J. Austral. Math. Soc. 19 (Series A) (1975), 505–510

STAR CENTER POINTS OF STARLIKE FUNCTIONS

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(Received 3 July 1973; revised 12 December 1973)

Communicated by E. Strzelecki

1. Introduction

Let

$$w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

be regular and univalent in the unit disk D and map D onto a region R. A point $w_0 \in R$ is called a *star-center point* of $f(z)$, or of R, if $tf(z) + (1-t)w_0 \in R$ for $z \in D$ and $0 \leqq t \leqq 1$. In this paper we consider only functions of the form (1) where $w = 0$ is a star-center point, i.e., those functions that are starlike with respect to the origin.

Given $w = f(z)$ as in (1), we define the *index of starlikeness* of $w = f(z)$ to be

$$\delta = \sup\{r \,|\, f(z) \text{ is a star-center point of } f(D) \text{ whenever } |z| \leqq r\}.$$

We denote by S_δ the class of all starlike functions whose index is equal to δ, $0 \leqq \delta \leqq 1$.

From the definition it follows that S_0 and S_1 are the classes of normalized starlike and convex univalent functions respectively. In this note we obtain estimates for $|a_n|$, $|f(x)|$ and $\mathrm{Re}[zf'(z)/f(z)]$ when $w = f(z) \in S_\delta$.

NOTATION: Let D_r denote the disk $|z| < r$. Let $\phi(z, a, \alpha) = e^{i\alpha}(z-a)(1-\bar{a}z)^{-1}$, where $-\pi < \alpha \leqq \pi$ and $|a| < 1$.

2. Preliminaries

In this section we give some necessary and sufficient conditions for a function to belong to the class S_δ.

* Supported in part by a National Science Foundation Grant.

LEMMA 1. *Let* $w = f(z)$ *be of the form* (1). *Then* $w = f(z) \in S_\delta$ *if and only if the function*

$$F(z) = tf(z) + (1-t)f(\delta z\phi(z,a,\alpha)) \tag{3}$$

is subordinate to $w = f(z)$ *in* D *for* $-\pi < \alpha \leqq \pi$, $|a| < 1$, $0 \leqq t \leqq 1$.

PROOF. If $w = f(z) \in S_\delta$, then the subordination holds because $|\phi(z,a,\alpha)| < 1$.

Suppose the subordination holds. For $\delta = 0$ the result is well-known, so we may suppose $\delta > 0$. Given z_0, $|z_0| < \delta$, we show $f(z_0)$ is a star-center point of $f(D_r)$ for $r > \delta^{-1}|z_0|$. For each z, $|z| > \delta^{-1}|z_0|$, let

$$A = \delta^{-1}\bar{z}^{-1}\bar{z}_0, \quad \alpha = \operatorname{Arg}(1-\bar{A}\bar{z})(1-Az)^{-1}, \quad a = -(\bar{A}-z)(1-\bar{A}\bar{z})^{-1}. \tag{4}$$

Then $z_0 = \delta z\phi(z,a,\alpha)$ and (3) yields $tf(z) + (1-t)f(z_0) = f(\zeta)$ for some ζ in D. By well-known properties of subordination, $|\zeta| < |z|$, (see [1, p. 227]).

LEMMA 2. *The function* $w = f(z) \in S_\delta$, $\delta > 0$, *if and only if*

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f(\zeta)} > 0 \tag{5}$$

for $|\zeta| < \delta$, $|\zeta| < |z| < 1$.

PROOF. Choose $\varepsilon > 0$ so that $|\zeta| + \varepsilon < \delta$, and $r > 0$ so that $|\zeta|\delta^{-1} < r < 1$. Since $f(D_\delta)$ is a convex set, (5) holds for $|z| = |\zeta| + \varepsilon$. By Lemma 1, (5) also holds for $|z| = r$. Using the, minimum modulus principle for harmonic functions (5) follows upon letting $\varepsilon \to 0$ and $r \to 1$.

LEMMA 3. *The function* $w = f(z)$, εS_δ *if and only if*

$$G(z,\phi) = \frac{zf'(z)}{f(z) - f(\delta z\phi(z,a,\alpha))} \tag{6}$$

has positive real part for $|z| < 1$, $|a| < 1$, $-\pi < \alpha \leqq \pi$.

PROOF. For $\delta = 0$ the result is well-known. If $\delta > 0$, then given z_0, $|z_0| < \delta$ and z, $|z| > \delta^{-1}|z_0|$, we use (4) to find a function $\phi(z,a,\alpha)$ that satisfies the equation $\delta z\phi(z,a,\alpha) = z_0$. For $\zeta = z_0$, (6) follows from (5).

