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ON THE SINGULARITIES OF PLANE CURVES

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Let Γ be a differentiable curve in a real projective plane P^2 met by every line of P^2 at a finite number of points. The singular points of Γ are inflections, cusps (cusps of the first kind) and beaks (cusps of the second kind). Let $n_1(\Gamma)$, $n_2(\Gamma)$ and $n_3(\Gamma)$ be the number of these points in Γ respectively. Then Γ is *non-singular* if

$$n(\Gamma) = n_1(\Gamma) + n_2(\Gamma) + n_3(\Gamma) = 0;$$

otherwise, Γ is *singular*.

We wish to determine when Γ is singular and then find the minimum value of $n(\Gamma)$. A history and an analysis of this problem were presented in [**1**] and [**2**]. It was shown that we may assume that Γ is a curve of even order (even degree if Γ is algebraic), met by every line in P^2. Then if Γ does not contain any multiple points or if Γ contains only a certain type of multiple point, Γ is singular. Presently, we complete this investigation.

We assume that P^2 has the usual topology. Let $p, q, \ldots$ and $L, M, \ldots$ denote the points and lines of P^2 respectively. Let $\langle p, L, \ldots \rangle$ denote the flat of P^2 spanned by $p, L, \ldots$. The other notations used are self-explanatory.

Differentiable curves. As we are presenting a theory already introduced in [**1**] and [**2**], we list only definitions and relevant results.

Let $T \subset P^2$ be an oriented line. For $t_0 \neq t_1$ in T, $[t_0, t_1]$ denotes the oriented closed line segment of T with initial point t_0 and terminal point t_1. We set

$$[t_0, t_1) = [t_0, t_1]\backslash\{t_1\}, (t_0 t_1] = [t_0, t_1]\backslash\{t_0\} \quad \text{and}$$
$$(t_0, t_1) = [t_0, t_1]\backslash\{t_0, t_1\}.$$

If $U(t) = (t_0, t_1)$ is a neighbourhood of t in T then

$$U^-(t) = (t_0, t), U^+(t) = (t, t_1) \quad \text{and} \quad U'(t) = U^-(t) \cup U^+(t).$$

A *curve* Γ in P^2 is a continuous map from T into P^2. Γ is *differentiable* if the *tangent line*

$$\Gamma_1(t) = \lim_{t \neq t' \to t} \langle \Gamma(t), \Gamma(t') \rangle$$

exists for each $t \in T$ and any line of P^2 meets $\Gamma(T)$ at a finite number of

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points. Henceforth Γ is differentiable and we identify $\Gamma(T)$ with Γ.

Let $\mathcal{M} \subseteq T$ be a segment. We call $\Gamma|_{\mathcal{M}}$ a *subarc* of Γ and identify $\Gamma(\mathcal{M})$ with $\Gamma|_{\mathcal{M}}$. If

$$n = \sup_{L \subset P^2} |L \cap \Gamma(\mathcal{M})|$$

is finite, we say that $\mathcal{M}$ is of *order n*. The *order of a point* $t \in T$, $\mathrm{ord}(t)$, is the minimum order which a $U(t)$ can possess. Clearly $\mathrm{ord}(t) \geqq 2$. A point t is *ordinary* if $\mathrm{ord}(t) = 2$; otherwise t is *singular*. $\mathcal{M}$ is *ordinary* if each point of $\mathcal{M}$ is ordinary.

