}3^2{\cdot}5,\ 2^6{\cdot}3^2,\ 2^6{\cdot}7,\ 2^3{\cdot}3^2{\cdot}5, 3^2{\cdot}5{\cdot}7,\ 2^5{\cdot}3^2\}$ |
| $M_{12}$ | $\{1,\ 2^6{\cdot}3^3{\cdot}5,\ 2^2{\cdot}3^3{\cdot}5{\cdot}11,\ 2^6{\cdot}3{\cdot}11,\ 2^5{\cdot}5{\cdot}11,\ 2^6{\cdot}3^3,\ 2^5{\cdot}3^2{\cdot}5,\ 2^6{\cdot}3{\cdot}5,\ 2^3{\cdot}3^2{\cdot}11,\ 2^2{\cdot}3{\cdot}5{\cdot}11,\ 2^2{\cdot}3^3{\cdot}5\}$ |
| $M_{23}$ | $\{1,\ 2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}23,\ 2^7{\cdot}7{\cdot}11{\cdot}23,\ 2^6{\cdot}3^2{\cdot}7{\cdot}11,\ 2^7{\cdot}3{\cdot}5{\cdot}23,\ 2^7{\cdot}3^2{\cdot}5{\cdot}7,\ 2^6{\cdot}3^2{\cdot}23,$ |
| | $3^2{\cdot}5{\cdot}11{\cdot}23,\ 2^6{\cdot}7{\cdot}23,\ 2^7{\cdot}7{\cdot}11,\ 2^4{\cdot}3^2{\cdot}5{\cdot}7\}$ |
| $\mathrm{PSL}(3,q)$, | $\{1,\ (q^2+q+1)(q^2-1)(q-1),\ q^2(q^2+q+1)(q-1)^2,\ q^3(q^2+q+1),$ |
| $4<q\not\equiv 1 \pmod 3$ | $q^2(q^2-1)(q-1), q^3(q^2-1), q^3(q^2-1)(q-1),\ q^3(q-1)^2\}$ |
| $\mathrm{PSL}(3,q)$, | $\{1,\ \frac{1}{3}(q^2+q+1)(q+1)(q-1)^2,\ \frac{1}{3}q^2(q^2+q+1)(q-1)^2,\ \frac{1}{3}q^3(q^2+q+1),$ |
| $4<q\equiv 1 \pmod 3$ | $\frac{1}{3}q^2(q+1)(q-1)^2,\ \frac{1}{3}q^3(q-1)(q+1),\ \frac{1}{3}q^3(q+1)(q-1)^2,\ \frac{1}{3}q^3(q-1)^2,\ q^3(q-1)^2\}$ |
| $\mathrm{PSU}(3,q)$, | $\{1,\ (q^2-q+1)(q+1)^2(q-1),\ q^3(q^2-q+1),\ q^2(q^2-q+1)(q+1)^2,$ |
| $4<q\not\equiv -1 \pmod 3$ | $q^3(q+1)^2(q-1),\ q^3(q+1)^2,\ q^2(q+1)^2(q-1),\ q^3(q-1)(q+1)\}$ |
| $\mathrm{PSU}(3,q)$, | $\{1,\ \frac{1}{3}(q^2-q+1)(q+1)^2(q-1),\ \frac{1}{3}q^3(q^2-q+1),\ \frac{1}{3}q^2(q^2-q+1)(q+1)^2,$ |
| $4<q\equiv -1 \pmod 3$ | $\frac{1}{3}q^3(q+1)^2(q-1),\ \frac{1}{3}q^3(q+1)^2,\ \frac{1}{3}q^2(q+1)^2(q-1),\ \frac{1}{3}q^3(q-1)(q+1),\ q^3(q+1)^2\}$ |
| ${}^2G_2(q), q=3^{2f+1}$ | $\{1,\ q^3(q^2-1),\ (q^2-1)(q^2-q+1),\ q^2(q^2-1),\ q^3(q-1),$ |
| $m=3^f, f\geq 1$ | $2{\cdot}3^{5f+3}(q+1)(q+1-3m),\ 2{\cdot}3^{5f+3}(q+1)(q+1+3m),\ 3^{5f+3}(q^2-q+1),$ |
| | $q^3(q+1),\ q^3(q+1-3m),\ q^3(q+1+3m)\}$ |