3. Distortion theorems

THEOREM 1. *If* $w = f(z) \in S_\delta$, $0 \leqq \delta \leqq 1$, *then*

$$\frac{1-|z|}{(1-\delta|z|)(1+|z|)} \leqq \operatorname{Re} \frac{zf'(z)}{f(z)} \leqq \frac{1+|z|}{(1+\delta|z|)(1-|z|)}, \tag{7}$$

equality holding in the cases $\delta = 0$ and $\delta = 1$.

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PROOF. By a well-known theorem (Nehari (1952); page 173),

$$[\operatorname{Re} G(0,\phi)]\frac{1-|z|}{1+|z|} \leqq \operatorname{Re} G(z,\phi) \leqq [\operatorname{Re} G(0,\phi)]\frac{1+|z|}{1-|z|}, \tag{8}$$

which becomes

$$\left[\operatorname{Re}\frac{1}{1+e^{i\alpha}a\delta}\right]\frac{1-|z|}{1+|z|} \leqq \operatorname{Re}\frac{zf'(z)}{f(z)-f(\delta z\phi(z,a,\alpha))} \leqq \left[\operatorname{Re}\frac{1}{1+e^{i\alpha}a\delta}\right]\frac{1+|z|}{1-|z|}. \tag{9}$$

Letting $\alpha = -\arg a + \pi$ and $z = a$ on the left side of (9) we obtain

$$\frac{1}{1-\delta|a|}\,\frac{1-|a|}{1+|a|} \leqq \operatorname{Re}\frac{af'(a)}{f(a)},$$

which is equivalent to the left side of (7). The right hand side of (7) is obtained similarly.

It is interesting to note that for $\delta = 1$, we obtain the well-known result that $\operatorname{Re}[zf'(z)/f(z)] \geqq (1+|z|)^{-1}$; see (Strohäcker (1933)) or more recently (Suffridge (1940));

THEOREM 2. *If $w = f(z) \in S_\delta$, then*

$$|z|\frac{(1-\delta|z|)^{(1-\delta)/(1+\delta)}}{(1+|z|)^{2/1+\delta}} \leqq |f(z)| \leqq |z|\frac{(1+\delta|z|)^{(1-\delta)/(1+\delta)}}{(1-|z|)^{2/1+\delta}}, \tag{10}$$

equality holding in the cases $\delta = 0$ and $\delta = 1$.

PROOF. Using the identity

$$\frac{\partial}{\partial|z|}\log\left|\frac{f(z)}{z}\right| = \frac{1}{|z|}\left|\operatorname{Re}\frac{zf'(z)}{f(z)} - 1\right|, \tag{11}$$

(10) is obtained upon integrating (7).

4. Coefficient estimates

We wish to give coefficient estimates for the expansion (1) when $w = f(z) \in S_\delta$. If $w = f(z) \in S_\delta$, then

$$\operatorname{Re}\frac{zf'(z)}{f(z)-f(-\delta z)} > 0 \tag{12}$$

for $z \in D$. This is so because (5) holds for all ζ, $|\zeta| < \delta$. If z is any point in D (5) holds for $\zeta = Rz$ where $-\delta < R < \delta$. Letting $R \to -\delta$ we obtain (12). Equation (12) also holds when $\delta = 0$. If we let

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(13)
$$F(z) = \frac{zf'(z)}{f(z)-f(-\delta z)} = \sum_{n=0}^{\infty} c_n z^n,$$

then $|c_n| \leqq 2\operatorname{Re} c_0 = 2(1+\delta)^{-1}$; (see (Robertson (1945))). We have the following theorem:

THEOREM 3. *If* $w = f(z) \in S_\delta$, *then*

(14)
$$|a_n| \leqq \prod_{k=1}^{n-1} \frac{k(1+\delta)+1-(-\delta)^k}{k(1+\delta)+1+(-\delta)^{k+1}},$$

equality holding in the cases $\delta = 0$ *and* $\delta = 1$.

PROOF. Equation (13) gives the following relationship between the coefficients of $w = f(z)$ and $F(z)$

(15)
$$(n - c_0(1-(-\delta)^n a_n = \sum_{k=1}^{n-1} c_k a_{n-k}(1-(-\delta)^{n-k}),$$

equality holding in the cases $\delta = 0$ and $\delta = 1$.