Let $t \in T$ and $\Gamma(t) \in L \subset P^2$. Then L *supports* Γ *at* t if there is an $L' \neq L$ with $\Gamma(t) \notin L'$ and a $U'(t)$ such that $\Gamma(U'(t))$ is contained in one of the open half-planes of P^2 determined by L and L'. If L does not support Γ at t then L *cuts* Γ *at* t. Let

$$S(t) = \{L \subset P^2 | \Gamma(t) \in L \neq \Gamma_1(t)\}.$$

Then either all $L \in S(t)$ support Γ at t or all $L \in S(t)$ cut Γ at t. Thus there are four types of points in T with respect to Γ: t is *regular* if $L \in S(t)$ [$\Gamma_1(t)$] cuts [supports] Γ at t; t is an *inflection* if $L \in S(t)$ and $\Gamma_1(t)$ both cut Γ at t; t is a *beak* if $L \in S(t)$ and $\Gamma_1(t)$ both support Γ at t; t is a *cusp* if $L \in S(t)$ [$\Gamma_1(t)$] supports [cuts] Γ at t. We note that an ordinary point is regular and hence inflections, cusps and beaks are singular.

Next we note that either every line of P^2 cuts Γ at an even number of points or every line of P^2 cuts Γ at an odd number of points. In the case of the former [latter], we say that Γ is of *even* [*odd*] *order*. Let $\mathcal{M} \subseteq T$. The *index* of $\Gamma(\mathcal{M})$, $\mathrm{ind}(\Gamma(\mathcal{M}))$, is the minimum number of points of $\Gamma(\mathcal{M})$ which can lie on any line of P^2. A point $t \in \mathcal{M}$ is a *simple point of* $\mathcal{M}$, if $\Gamma(t') \neq \Gamma(t)$ for $t' \in \mathcal{M} \setminus \{t\}$; otherwise, t is a *multiple point of* $\mathcal{M}$. Let $m(\Gamma(\mathcal{M}))$ be the number of multiple points of $\mathcal{M}$. We say that $\mathcal{M}$ is *simple* if $m(\Gamma(\mathcal{M})) = 0$. A point $p \in \Gamma(T)$ is *strong* if there exist $t_i \neq t_j$ such that

$$p = \Gamma(t_i) = \Gamma(t_j) \quad \text{and} \quad \mathrm{ind}(\Gamma[t_i, t_j]) = 0.$$

Let $s(\Gamma(\mathcal{M}))$ be the number of strong points of Γ contained in $\Gamma(\mathcal{M})$. Since a simple point of $\mathcal{M}$ need not be a simple point of Γ, we note that $m(\Gamma(\mathcal{M})) = 0$ does not imply that $s(\Gamma(\mathcal{M})) = 0$. If $s(\Gamma) = 0$, we say that Γ is *almost simple*.

Finally let $\mathcal{R}$ be a connected compact set in P^2 such that $\mathcal{R}$ is *bounded*; that is, there is an $L \subset P^2$ not meeting $\mathcal{R}$. We denote by $H(\mathcal{R})$ the convex hull of $\mathcal{R}$ in the affine plane $P^2 \setminus L$. It is clear that if $N \cap \mathcal{R} = \emptyset$ then $H(\mathcal{R})$ is also the convex hull of $\mathcal{R}$ in $P^2 \setminus N$. We say that $\mathcal{R}$ is *convex* if $\mathcal{R} = H(\mathcal{R})$.

As indicated in the introduction, we wish to determine when $n(\Gamma) > 0$ and then find the minimum value of $n(\Gamma)$. Hence we restrict our attention to curves Γ with the property that $n(\Gamma) < \infty$ and (cf. [**1**]) $m(\Gamma) < \infty$. Thus Γ will be the differentiable union of a finite number of simple regular arcs. As such arcs are ordinary (cf. [**4**], p.148), we have that $\Gamma_1(t)$ depends continuously on $t \in T$, a regular point is ordinary (hence a singular point is an inflection, a cusp or a beak) and $\bar{n}(\Gamma) \equiv 0 \pmod 2$ if and only if Γ is of even order where

$$\bar{n}(\Gamma) = n_1(\Gamma) + 2n_2(\Gamma) + n_3(\Gamma).$$

The main theorems. Henceforth we assume that Γ is a differentiable curve of even order with $\mathrm{ind}(\Gamma) > 0$, $n(\Gamma) < \infty$ and $m(\Gamma) < \infty$. We note the following results regarding the minimum number of singular points of Γ.

