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Part 1. Risk theory

ASYMPTOTICS OF CONDITIONAL MOMENTS OF THE SUMMAND IN POISSON COMPOUNDS

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ASYMPTOTICS OF CONDITIONAL MOMENTS OF THE SUMMAND IN POISSON COMPOUNDS

BY TOMASZ ROLSKI AND AGATA TOMANEK

Abstract

Suppose that N is a $\mathbb{Z}_+$-valued random variable and that $X, X_1, X_2, \ldots$ is a sequence of independent and identically distributed $\mathbb{Z}_+$ random variables independent of N. In this paper we are interested in properties of the conditional variable $N_k \stackrel{\mathrm{D}}{=} (N \mid \sum_{j=1}^{N} X_j = k)$. In particular, we want to know the asymptotic behavior of the conditional mean $\mathrm{E}\, N_k$ or the conditional variance $\operatorname{var} N_k$ as $k \to \infty$. We consider the cases when X is Poisson and when X is mixed Poisson. The problem is motivated by modeling loss reserves in nonlife insurance.

Keywords: Compound Poisson; conditional expectation; loss reserving

2010 Mathematics Subject Classification: Primary 62E20
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1. Introduction

Recently, the following mathematical problem appeared in modeling of loss reserves; see, e.g. [4], [6], and also the older papers [7] and [8]. Suppose that N is a $\mathbb{Z}_+$-valued random variable and that $X, X_1, X_2, \ldots$ is a sequence of independent and identically distributed (i.i.d.) $\mathbb{Z}_+$ random variables independent of N. In the abovementioned papers, for a prediction of future payments under the condition of known old payments, we need an estimation of the number of old payments. Hence, we are interested in properties of the conditional variable

$$N_k \stackrel{\mathrm{D}}{=} \left(N \,\middle|\, \sum_{j=1}^{N} X_j = k\right).$$

In particular, we want to know the conditional mean $\mathrm{E}\, N_k$ or the conditional variance $\operatorname{var} N_k$ and their asymptotics as $k \to \infty$.

It turns out that the simplest case when N is Poisson with mean a and X is Poisson with mean b leads to interesting mathematical problems, with roots in nineteenth-century mathematics. We refer to this case as the $(\mathrm{Poi}(a), \mathrm{Poi}(b))$ case. Compute

$$\mathrm{E}\, N_k = \sum_{m=0}^{\infty} m \frac{a^m}{m!} \mathrm{e}^{-a} \frac{(mb)^k}{k!} \mathrm{e}^{-bm} \Big/ \sum_{m=0}^{\infty} \frac{a^m}{m!} \mathrm{e}^{-a} \frac{(mb)^k}{k!} \mathrm{e}^{-bm} = \frac{B^c(k+1)}{B^c(k)},$$

where

$$B^c(k) = \sum_{m=1}^{\infty} m^k \frac{c^m}{m!} \mathrm{e}^{-c}$$

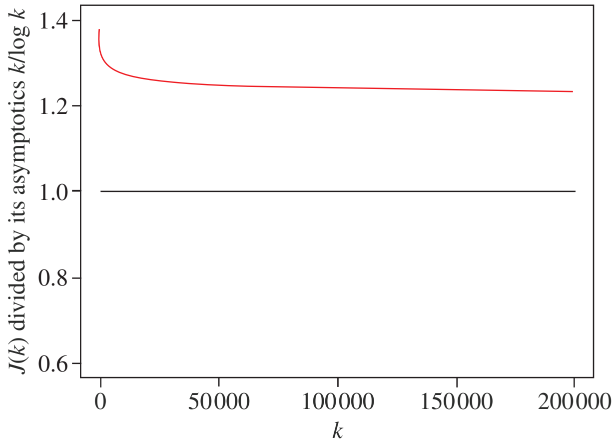

FIGURE 1: The ratio $J(k)/(k/\log k)$.

is the kth moment of the Poisson distribution and $c = a\mathrm{e}^{-b}$. More generally,

$$\mathrm{E}(N_k)^l = \frac{B^c(k+l)}{B^c(k)}, \qquad \operatorname{var} N_k = \frac{B^c(k+1)}{B^c(k)}\bigg(\frac{B^c(k+2)}{B^c(k+1)} - \frac{B^c(k+1)}{B^c(k)}\bigg).$$

Therefore, of particular interest is the ratio $J^c(k) = B^c(k+1)/B^c(k)$. In practice, we want to obtain the properties of N_k for large k and, therefore, asymptotic formulae can be helpful. Jessen *et al.* [4] showed that $J^c(k) \sim k/\log k$ as $k \to \infty$. For $c = 1$, the asymptotics of $J^1(k) = J(k)$ were given earlier in [3], where a redundant e appeared in the denominator. Unfortunately, these asymptotics are extremely slow, as illustrated in Figure 1.