TABLE 2. Maximal subgroups of ${}^2G_2(q)$, $q \geq 27$.

| Maximal subgroup | Structure | Remark |
|---|---|---|
| $P$ | $[q^3] : Z_{q-1}$ | |
| $C_{R_0(i)}$ | $2 \times L_2(q)$ | |
| $N_{R_0\langle i,j\rangle}$ | $(2^2 \times D_{(1/2)(q+1)}) : 3$ | |
| $M^+W^+$ | $Z_{q+\sqrt{3q}+1} : Z_6$ | |
| $M^-W^-$ | $Z_{q-\sqrt{3q}+1} : Z_6$ | |
| $C_{R_0(\Psi_\alpha)}$ | ${}^2G_2(q_0)$ | $q = q_0^\alpha$, $\alpha$ is a prime |

## 3. Main result for ${}^2G_2(3^{2f+1})$

Sometimes it is convenient to consider the ratio of a set. Let $A$ be a set of nonzero integers. We define $\mathrm{ratio}(A) = \{a/b \mid a, b \in A, a \neq b \text{ and } b \neq 1\}$.

LEMMA 3.1. *Let $G$ be a finite group with $\mathrm{cod}(G) = \mathrm{cod}({}^2G_2(3^{2f+1}))$, where $f \geq 1$. If $N$ is a maximal normal subgroup of $G$, then $G/N \cong {}^2G_2(3^{2f+1})$.*

PROOF. Let $N$ be a maximal normal subgroup of $G$. We set $q = 3^{2f+1}$. Since $\mathrm{cod}(G) = \mathrm{cod}({}^2G_2(q))$, we see that $G$ is perfect. Then $G/N$ is a nonabelian simple group. Since $\mathrm{cod}(G/N) \subseteq \mathrm{cod}(G)$, we see that $|\mathrm{cod}(G/N)|$ is either 4, 5, 6, 7, 8, 9, 10 or 11.

Suppose $|\mathrm{cod}(G/N)| = 4$. Then $G/N \cong \mathrm{PSL}(2, k)$, where $k = 2^t \geq 4$, and $\mathrm{cod}(G/N) = \{1, k(k-1), k(k+1), k^2-1\}$. Since $k^2 - 1$ is the only nontrivial odd codegree, it must be the same as either $3^{5f+3}(q^2 - q + 1)$ or $q^3(q + 1 - 3m)$ or $q^3(q + 1 + 3m)$. Note that $(k + 1, k - 1) = 1$. Suppose that $k^2 - 1 = 3^{5f+3}(q^2 - q + 1)$. If $3^{5f+3} | k - 1$, then $k + 1 | q^2 - q + 1$. Clearly $q^2 - q + 1 < 3^{5f+3}$, which implies that $k + 1 < k - 1$, which is a contradiction. If $3^{5f+3} | k + 1$, then $k - 1 | q^2 - q + 1$. Clearly $3^{5f+3} - (q^2 - q + 1) = 3^{5f+3} - 3^{4f+2} + 3^{2f+1} - 1 > 2$, which implies that $(k + 1) - (k - 1) > 2$, which is a contradiction. Suppose $k^2 - 1 = q^3(q + 1 - 3m)$. If $q^3 | k - 1$, then $k + 1 | (q + 1 - 3m)$. Obviously, we have $q^3 > q + 1 - 3m$, which implies that $k - 1 > k + 1$, which is a contradiction. If $q^3 | k + 1$, then $k - 1 | (q + 1 - 3m)$. Clearly $q^3 - (q + 1 - 3m) = (3^{6f+3} - 3^{2f+1} - 1) + 3^{f+1} > 2$, which implies that $k + 1 - (k - 1) > 2$, which is a contradiction. Suppose $k^2 - 1 = q^3(q + 1 + 3m)$. If $q^3 | k - 1$, then $k + 1 | (q + 1 + 3m)$. Since $q^3 > q + 1 + 3m$, we have $k - 1 > k + 1$, which is a contradiction. If $q^3 | k + 1$, then $k - 1 | (q + 1 + 3m)$. Clearly $q^3 - (q + 1 + 3m) = 3^{6f+3} - (3^{2f+1} + 1 + 3^{f+1}) > 2$, which implies that $k + 1 - (k - 1) > 2$, which is a contradiction.