1. If $m(\Gamma) = 0$ then $n(\Gamma) \geqq 3$ and if $n_2(\Gamma) > 0$ [$n_2(\Gamma) = 0$] then $\bar{n}(\Gamma) \geqq 6$ [4]. ([**2**], 4.)

2. If $s(\Gamma) = 0$ then $n(\Gamma) \geqq 2$ and $\bar{n}(\Gamma) \geqq 4$. ([**1**], 3.1)

Thus it remains to determine the minimum values of $n(\Gamma)$ and $\bar{n}(\Gamma)$ when $s(\Gamma) > 0$ and $m(\Gamma) > 0$. In Figure 1 of [**1**], we presented a non-singular Γ with $s(\Gamma) = m(\Gamma) = 3$ and each strong point, a double point. From that example, it is readily seen that there exists a non-singular Γ with $s(\Gamma) = m(\Gamma) = 1$ and the strong point, a triple point. Finally in Figure 1, we present a non-singular Γ with $m(\Gamma) > s(\Gamma) = 2$.

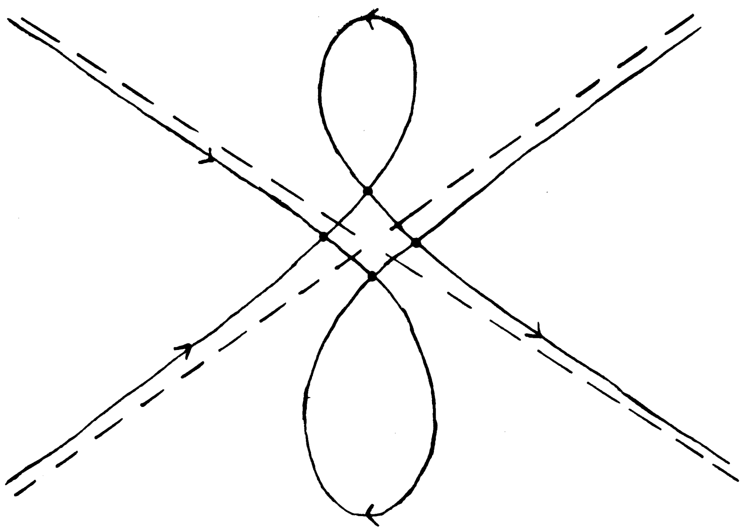

Figure 1.

We now state the main theorems and list the results required for the proofs. By the preceding, we of course assume that every strong point Γ is a double point.

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3. THEOREM. *If* $s(\Gamma) = 1$ *then* Γ *is singular.*

4. THEOREM. *If* $s(\Gamma) = m(\Gamma) = 1$ *then* $n(\Gamma) \geqq 2$ *and* $\bar{n}(\Gamma) \geqq 4$.

5. THEOREM. *If* $s(\Gamma) = m(\Gamma) = 2$ *then* Γ *is singular.*

6. Let (s_1, s_2) be a subarc of order two. Then (s_1, s_2) is simple and ordinary, $\Gamma_1(s) \cap \Gamma[s_1, s_2] = \{\Gamma(s)\}$ for $s \in (s_1, s_2)$ and there is a line $L \subset P^2$ such that

$$L \cap \Gamma[s_1, s_2] = \emptyset \text{ and } \operatorname{ind} \Gamma[s_1, s_2] = 0.$$

Let (t_1, t_2) be ordinary and simple.

7. There exist $s_1 < s_2$ (s_1 preceding s_2) in $[t_1, t_2]$ such that (s_1, s_2) is of order two and

$$\Gamma[t_1, t_2] \subset H(\Gamma[s_1, s_2]).$$

We call $[s_1, s_2]$, the (unique) *convex cover of* $[t_1, t_2]$. ([**1**], 3.15).