In this paper we propose two other asymptotics for $\mathrm{E}\, N_k$ in the ($\mathrm{Poi}(a)$, $\mathrm{Poi}(b)$) case. We also discuss the asymptotics when X is mixed Poisson.

It turns out that the study of the asymptotics of $B(k) = B^1(k)$ has a long history. It was discovered by Dobinski that $B(k)$ is equal to the kth Bell number. De Bruijn [2, Chapter 2.4] gave the asymptotic formula

$$\frac{\log B(n)}{n} = \log n - \log\log n - 1 + o\bigg(\frac{\log\log n}{\log n}\bigg).$$

However, for our purposes, the prototype is a result from Lovász [5, Problem 9 on page 17, solved on page 166] (who quoted Moser and Wyman),

$$B(k) \sim k^{-1/2}[\Lambda(k)]^{k+1/2}\mathrm{e}^{\Lambda(k)-k-1},$$

where $\Lambda(x)$ is the function defined by $\Lambda(x)\log\Lambda(x) = x$. The function Λ is related to the Lambert W-function by $W(x) = x/\Lambda(x)$. From [2, Equation (2.4.3)], we know that

$$W(x) = \log x - \log\log x + O\bigg(\frac{\log\log x}{\log x}\bigg),$$

and, hence,

$$\Lambda(x) \sim \frac{x}{\log x}\bigg(1 + \frac{\log\log x}{\log x} + O\bigg(\bigg(\frac{\log\log x}{\log x}\bigg)^2\bigg)\bigg).$$

We also refer the reader to [9] for interesting connections between Bell numbers and Poisson distributions.

Recently, Jessen *et al.* [4] showed that

$$B^c(k) = (1 + o(1)) \sum_{m \in [k(1-\varepsilon)/\log k, k(1+\varepsilon)/\log k]} m^k \mathrm{e}^{-c} \frac{c^m}{m!},$$

from which they concluded that $J^c(k) = B^c(k+1)/B^c(k) \sim k/\log k$. We will utilize their method of proof for other cases.

2. The (Poi(a), Poi(b)) case

The following result can be proved similarly as in [5].

Lemma 1. *For $c > 0$, an asymptotic evaluation for $B^c(n)$ is given by*

$$B^c(n) \sim n^{-1/2} \Lambda^c(n)^{n+1/2} \mathrm{e}^{\Lambda^c(n)-n-c},$$

where $\Lambda^c(n) \log(\Lambda^c(n)/c) = n$.

Proof. The proof follows almost exactly as in [5]; however, instead of the functions g_n, we have

$$g_n^c(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} x^{n-x-1/2} \mathrm{e}^{x-c} c^x & \text{for } x \geq 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

We can show that $g_n^c(x)$ has a unique maximum at the point $\Lambda^c(n)$, where

$$\Lambda^c(n) \log\left(\frac{\Lambda^c(n)}{c}\right) = n.$$

For later reference, note that $\Lambda(n) \sim \Lambda^c(n) \sim n/\log n$. Now, using Lemma 1, we will prove the following proposition.

Proposition 1. *For $c > 0$ and $k \to \infty$, we have*

$$\mathrm{E}\, N_k \sim \Lambda^c(k+1), \qquad \operatorname{var} N_k \sim \frac{\Lambda^c(k+1)^2}{\Lambda^c(k+1) + k}.$$

Hence, $\operatorname{var} N_k/(\mathrm{E}\, N_k)^2 \to 0$.

Proof. Applying Lemma 1, we have

$$J^c(k) \sim \frac{(k+1)^{-1/2} \Lambda^c(k+1)^{k+3/2} \mathrm{e}^{\Lambda^c(k+1)-k-1-c}}{k^{-1/2} \Lambda^c(k)^{k+1/2} \mathrm{e}^{\Lambda^c(k)-k-c}}$$

$$= \left(\frac{k}{k+1}\right)^{-1/2} \left(\frac{\Lambda^c(k+1)}{\Lambda^c(k)}\right)^{k+1/2} \Lambda^c(k+1) \mathrm{e}^{-1} \mathrm{e}^{\Lambda^c(k+1)-\Lambda^c(k)}.$$