Suppose $|\mathrm{cod}(G/N)| = 5$. Then $G/N \cong \mathrm{PSL}(2, k)$, where $k > 5$ is an odd prime power. We note that $2 \in \mathrm{ratio}(\mathrm{cod}(G/N))$. This is a contradiction since the smallest nontrivial integer in $\mathrm{ratio}(\mathrm{cod}(G))$ is $q - 1$, where $q \geq 27$.

Suppose $|\mathrm{cod}(G/N)| = 6$. Then $G/N \cong {}^2B_2(2^{2t+1})$ or $\mathrm{PSL}(3, 4)$.

Suppose $G/N \cong {}^2B_2(s)$ with $s = 2^{2t+1}$ and $r = 2^t$, $t \geq 1$. The only nontrivial odd codegree in $\mathrm{cod}({}^2B_2(s))$ is $(s - 1)(s^2 + 1)$. Thus, $(s - 1)(s^2 + 1)$ could be equal to either

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$3^{5f+3}(q^2-q+1)$, $q^3(q+1-3m)$ or $q^3(q+1+3m)$. Note that $(s-1, s^2+1)=1$. Suppose $(s-1)(s^2+1)=3^{5f+3}(q^2-q+1)$. If $3^{5f+3}|s-1$, then $s^2+1|q^2-q+1$. Clearly $3^{5f+3}>q^2-q+1$, which implies that $s-1>s^2+1$, which is a contradiction. If $3^{5f+3}|s^2+1$, we have a contradiction since $s^2-1$ is divisible by 3. Suppose $(s-1)(s^2+1)=q^3(q+1-3m)$. If $q^3|s-1$, then $s^2+1|q+1-3m$. Clearly $q^3>q+1-3m$, which implies that $s-1>s^2+1$, which is a contradiction. If $q^3|s^2+1$, then $s-1|q+1-3m$. This implies that $q^3\leq(q+2-3m)^2+1$, which is a contradiction. Suppose $(s-1)(s^2+1)=q^3(q+1+3m)$. If $q^3|s-1$, then $s^2+1|q+1+3m$. Clearly $q^3>q+1+3m$, which implies that $s-1>s^2+1$, which is a contradiction. If $q^3|s^2+1$, then $s-1|q+1+3m$. This implies that $q^3\leq(q+2+3m)^2+1$, which is a contradiction.

Suppose $G/N\cong \mathrm{PSL}(3,4)$. Note that $3^2{\cdot}5{\cdot}7$ is the only nontrivial odd codegree in $\mathrm{cod}(G/N)$. Thus, $3^2{\cdot}5{\cdot}7$ could be equal to either $3^{5f+3}(q^2-q+1)$, $q^3(q+1-3m)$ or $q^3(q+1+3m)$. By considering the power of 3 in those numbers, we see that each is impossible.

Suppose $|\mathrm{cod}(G/N)|=7$. Then $G/N\cong \mathrm{PSL}(3,3)$, $\mathrm{Alt}_7$, $M_{11}$ or $J_1$.

Suppose $G/N\cong\mathrm{PSL}(3,3)$. Note that $3\in\mathrm{ratio}(\mathrm{cod}(G/N))$. This is a contradiction since the smallest nontrivial integer in $\mathrm{ratio}(\mathrm{cod}(G))$ is $q-1$, where $q\geq 27$.

Suppose $G/N\cong\mathrm{Alt}_7$. Then $2^3{\cdot}3^2\in\mathrm{cod}(G/N)$. Since the powers of 3 of all the terms that are divisible by 3 in $\mathrm{cod}(G)$ are greater than 2, this is a contradiction.

Suppose $G/N\cong M_{11}$. Note that $5\in\mathrm{ratio}(\mathrm{cod}(G/N))$. This is a contradiction since the smallest nontrivial integer in $\mathrm{ratio}(\mathrm{cod}(G))$ is $q-1$, where $q\geq 27$.

Suppose $G/N\cong J_1$. Note that $\mathrm{cod}(J_1)$ has two nontrivial odd codegrees $7{\cdot}11{\cdot}19$ and $3{\cdot}5{\cdot}11{\cdot}19$. The odd codegrees in $\mathrm{cod}(G)$ are $3^{5f+3}(q^2-q+1)$, $q^3(q+1-3m)$ and $q^3(q+1+3m)$. By considering the power of 3 in those numbers, we see that each is impossible.