8. If $\Gamma(t_1) \neq \Gamma(t_2)$ and $\langle \Gamma(t_1), \Gamma(t_2) \rangle \cap \Gamma(t_1, t_2) = \emptyset$ then (t_1, t_2) is of order two. ([**1**], 3.13).

9. For any $t \in (t_1, t_2)$

$$\Gamma_1(t) \cap \Gamma[t_1, t) = \emptyset \quad \text{or} \quad \Gamma_1(t) \cap \Gamma(t, t_2] = \emptyset.$$

([**1**], 3.12).

10. Let $s(\Gamma(t_1, t_2)) = 0$. If $L \cap \Gamma[t_2, t_1] = \emptyset$ then L meets, and cuts, Γ in exactly two points. If L' is a limit of lines, none of which meets $\Gamma[t_2, t_1]$, then L' cuts Γ in at most two points and these points lie in $[t_1, t_2]$. ([**1**], 3.17).

Finally we note some elementary facts about the convex hull of a subarc of Γ.

11. LEMMA. *Let* $\mathscr{R}^* = H(\Gamma[u, v])$, $u \neq v$ *and* $\operatorname{ind}(\Gamma[u, v]) = 0$. *Let* $t \in (u, v)$ *be an ordinary point with the property that* t *is simple in* $[u, v]$ *and* $\Gamma(t) \in \operatorname{bd}(\mathscr{R}^*)$.

1. *The only supporting line of* $\mathscr{R}^*$ *through* $\Gamma(t)$ *is* $\Gamma_1(t)$.
2. *If* $|\Gamma_1(t) \cap \Gamma[u, v]| = 1$ *then there is a* $U(t)$ *such that*

$$\Gamma(U(t)) \subset \operatorname{bd}(\mathscr{R}^*).$$

3. *If* $|\Gamma_1(t) \cap \Gamma[u, v]| = 2$ *then there is a* $U(t)$ *such that either*

$$\Gamma(U^+(t)) \subset \operatorname{bd}(\mathscr{R}^*) \quad \textit{or} \quad \Gamma(U^-(t)) \subset \operatorname{bd}(\mathscr{R}^*).$$

Proof. 1. Since $\Gamma(t) \in \operatorname{bd}(\mathscr{R}^*)$, there is a line L through $\Gamma(t)$ which supports $\mathscr{R}^*$. Since $\Gamma(t) \in \Gamma(u, v) \subset \mathscr{R}^*$, it follows that L also supports Γ at t and thus $L = \Gamma_1(t)$.

2. Let

$$|\Gamma_1(t) \cap \Gamma[u, v]| = 1.$$

It is clear that there is a $U(t) \subset (u, v)$ such that $U(t)$ is order two and

$$|\Gamma_1(s) \cap \Gamma[u, v]| = 1 \quad \text{for all } s \in U(t).$$

Since each $s \in U(t)$ is ordinary, $\Gamma_1(s)$ supports $\Gamma[u, v]$ at s. As

$$\mathscr{R}^* = H(\Gamma[u, v]),$$

it follows that $\Gamma(s) \in \mathrm{bd}(\mathscr{R}^*)$ for $s \in U(t)$.

3. Since $\Gamma_1(t)$ is a line of support of $\mathscr{R}^*$,

$$|\Gamma_1(t) \cap \Gamma[u, v]| = 2$$

implies that $\Gamma_1(t)$ supports Γ at a point $t' \neq t$ in $[u, v]$. Hence the continuity of tangents yields that there is either a $U^+(t)$ or a $U^-(t)$ in (u, v) with the property that

$$|\Gamma_1(s) \cap \Gamma[u, v]| = 1 \text{ for } s \in U^+(t) \text{ or } s \in U^-(t).$$

Now 11.2 yields 11.3.