We now focus on the second factor. Note that

$$(\Lambda^c(k))' = \frac{1}{1 + \log(\Lambda^c(k)/c)}.$$

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From the mean value theorem, there exists a point $\theta_k \in (k, k+1)$ such that

$$\Lambda^c(k+1) - \Lambda^c(k) = (\Lambda^c(\theta_k))' = \frac{1}{1+\log(\Lambda^c(\theta_k)/c)} \to 0 \quad \text{as } k \to \infty.$$

Therefore,

$$\left(\frac{\Lambda^c(k+1)}{\Lambda^c(k)}\right)^{k+1/2} = \left(1 + \frac{1}{k}\frac{\Lambda^c(\theta_k)}{\Lambda^c(k)}\frac{k}{\Lambda^c(\theta_k)+\theta_k}\right)^{k+1/2}.$$

In order to show that

$$\left(\frac{\Lambda^c(k+1)}{\Lambda^c(k)}\right)^{k+1/2} \to \mathrm{e} \quad \text{as } k \to \infty,$$

we have to check that

$$\frac{\Lambda^c(\theta_k)}{\Lambda^c(k)}\frac{k}{\Lambda^c(\theta_k)+\theta_k} \to 1.$$

Using the mean value argument once again, it is easy to show that $\Lambda^c(\theta_k)/\Lambda^c(k) \to 1$. To deal with the second part, we write

$$\frac{k}{\Lambda^c(\theta_k)+\theta_k} = \frac{k}{\theta_k/\log(\Lambda^c(\theta_k)/c)+\theta_k} = \frac{k}{\theta_k}\frac{1}{1+1/\log(\Lambda^c(\theta_k)/c)} \to 1.$$

Hence, $J^c(k) = \Lambda^c(k+1) + o(1)$, which is our claim.

We now turn to the variance. It was already mentioned that

$$\operatorname{var} N_k = J^c(k)(J^c(k+1) - J^c(k)).$$

From this, it follows that, for $\theta_{k+1} \in (k+1, k+2)$,

$$\begin{aligned} \operatorname{var} N_k &\sim \Lambda^c(k+1)(\Lambda^c(k+2) - \Lambda^c(k+1)) \\ &= \Lambda^c(k+1)\frac{1}{1+\log(\Lambda^c(\theta_{k+1})/c)} \\ &\sim \frac{\Lambda^c(k+1)^2}{\Lambda^c(k+1)+k}, \end{aligned}$$

which completes the proof.

In Figure 2 we present the results of a numerical experiment which confirms the good quality of the approximation given in Proposition 1.

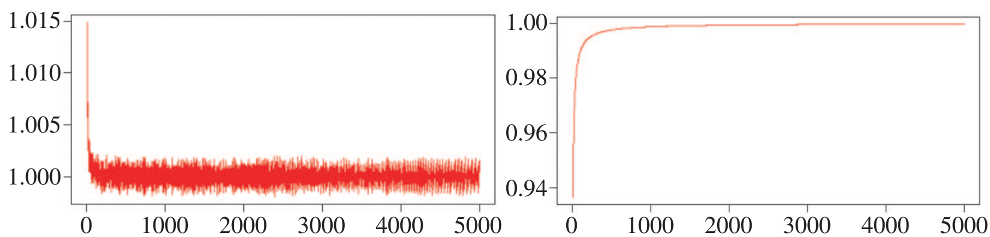

FIGURE 2: The ratios $\mathrm{E}\, N_k/\Lambda^c(k+1)$ (*left*) and $\operatorname{var} N_k/(\Lambda^c(k+1)^2/(\Lambda^c(k+1)+k))$ (*right*) for $c = 5$.

3. General scheme

In this section we follow [4]. Consider

$$F_k = \sum_{m\ge 1} f_k(m), \qquad G_k = \sum_{m\ge 1} m f_k(m),$$

and the ratio $\iota_k = G_k/F_k$. Our aim is to work out the asymptotics for ι_k as $k\to\infty$. The idea is to find $\lambda(k)$, where the sequence $f_k(m)$ achieves its maximum. However, for practical situations, it is sometimes better to use the following approximation. Define the ratio

$$q_k(m) = \frac{f_k(m+1)}{f_k(m)}.$$

Suppose that $q_k(l)$ can be considered as a function $q_y(\lambda)$ with $(y,\lambda)\in[0,\infty)^2$, and let $\lambda(y)$ be the unique solution of the so-called λ-equation

$$q_y(\lambda) = 1.$$

For $\varepsilon > 0$, let $l^* = l^*(k) = \lfloor(1-\varepsilon)\lambda(k)\rfloor$ and $r^* = r^*(k) = \lceil(1+\varepsilon)\lambda(k)\rceil$.