Suppose $|\mathrm{cod}(G/N)|=8$. Then $G/N\cong\mathrm{PSL}(3,s)$, where $4<s\not\equiv 1 \pmod 3$, $\mathrm{PSU}(3,s)$, where $4<s\not\equiv -1 \pmod 3$, or $G_2(2)'$.

Suppose $\mathrm{cod}(G/N)=\mathrm{cod}(\mathrm{PSU}(3,s))$ with $4<s\not\equiv -1 \pmod 3$. Note that $3\in\mathrm{ratio}(\mathrm{cod}(G/N))$. This is a contradiction since the smallest nontrivial integer in $\mathrm{ratio}(\mathrm{cod}(G))$ is $q-1$, where $q\geq 27$.

Suppose $G/N\cong G_2(2)'$. Note that $3\in\mathrm{ratio}(\mathrm{cod}(G/N))$. This is a contradiction since the smallest nontrivial integer in $\mathrm{ratio}(\mathrm{cod}(G))$ is $q-1$, where $q\geq 27$.

Suppose $G/N\cong\mathrm{PSL}(3,s)$ for some $4<s\not\equiv 1 \pmod 3$. Note that $s-1, s+1\in\mathrm{ratio}(\mathrm{cod}(G/N))$, and the only nontrivial integers in $\mathrm{ratio}(\mathrm{cod}(G))$ are $q-1$, $q$, $q+1$, where $q$ is a power of 3 and $q\geq 27$. This will force $s=q$, which is a contradiction since $q^3(q-1)^2\notin\mathrm{cod}(G)$.

Suppose $|\mathrm{cod}(G/N)|=9$. Then $G/N\cong\mathrm{PSL}(3,s)$, where $4<s\equiv 1 \pmod 3$, or $\mathrm{PSU}(3,s)$, where $4<s\equiv -1 \pmod 3$.

Suppose $G/N\cong\mathrm{PSL}(3,s)$ with $4<s\equiv 1 \pmod 3$. Note that $3\in\mathrm{ratio}(\mathrm{cod}(G/N))$. This is a contradiction since the smallest nontrivial integer in $\mathrm{ratio}(\mathrm{cod}(G))$ is $q-1$, where $q\geq 27$.

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Suppose $G/N \cong \mathrm{PSU}(3,s)$ with $4 < s \equiv -1 \pmod 3$. Note that $s-1, s+1 \in \mathrm{ratio}(\mathrm{cod}(G/N))$, and the only nontrivial integers in $\mathrm{ratio}(\mathrm{cod}(G))$ are $q-1$, $q$, $q+1$, where $q$ is a power of 3 and $q \geq 27$. This will force $s = q$, which is a contradiction since $q^3(q+1)^2 \notin \mathrm{cod}(G)$.

Suppose $|\mathrm{cod}(G/N)| = 10$. Then $G/N \cong M_{22}$. Note that $11 \in \mathrm{ratio}(\mathrm{cod}(G/N))$. This is a contradiction since the only nontrivial integer in $\mathrm{ratio}(\mathrm{cod}(G))$ is $q$, where $q \geq 27$.

Suppose $|\mathrm{cod}(G/N)| = 11$. Then $G/N \cong \mathrm{PSL}(4,2), M_{12}, M_{23}$ or ${}^2G_2(3^{2s+1})$, where $s \geq 1$.

Suppose $G/N \cong \mathrm{PSL}(4,2)$. Note that $2 \in \mathrm{ratio}(\mathrm{cod}(G/N))$. This is a contradiction since the smallest nontrivial integer in $\mathrm{ratio}(\mathrm{cod}(G))$ is $q-1$, where $q \geq 27$.

Suppose $G/N \cong M_{12}$. Note that $5 \in \mathrm{ratio}(\mathrm{cod}(G/N))$. This is a contradiction since the smallest nontrivial integer in $\mathrm{ratio}(\mathrm{cod}(G))$ is $q-1$, where $q \geq 27$.

Suppose $G/N \cong M_{23}$. Note that $8 \in \mathrm{ratio}(\mathrm{cod}(G/N))$. This is a contradiction since the smallest nontrivial integer in $\mathrm{ratio}(\mathrm{cod}(G))$ is $q-1$, where $q \geq 27$.