Let

$$P_k = k + \sum_{j=0}^{k-1} \rho^j,$$

$$Q_k = \sum_{j=0}^{k-1} (k-j)\rho^j, \qquad k = 1, 2, \cdots.$$

Let $S_1 = (1-\rho)$, and define S_n recursively by

$$S_n = S_{n-1} + \frac{1-\rho^n}{(1-\rho)^{n-1}} \prod_{k=1}^{n-1} \frac{P_k}{Q_n}, \quad n = 2, 3, \cdots.$$

Set

$$T_n = \frac{1}{(1-\rho)^{-1}} \frac{Q_n}{2} \prod_{k=1}^{n} \frac{P_k}{Q_k}.$$

We will prove the following identity;

(16)
$$S_n = T_n, \; n = 1, 2 \cdots.$$

Assuming (16) for $n = m-1$, we have

$$S_m = \frac{1}{(1-\rho)^{m-3}} \frac{Q_{m-1}}{2} \prod_{k=1}^{m-1} \frac{P_k}{Q_k} + \frac{1-\rho^m}{(1-\rho)^{m-1}} \prod_{k=1}^{m-1} P_k/Q_k$$

$$= T_m\left[(1-\rho)Q_{m-1}/P_m + 2\frac{(1-\rho^m)}{(1-\rho)} P_m\right] = T_m.$$

Since (16) is easily verified for $n = 1$, (16) follows.

Let $\rho = -\delta$. We will prove that

$$|a_n| \leqq \frac{1}{(1-\rho)^{n-1}} \prod_{k=1}^{n-1} \frac{P_k}{Q_k}, \quad n = 2, 3, \cdots. \tag{17}$$

These estimates are obtained from (15) by induction when we use $c_0 = (1+\delta)^{-1}$, the bounds $|c_n| \leqq 2(1+\delta)^{-1}$ and the estimates for $|a_{n-1}|, |a_{n-2}|$, etc.

For $n = 2$, (15) gives $a_2 = c_1(1+\delta)(2 - c_0(1-\delta^2))^{-1}$. Hence $|a_2| \leqq 2(1+\delta)^{-1}$, which gives (14) for $n = 2$. Assume now that (15) holds for $k \leqq n-1$. Then (15) gives

$$a_n = \frac{1}{n - c_0(1-\rho^n)} \sum_{k=1}^{n-1} c_k a_{n-k}(1-\rho^{n-k}).$$

Hence,

$$|a_n| \leqq \frac{2 \sum_{k=1}^{n-1} |a_k|(1-\rho^k)}{n(1-\rho) - (1-\rho^n)} = \frac{2 \sum_{k=1}^{n-1} a_k \;|(1-\rho^k)}{(1-\rho)^2 Q_{n-1}}.$$

By the induction hypothesis and (16) we have

$$|a_n| \leqq \frac{2S_{n-1}}{(1-\rho)^2 Q_{n-1}} = \frac{2T_{n-1}}{(1-\rho)^2 Q_{n-1}} = \frac{1}{(1-\rho)^{\alpha-1}} \prod_{k=1}^{n-1} P_k/Q_k,$$

and (17) is satisfied.

We now show that (17) is equivalent to (14). Note that

$$P_k = k + (1-\rho^k)/(1-\rho)$$

and

$$(1-\rho)Q_k = k - \rho(1-\rho^k)/(1-\rho).$$

Hence,

$$\frac{P_k}{1(-\rho)\,Q_k} = [k(1-\rho) + 1 - \rho^k][k(1-\rho) + \rho + \rho^{k+1}]^{-1}. \tag{18}$$

If we combine (17) and (18) we obtain (14).

We conclude with an example. It has been suggested that if $w = f(z) \in S_\delta$, then there may exist some $\beta > 0$, depending on δ, such that

$$\inf_{z \in D} \operatorname{Re} \frac{zf'(z)}{f(z)} > \beta.$$

The functions

$$f_\beta(z) = \frac{1}{2\beta}\left[1 - \left(\frac{1+z}{1-z}\right)^\beta\right]$$

https://doi.org/10.1017/S1446788700034522 Published online by Cambridge University Press

serve as a counterexample in the following sense. As β varies in the interval [0, 1], $f_\beta(z)$ has an index that decreases with respect to β in the interval [0, 1]. Furthermore

$$\inf_{z \in D} \operatorname{Re} \frac{zf'_\beta(z)}{f_\beta(z)} = 0$$

for all β, $1 \leqq \beta \leqq 2$.

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https://doi.org/10.1017/S1446788700034522 Published online by Cambridge University Press