We now list some assumptions needed for the sequence $(f_k(m))$.

(A1) For large enough y, the λ-equation has a unique solution.

(A2) As $k\to\infty$,

$$\frac{f_k(l^*)}{f_k(\lambda)} \to 0, \qquad \frac{f_k(r^*)}{f_k(\lambda)} \to 0.$$

(A3) We have

$$\rho_k = \sup_{m\ge r^*} q_k(m), \qquad \rho'_k = \sup_{m\le l^*} \frac{1}{q_k(m)},$$

and

$$\limsup_k \rho_k < 1, \qquad \limsup_k \rho'_k < 1.$$

Proposition 2. *If assumptions (A1)–(A3) hold then $\iota_k \sim \lambda(k)$ for $k\to\infty$.*

Proof. Decompose F_k as

$$F_k = \sum_{m\le l^*} f_k(m) + \sum_{m\in(l^*,r^*)} f_k(m) + \sum_{r^*\le m} f_k(m) =: I_1(k) + I_2(k) + I_3(k).$$

Now

$$\begin{aligned} I_3 &= f_k(r^*) + f_k(r^*)q_k(r^*) + f_k(r^*)q_k(r^*)q_k(r^*+1) + \cdots \\ &\le f_k(r^*)(1 + \rho_k(r^*) + \rho_k(r^*)^2 + \cdots) \\ &= f_k(r^*)\frac{1}{1-\rho_k(r^*)}. \end{aligned}$$

Since $I_2 \ge f_k(\lambda)$, by assumptions (A2) and (A3) we have $I_3/I_2 \to 0$. Similarly, we show that $I_1/I_2 \to 0$, and, hence, $F_k = I_2(1+o(1))$. In a similar way as before we can demonstrate that $G_k = I'_2(1+o(1))$, where $\sum_{m\in(l^*,r^*)} m f_k(m)$. Now

$$l^* \le \frac{\sum_{l^*<m<r^*} m f_k(m)}{\sum_{l^*<m<r^*} f_k(m)} \le r^*.$$

Since ε is arbitrary, the proof is complete.

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4. Special cases

4.1. Another approach for the (Poi(*a*), Poi(*b*)) case

In this case

$$f_k(l) = l^k \frac{c^l}{l!}, \qquad q_k(l) = \frac{c}{l+1}\left(\frac{l+1}{l}\right)^k,$$

and, hence, the λ-equation is

$$\frac{c}{\lambda+1}\left(\frac{\lambda+1}{\lambda}\right)^y = 1. \tag{1}$$

Proposition 3. *It holds that* $\lambda(y) \sim \Lambda^c(y)$ *as* $y \to \infty$.

Proof. We first note that $\lambda(y) \to \infty$ for $y \to \infty$. Hence, from

$$\log\left(\frac{c}{\lambda+1}\right) + y\log\left(1+\frac{1}{\lambda}\right) = 0,$$

we have

$$\log\left(\frac{c}{\lambda+1}\right) + \frac{y}{\lambda} + o\left(\frac{1}{\lambda}\right).$$

Now

$$y + \lambda o\left(\frac{1}{\lambda}\right) = \lambda\log\frac{\lambda}{c} + \lambda\log\left(1+\frac{1}{\lambda}\right).$$

Thus, $y + O(y) = \lambda\log(\lambda/c)$, which implies that $\lambda(y) = \Lambda^c(y+O(y))$. Simple calculations show that $\Lambda^c(y+O(y)) \sim \Lambda^c(y)$.

From Proposition 3 we see that $\lambda(k) \sim k/\log k$. Using this fact and Stirling's formula, we can verify that assumptions (A2) and (A3) do hold in Proposition 2 (see Appendix A for details of these computations). We thus conclude with the following proposition.

Proposition 4. *It holds that* $\mathrm{E}\, N_k \sim \lambda(k)$ *as* $k \to \infty$.