Suppose $G/N \cong {}^2G_2(3^{2s+1})$, $s \geq 1$. Comparing the smallest nontrival odd codegree of each set, we see that $q = 3^{2s+1}$. Thus, $G/N \cong {}^2G_2(3^{2f+1})$. □

We now prove the main result of this paper.

THEOREM 3.2. *Let $G$ be a group such that* $\mathrm{cod}(G) = \mathrm{cod}({}^2G_2(q))$, *where* $q = 3^{2f+1}$, $f \geq 1$. *Then* $G \cong {}^2G_2(q)$.

PROOF. Let $G$ be a group with $\mathrm{cod}(G) = \mathrm{cod}({}^2G_2(q))$. Let $N$ be a maximal normal subgroup of $G$. Then, $G/N \cong {}^2G_2(q)$ by Lemma 3.1. Assume to the contrary that $G$ is a minimal counterexample. By the choice of $G$, $N$ is a minimal normal subgroup of $G$. Otherwise there exists a nontrivial normal subgroup $L$ of $G$ such that $L$ is included in $N$. Then $\mathrm{cod}(G/L) = \mathrm{cod}(G)$ for $\mathrm{cod}(G) = \mathrm{cod}(G/N) \subseteq \mathrm{cod}(G/L) \subseteq \mathrm{cod}(G)$ and $G/L \cong {}^2G_2(q)$ because $G$ is a minimal counterexample, which is a contradiction.

*Step 1: $N$ is the unique minimal normal subgroup of $G$.* Otherwise, assume $M$ is another minimal normal subgroup of $G$. Then $G = N \times M$ because $G/N$ is simple and $N \cong M \cong {}^2G_2(q)$ because $M$ is also a maximal normal subgroup of $G$. Choose $\psi_1 \in \mathrm{Irr}(N)$ and $\psi_2 \in \mathrm{Irr}(M)$ such that $\mathrm{cod}(\psi_1) = \mathrm{cod}(\psi_2) = \max(\mathrm{cod}({}^2G_2(q)))$. Set $\chi = \psi_1 \cdot \psi_2 \in \mathrm{Irr}(G)$. Then $\mathrm{cod}(\chi) = (\max(\mathrm{cod}({}^2G_2(q))))^2 \notin \mathrm{cod}(G)$, which is a contradiction.

Set $\mathrm{Irr}(G|N) = \{\chi \in \mathrm{Irr}(G) \mid N \text{ is not contained in the kernel of } \chi\}$.

*Step 2: $\chi$ is faithful for each $\chi \in \mathrm{Irr}(G|N)$.* Since $N$ is not contained in the kernel of $\chi$ for each $\chi \in \mathrm{Irr}(G|N)$, the kernel of $\chi$ is trivial by Step 1.

*Step 3: $N$ is elementary abelian.* Assume to the contrary that $N$ is not abelian. Then $N = S^n$, where $S$ is a nonabelian simple group and $n \in \mathbb{N}$. By Theorems 2, 3 and 4 and Lemma 5 in [5], we see that there exists a nonprincipal character $\psi \in \mathrm{Irr}(N)$ that extends to some $\chi \in \mathrm{Irr}(G)$. Then $\ker(\chi) = 1$ by Step 2 and $\mathrm{cod}(\chi) = |G|/\chi(1) = |G/N| \cdot |N|/\psi(1)$. This is a contradiction since $|G/N|$ is divisible by $\mathrm{cod}(\chi)$.

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*Step 4: It is enough to assume that* $\mathbf{C}_G(N) = N$. We note that $\mathbf{C}_G(N) \trianglelefteq G$. As $N$ is abelian by Step 3, either $\mathbf{C}_G(N) = G$ or $\mathbf{C}_G(N) = N$.

We now prove the result in the case $\mathbf{C}_G(N) = G$. Assume so. Then $N$ is contained in the centre $\mathbf{Z}(G)$ of $G$. Since $G$ is perfect, $\mathbf{Z}(G) = N$ and $N$ is isomorphic to a subgroup of the Schur multiplier of $G/N$ [10, Corollary 11.20]. This forces $N$ to be trivial as ${}^2G_2(q)$ has trivial Schur multiplier, and we are done.