12. LEMMA. *Let $\mathscr{R}$ be a closed bounded region bounded by the simple subarc $\Gamma[r, r']$ and the line $L = \langle \Gamma(r), \Gamma(r') \rangle$, $L \cap \Gamma(r, r') = \emptyset$. Let r be ordinary, $\Gamma(v, r) \subset \mathscr{R}$; $\Gamma(v) \in L \backslash \{\Gamma(r)\}$. Then $\Gamma(v, r)$ is not both simple and ordinary.*

Proof. Let $\Gamma(v, r)$ be simple. As $|L \cap \Gamma| < \infty$ and v may be replaced by any $v' \in (v, r)$ satisfying $\Gamma(v') \in L \backslash \{\Gamma(r)\}$, we may assume that

$$\Gamma(v, r) \subset \mathrm{int}(\mathscr{R}).$$

Since $\Gamma(r) \in \mathrm{bd}(\mathscr{R})$ and $\Gamma(r)$ is ordinary, $L = \Gamma_1(r)$ by 11.1.

Put $\mathscr{R}^* = H(\mathscr{R})$ and thus

$$\mathscr{R}^* = H(\Gamma[r, r']) = H(\Gamma[v, r']).$$

For $w \in (v, r)$, we note that

$$\mathscr{R}^* = H(\Gamma[w, r']) \quad \text{and} \quad |\Gamma_1(r) \cap \Gamma[w, r']| = 2.$$

Hence by 11.3, there is a $U(r) \subset (v, r')$ such that either $\Gamma(U^+(r))$ or $\Gamma(U^-(r))$ lies in $\mathrm{bd}(\mathscr{R}^*)$. Since

$$\Gamma(U^-(r)) \subset \Gamma(v, r) \subset \mathrm{int}(\mathscr{R}) \subseteq \mathrm{int}(\mathscr{R}^*),$$

we have

(i) $\Gamma(U^+(r)) \subset \mathrm{bd}(\mathscr{R}^*)$.

Suppose that $\Gamma(v, r)$ is ordinary. Then (v, r) is of order two by 8. Since

$$L \cap \Gamma[v, r] = \{\Gamma(v), \Gamma(r)\},$$

L and $\Gamma(v, r)$ bound a bounded closed region $\mathscr{R}'$. Clearly,

$$\mathscr{R}' = H(\Gamma[v, r]) \subset \mathscr{R}^*.$$

Since $\mathscr{R}^*$ is a closed bounded region in P^2, $\mathscr{R}^*$ is contained in some affine restriction A^2 of P^2. In A^2, we note that $\Gamma(r)$ is the initial point of two opposite rays $\mathscr{L}$ and $\mathscr{L}'$ on L, say $\Gamma(v) \in \mathscr{L}$. Since (v, r) is of order two, every ray from $\Gamma(r)$ meets $\Gamma(v, r)$ in at most one point. Let $w \in [v, r]$ move from v to r. Then the ray from $\Gamma(r)$ through $\Gamma(w)$ rotates monotonically about $\Gamma(r)$ starting from $\mathscr{L}$ and hence, ending at $\mathscr{L}'$.

Next let $s \in (r, r')$ tend to r. Since Γ is ordinary at r, the ray from $\Gamma(r)$ through $\Gamma(s)$ necessarily converges to the ray opposite $\mathscr{L}'$; that is, $\mathscr{L}$. Since L supports Γ at r, this implies that $\Gamma(s) \in \text{int}(\mathscr{R}^*)$. This is a contradiction by i).