4.2. (Poi(*a*), mixPoi(*F*))

If ξ is a random variable with distribution F then mixPoi(F) is a mixed Poisson distribution with mixing distribution F. Thus, if $\xi \sim F$ then $X \sim \text{mixPoi}(F)$ if

$$\mathrm{P}(X=k) = \mathrm{E}\left[\frac{\xi^k}{k!}\mathrm{e}^{-\xi}\right].$$

Let

$$C_k(m) = \mathrm{E}(\xi_1 + \cdots + \xi_m)^k \mathrm{e}^{-(\xi_1+\cdots+\xi_m)}.$$

In this case

$$f_k(m) = \frac{a^m}{m!}\mathrm{e}^{-a} C_k(m).$$

In the next lemma we use the standard change-of-measure argument. For this, suppose that the basic probability measure is $\mathrm{P} = P \times P \times \cdots$ on $\Omega = \mathbb{R}\times\mathbb{R}\times\cdots$ and let $\mathrm{P}^{(s)} = P^{(s)} \times P^{(s)} \times \cdots$ be the probability measure defined by $P^{(s)} = \mathrm{e}^{-s\xi}\,\mathrm{d}P/\phi(s)$, where $\phi(s) = \mathrm{E}\,\mathrm{e}^{-\xi s}$ is the Laplace transform and $\mathrm{P}^{(0)} = \mathrm{P}$. Furthermore, for short, we will write $\tilde{\mathrm{E}} = \mathrm{E}^{(1)}$.

Lemma 2. *We have*

$$\mathrm{E}(\xi_1 + \cdots + \xi_m)^k \mathrm{e}^{-(\xi_1+\cdots+\xi_m)} = \phi^m(1)\tilde{\mathrm{E}}(\xi_1 + \cdots + \xi_m)^k.$$

Proof. We have

$$\mathrm{d}\underbrace{P \times P \times \cdots \times P}_{m \text{ times}} = \phi^m(s)\mathrm{e}^{s(\xi_1+\cdots+\xi_m)}\,\mathrm{d}\underbrace{P^{(s)} \times P^{(s)} \times \cdots \times P^{(s)}}_{m \text{ times}}.$$

4.3. (Poi(a), mixPoi(F)): the case of exponential ξ

Recall that if $\xi \sim \mathrm{Exp}(b)$ then

$$\mathrm{E}(\xi_1 + \cdots + \xi_m)^k = \frac{(m+k-1)!}{b^k(m-1)!}.$$

Since

$$\phi^{(s)}(t) = \frac{\phi(t+s)}{\phi(s)},$$

we have, under $\tilde{\mathrm{P}}$, $\xi \sim \mathrm{Exp}(b+1)$. Hence,

$$C_k(m) = \tilde{\mathrm{E}}(\xi_1 + \cdots + \xi_m)^k = \frac{(m+k-1)!}{(b+1)^k(m-1)!}\frac{b}{b+1}.$$

In this case we have

$$f_k(m) = \frac{1}{m!}\left(\frac{ab}{b+1}\right)^m \mathrm{e}^{-a}\left(\frac{1}{b+1}\right)^k \frac{(k+m-1)!}{(m-1)!}$$

and

$$q_k(m) = \frac{ab}{b+1}\frac{k+m}{m(m+1)} = C\frac{k+m}{m(m+1)},$$

where $C = ab/(b+1)$. It is easy to check that the solution to the λ-equation is

$$\lambda(k) = \frac{C - 1 + \sqrt{(1-C)^2 + 4Ck}}{2}.$$

Hence, we can use the approximation

$$\lambda_{\mathrm{approx}}(k) \sim \sqrt{Ck} \quad \text{as } k \to \infty. \tag{2}$$

To check that assumptions (A2) and (A3) hold in Proposition 2, we have to use Stirling's formula and asymptotic relation (2) (see Appendix A for some details of these computations). We thus conclude with the following proposition.

Proposition 5. *It holds that* $\mathrm{E}\,N_k \sim \lambda_{\mathrm{approx}}$ *as* $k \to \infty$.

In Figure 3 we present the results of a numerical experiment which confirms the good quality of the approximation given in Proposition 5.

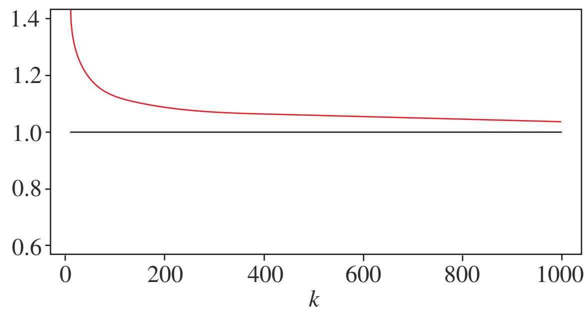

FIGURE 3: The ratio $\mathrm{E}N_k/\sqrt{Ck}$ for $C = 5$.