*Step 5: Let* $\lambda$ *be a nonprincipal character in* $\mathrm{Irr}(N)$ *and* $\theta \in \mathrm{Irr}(I_G(\lambda)|\lambda)$. *We show that* $|I_G(\lambda)|/\theta(1) \in \mathrm{cod}(G)$. *Also,* $\theta(1)$ *divides* $|I_G(\lambda)/N|$ *and* $|N|$ *divides* $|G/N|$. Let $\lambda$ be a nonprincipal character in $\mathrm{Irr}(N)$. Given $\theta \in \mathrm{Irr}(I_G(\lambda)|\lambda)$. Note that $\chi = \theta^G \in \mathrm{Irr}(G)$ and $\chi(1) = |G : I_G(\lambda)| \cdot \theta(1)$ by Clifford theory (see [10, Ch. 6]). Then $\ker(\chi) = 1$ by Step 2 and $\mathrm{cod}(\chi) = |I_G(\lambda)|/\theta(1)$. In particular, $\theta(1)$ divides $|I_G(\lambda)/N|$, and then $|N|$ divides $|I_G(\lambda)|/\theta(1)$. Since $\mathrm{cod}(G) = \mathrm{cod}(G/N)$ and $|G/N|$ is divisible by every element in $\mathrm{cod}(G/N)$, we have $|N| \mid |G/N|$.

*Step 6: Final contradiction.* By Step 3, $N$ is an elementary abelian $r$-subgroup for some prime $r$ and we assume $|N| = r^n$, $n \in \mathbb{N}$. By the normaliser–centraliser theorem, $n > 1$.

Let $\lambda \in \mathrm{Irr}(N)$ be a nonprincipal character and $T := I_G(\lambda)$. By Step 5, $|T|/\theta(1) \in \mathrm{cod}(G)$ for all $\theta \in \mathrm{Irr}(T|\lambda)$.

Since $N$ is abelian by Step 1, $|\mathrm{Irr}(N)| = |N|$. Therefore, $|N| = |\mathrm{Irr}(N)| > |G : T|$ since $|G : T|$ is the number of conjugates of $\lambda$ in $G$ which are all contained in $\mathrm{Irr}(N)$.

We now show that $q^3$ is the largest power of a prime that divides the order of ${}^2G_2(q)$. Note that the order of ${}^2G_2(q)$ is $q^3(q^3 + 1)(q - 1)$. We first observe that $q^3 + 1 = 2^k$ has no integer solution, and thus $q^3 + 1$ is divisible by a prime greater than or equal to 5. We also note that either $q^3 + 1$ or $q - 1$ is divisible by 4 but not both, and thus the largest power of 2 that divides $(q^3 + 1)(q - 1)$ is less than $q^3$. Suppose that $r$ is an odd prime that divides $(q^3 + 1)(q - 1)$. Since $\gcd(q + 1, q - 1) = 2$, the largest power of $r$ that divides $(q^3 + 1)(q - 1)$ is also less than $q^3$. Thus, $q^3$ is the largest power of a prime that divides the order of ${}^2G_2(q)$. Thus, $|N| \leq q^3$.

Let $K$ be a maximal subgroup of ${}^2G_2(q)$ such that $T/N \leq K$. If $K$ is of type $P$ in Lemma 2.3, then $|G : T| \geq q^3 + 1$, and it is clear that $q^3 + 1 > q^3$, which is a contradiction. If $K$ is of type $C_{R_0(i)}$, then $|G : T| \geq q^2(q^2 - q + 1) > q^3$, which is a contradiction. If $K$ is of type $N_{R_0\langle i,j\rangle}$, then $|G : T| \geq q^3(q^2 - q + 1)(q - 1)/12(q + 1)$, thus $|G : T| > q^3$, which is a contradiction. If $K$ is of type $M^+W^+$, then $|G : T| \geq q^3(q^2 - q + 1)(q - 1)/6(q + \sqrt{3q} + 1)$, and thus $|G : T| > q^3$, which is a contradiction. If $K$ is of type $M^-W^-$, then $|G : T| \geq q^3(q^2 - q + 1)(q - 1)/6(q - \sqrt{3q} + 1)$, and thus $|G : T| > q^3$, which is a contradiction. If $K$ is of type $C_{R_0}(\Psi_\alpha)$, then $|G : T| \geq q^3(q^3 + 1)(q - 1)/q_0^3(q_0^3 + 1)(q_0 - 1)$, where $q = q_0^\alpha$, $\alpha$ a prime. Thus, $|G : T| > q^3$, which is a contradiction. □

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