4.4. (Poi(a), mixPoi(F)): the case of bounded ξ

Let r be the right end of the support of Z. It is known (see, e.g. [1, Problem 5.11]) that $(\tilde{\mathrm{E}}Z^k)^{1/k} = r - \alpha(k)$, where $\lim_{k\to\infty} \alpha(k) = 0$. We need a finer result for $\alpha_m(k)$ for $Z = \bar{S}_m$.

Suppose that $f(x),\ 0 \le x \le r$, is the density function of ξ. We assume that $f(r) = f(r-)$. Then

$$\bar{S}_m = \frac{\xi_1 + \cdots + \xi_m}{m}$$

has density $mf^{*m}(mx)$. Note that $f^{*m}(mr) = 0$ for $m > 1$. Let $\tilde{f}$ be the density function of ξ under $\tilde{\mathrm{P}}$.

Proposition 6. *We have*

$$\tilde{\mathrm{E}}\bar{S}_m^k \sim \frac{r^{k+2}m^2}{(k+1)(k+2)}\tilde{f}^m(r).$$

Hence,

$$\frac{l}{l+1}\frac{\tilde{\mathrm{E}}\bar{S}_{l+1}^k}{\tilde{\mathrm{E}}\bar{S}_l^k} = \tilde{f}(r)(1 + o(1)). \tag{3}$$

The proof of this proposition can be deduced from the following lemmas.

Lemma 3. *Suppose that ξ has density function f. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}f^{*m}(x)\bigg|_{x=mr} = f^m(r),$$

and, hence, for the density $h_m(x)$ of $\bar{S}_m$, we have

$$h_m(r) = m\frac{\mathrm{d}}{\mathrm{d}x}f^{*m}(mx)\bigg|_{x=r} = m^2 f^m(r).$$

Proof. We use induction with respect m. For two density functions f_1 and f_2 on $[0, r_i]$ $(i = 1, 2)$, which are right continuous at r_1 and r_2, respectively,

$$\begin{aligned} f_1 * f_2(x) &= \int_0^{r_2} f_1(x-y)f_2(y)\,\mathrm{d}y \\ &= \int_{x-r_1}^{r_2} f_1(x-y)f_2(y)\,\mathrm{d}y, \qquad r_1 < x < r_1 + r_2. \end{aligned}$$

Hence, for $r_1 < x < r_1 + r_2$,

$$\frac{\mathrm{d}}{\mathrm{d}x} f_1 * f_2(x) = \int_{x-r_1}^{r_2} f_1'(x-y) f_2(y)\,\mathrm{d}y - f_1(r_1) f_2(x-r_1)$$

and

$$\lim_{x\to r_1+r_2} \frac{\mathrm{d}}{\mathrm{d}x} f_1 * f_2(x) = -f_1(r_1-) f_2(r_2-).$$

Lemma 4. *Let g be a probability density function on $[0, r]$ such that $g(r-) = g(r) = 0$ and $g'(r-) < 0$. Then, for $k \to \infty$,*

$$\int_0^r x^k g(x)\,\mathrm{d}x \sim \frac{r^{k+2}}{(k+1)(k+2)}(-g'(r-)).$$

Proof. Integration by parts followed by a substitution yields

$$\begin{aligned}\int_0^r x^k g(x)\,\mathrm{d}x &= -\frac{1}{k+1}\int_0^r x^{k+1} g'(x)\,\mathrm{d}x \\ &= r^{k+2}\frac{1}{(k+1)(k+2)}\int_0^1 (-g'(rx^{1/(k+2)}))\,\mathrm{d}x \\ &\sim \frac{r^{k+2}}{(k+1)(k+2)}(-g'(r-))\end{aligned}$$

for $k \to \infty$.

Our interest is in $\iota_k = F_k/G_k$, where

$$f_k(m) = m^k \frac{c^m}{m!}\tilde{\mathrm{E}}\bar{S}_m^k \quad \text{and} \quad c = a\phi(1).$$

Then the λ-equation is given by

$$\frac{c}{l+1}\left(\frac{l+1}{l}\right)^k \frac{\tilde{\mathrm{E}}\bar{S}_{l+1}^k}{\tilde{\mathrm{E}}\bar{S}_l^k} = 1.$$

Therefore, for large k, we have the following approximation for the λ-equation:

$$\frac{c'}{l+1}\left(\frac{l+1}{l}\right)^{k+2} = 1 \quad \text{and} \quad c' = c\tilde{f}(r). \tag{4}$$

Clearly, the solution of (4) is asymptotically consistent with the solution of (1).

Unfortunately, the convergence in (3) does not seem to be uniform, and, therefore, we cannot conclude with a proposition, only with the following conjecture.

Conjecture 1. *Suppose that $0 < \tilde{f}(r) < \infty$. The solution λ of (4) is asymptotically equivalent with $\Lambda^{c'}(k)$. Furthermore, as $k \to \infty$,*

$$\lambda(k) \sim \mathrm{E}\,N_k.$$

5. Central limit theorem

Suppose now that (Y_i) is a sequence of i.i.d. random variables independent of N_k. We assume that $\mathrm{E}\,Y_1 = 0$ and $\operatorname{var} Y_i = 1$. Our aim is to consider a limit theorem for

$$Z_k = \frac{1}{\sqrt{N_k}} \sum_{i=1}^{N_k} Y_i.$$

We have to prove the following proposition.

Proposition 7. *If, for $k \to \infty$,*

$$\frac{\operatorname{var} N_k}{(\mathrm{E}\,N_k)^2} \to 0 \tag{5}$$

then $Z_k \xrightarrow{\mathrm{D}} \mathcal{N}(0, 1)$.

Proof. For each $a > 0$, because $\mathrm{E}\,N_k \to \infty$,

$$\begin{aligned} \mathrm{P}(N_k < a) &= \mathrm{P}(N_k - \mathrm{E}\,N_k < a - \mathrm{E}\,N_k) \\ &= \mathrm{P}(|N_k - \mathrm{E}\,N_k| > \mathrm{E}\,N_k - a) \\ &\le \frac{\operatorname{var} N_k}{(\mathrm{E}\,N_k - a)^2}. \end{aligned}$$

Hence, $N_k \to \infty$ in probability. Now we can use a classical result (see, e.g. [10, p. 471]) to complete the proof.

Note that condition (5) holds in the $(\mathrm{Poi}(a), \mathrm{Poi}(b))$ case (see Proposition 1).

Appendix A. Details of computations

Formally, we understand $x! = \Gamma(x+1)$. To check assumption (A1), we write

$$\frac{\lambda!}{((1-\varepsilon)\lambda)!} = (1-\varepsilon)^{-1/2-(1-\varepsilon)\lambda} \mathrm{e}^{-\varepsilon\lambda(1-\log\lambda)}(1+o(1)),$$

where we used Stirling's formula, and

$$\begin{aligned} \frac{f_k(l^*)}{f_k(\lambda)} &= \frac{c^{(1-\varepsilon)\lambda}((1-\varepsilon)\lambda)^k}{((1-\varepsilon)\lambda)!} \left(\frac{c^\lambda \lambda^k}{\lambda!}\right)^{-1} (1+o(1)) \\ &= \exp\bigl\{-\varepsilon\lambda(\log c + 1 - \log\lambda) + \bigl(k - \tfrac{1}{2} - (1-\varepsilon)\lambda\bigr)\log(1-\varepsilon)\bigr\}(1+o(1)) \\ &\to 0, \end{aligned}$$

because the dominating terms are $\varepsilon\lambda\log\lambda + k\log(1-\varepsilon)$. Similarly,

$$\frac{\lambda!}{((1+\varepsilon)\lambda)!} = (1+\varepsilon)^{-1/2-(1+\varepsilon)\lambda} \mathrm{e}^{\varepsilon\lambda(1-\log\lambda)}(1+o(1))$$

and

$$\begin{aligned} \frac{f_k(r^*)}{f_k(\lambda)} &= \frac{c^{(1+\varepsilon)\lambda}((1+\varepsilon)\lambda)^k}{((1+\varepsilon)\lambda)!} \left(\frac{c^\lambda \lambda^k}{\lambda!}\right)^{-1} (1+o(1)) \\ &= \exp\bigl\{\varepsilon\lambda(1+\log c) - \varepsilon\lambda\log\lambda - (1+\varepsilon)\lambda\log(1+\varepsilon) + \bigl(k - \tfrac{1}{2}\bigr)\log(1+\varepsilon)\bigr\} \\ &\quad \times (1+o(1)) \\ &\to 0, \end{aligned}$$

because the dominating terms are $-\varepsilon\lambda\log\lambda + k\log(1+\varepsilon)$.

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To check assumption (A2), we write

$$\begin{aligned}
\frac{f_k(r^*)}{f_k(\lambda)} &= \frac{1}{((1+\varepsilon)\lambda)!} C^{(1+\varepsilon)\lambda} \frac{(k+(1+\varepsilon)\lambda-1)!}{((1+\varepsilon)\lambda-1)!} \left(\frac{1}{\lambda!} C^{\lambda} \frac{(k+\lambda-1)!}{(\lambda-1)!}\right)^{-1} (1+o(1)) \\
&= \left(\frac{\lambda!}{((1+\varepsilon)\lambda)!}\right)^2 (1+\varepsilon) \frac{(k+(1+\varepsilon)\lambda-1)!}{(k+\lambda-1)!} (1+o(1)) \\
&= (1+\varepsilon)^{-2(1+\varepsilon)\lambda} \mathrm{e}^{\varepsilon\lambda - 2\varepsilon\lambda\log\lambda} C^{\varepsilon\lambda} \left(1+\frac{\varepsilon\lambda}{k+\lambda-1}\right)^{k+\lambda-1/2} \\
&\quad \times (k+(1+\varepsilon)\lambda-1)^{\varepsilon\lambda}(1+o(1)) \\
&= \exp\{-2(1+\varepsilon)\lambda\log(1+\varepsilon) - 2\varepsilon\lambda\log\lambda + 2\varepsilon\lambda + \varepsilon\lambda\log C + \varepsilon\lambda\log k\}(1+o(1)) \\
&= \exp\{(-2(1+\varepsilon)\log(1+\varepsilon) + 2\varepsilon)(Ck)^{1/2}\}(1+o(1)) \\
&\to 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{f_k(l^*)}{f_k(\lambda)} &= \frac{1}{((1-\varepsilon)\lambda)!} C^{(1-\varepsilon)\lambda} \frac{(k+(1-\varepsilon)\lambda-1)!}{((1-\varepsilon)\lambda-1)!} \left(\frac{1}{\lambda!} C^{\lambda} \frac{(k+\lambda-1)!}{(\lambda-1)!}\right)^{-1} (1+o(1)) \\
&= \left(\frac{\lambda!}{((1-\varepsilon)\lambda)!}\right)^2 (1-\varepsilon) \frac{(k+(1-\varepsilon)\lambda-1)!}{(k+\lambda-1)!} (1+o(1)) \\
&= (1-\varepsilon)^{-2(1-\varepsilon)\lambda} \mathrm{e}^{-\varepsilon\lambda + 2\varepsilon\lambda\log\lambda} C^{-\varepsilon\lambda} \left(1-\frac{\varepsilon\lambda}{k+\lambda-1}\right)^{k+\lambda-1/2} \\
&\quad \times (k+(1-\varepsilon)\lambda-1)^{-\varepsilon\lambda}(1+o(1)) \\
&= \exp\{-2(1-\varepsilon)\lambda\log(1-\varepsilon) + 2\varepsilon\lambda\log\lambda - 2\varepsilon\lambda - \varepsilon\lambda\log C - \varepsilon\lambda\log k\}(1+o(1)) \\
&= \exp\{(-2(1-\varepsilon)\log(1-\varepsilon) - 2\varepsilon)(Ck)^{1/2}\} \\
&\to 0.
\end{aligned}$$

To check assumption (A3), we write

$$\begin{aligned}
\rho_k &= \sup_{m \ge r^*} \frac{C}{m+1} \frac{m+k}{m} \\
&\le \frac{C}{(1+\varepsilon)\sqrt{kC}+1} \left(1 + \frac{k}{(1+\varepsilon)\sqrt{kC}}\right) \\
&= \frac{C(1+\varepsilon)+\sqrt{Ck}}{(1+\varepsilon)+(1+\varepsilon)^2\sqrt{kC}} \\
&\to \frac{1}{(1+\varepsilon)^2} \\
&< 1 \quad \text{as } k \to \infty
\end{aligned}$$

and

$$\begin{aligned}
\rho'_k &= \sup_{m \le l^*} \frac{m+1}{C} \frac{m}{m+k} \\
&\le \frac{(1-\varepsilon)\sqrt{kC}+1}{C} \left(1 - \frac{k}{(1-\varepsilon)\sqrt{kC}+k}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{Ck}(1-\varepsilon) + (1-\varepsilon)^2 kC}{kC + (1-\varepsilon)C\sqrt{kC}} \\
&\to (1-\varepsilon)^2 \\
&< 1 \quad \text{as } k \to \infty.
\end{aligned}
$$

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