13. Lemma. *Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be convex sets in P^2 such that $\mathscr{R} = \mathscr{R}_1 \cap \mathscr{R}_2$ is connected. Then there is a line $L \subset P^2$ such that*

$$L \cap (\mathscr{R}_1 \cup \mathscr{R}_2) = \emptyset.$$

Proof. Since $\mathscr{R}_i$ is convex, there is an L_i such that $L_i \cap \mathscr{R}_i = \emptyset$; $i = 1, 2$. We may assume that $L_1 \neq L_2$, $\mathscr{R}$, $\mathscr{R}_1$ and $\mathscr{R}_2$ are mutually distinct and L_1 $[L_2]$ is not a supporting line of $\mathscr{R}_2$ $[\mathscr{R}_1]$. Let $\{q\} = L_1 \cap L_2$ and denote by $\mathscr{Q}$ and $\mathscr{Q}'$ the closed half-planes of P^2 determined by L_1 and L_2. Since

$$(L_1 \cup L_2) \cap \mathscr{R} = \emptyset,$$

$\mathscr{R} \subset \text{int}(\mathscr{Q}')$ say. Let $L^* \subset \mathscr{Q}'$, $L_1 \neq L^* \neq L_2$ and set

$$\mathscr{R}_i^* = \mathscr{Q} \cap \mathscr{R}_i; \ i = 1, 2.$$

As $\mathscr{R} \subset \text{int}(\mathscr{Q}')$,

$$L_1 \cap \text{int}(\mathscr{R}_2) \neq \emptyset \quad \text{and} \quad L_2 \cap \text{int}(\mathscr{R}_1) \neq \emptyset$$

imply that $\mathscr{R}_1^*$ and $\mathscr{R}_2^*$ are non-empty, disjoint convex sets in $P^2 \backslash L^*$. Hence (cf. [**3**]) there exist two distinct lines N_1 and N_2 such that $N_i \backslash L^*$ supports and separates $\mathscr{R}_1^*$ and $\mathscr{R}_2^*$ in $P^2 \backslash L^*$; $i = 1, 2$. Put $N_1 \cap N_2 = \{p\}$. Let $\mathscr{P}$ and $\mathscr{P}'$ be the closed half-planes in P^2 determined by N_1 and N_2. Then by our construction, $q \neq p \in \text{int}(\mathscr{Q})$ and $\mathscr{R}_1^*$ and $\mathscr{R}_2^*$ are both contained in $\mathscr{P}'$ say.

If $q \in \mathscr{P}$ then $\langle p, q\rangle \subset \mathscr{P} \cap \mathscr{Q}$ and it follows that either

$$\langle p, q\rangle \cap (\mathscr{R}_1 \cup \mathscr{R}_2) = \emptyset \quad \text{or} \quad \langle p, q\rangle \cap (\mathscr{R}_1 \cup \mathscr{R}_2) = \{p\}.$$

In the latter case, $\langle p, q\rangle$ is disjoint from say $\mathscr{R}_2$ and supports $\mathscr{R}_1$ at p. Hence a suitable line through q close to $\langle p, q\rangle$ is disjoint from $\mathscr{R}_1 \cup \mathscr{R}_2$.

Let $q \in \mathscr{P}'$. As N_1 and N_2 separate $\mathscr{R}_1^*$ and $\mathscr{R}_2^*$ in $\mathscr{Q}$, this readily yields that N_1 or N_2, say N_1, does not meet $(L_1 \cap \mathscr{R}_2^*) \cup (L_2 \cap \mathscr{R}_1^*)$. Since

$$L_2 \cap N_1 \in \mathscr{Q} \backslash \mathscr{R}_1^*,$$

we obtain that $L_2 \cap N_1 \notin \mathscr{R}_1$. As $\mathscr{R}_1$ and $\mathscr{R}_1 \cap N_1$ are convex in $P^2 \backslash L_1$,

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this implies that $\mathscr{R}_1 \cap N_1 \cap \mathscr{Q}' = \emptyset$ and hence $N_1 \backslash L_1$ supports $\mathscr{R}_1$ in $P^2 \backslash L_1$. Similarly, $N_1 \backslash L_2$ supports $\mathscr{R}_2$ in $P^2 \backslash L_2$. Altogether, N_1 supports both $\mathscr{R}_1$ and $\mathscr{R}_2$.

We note that the closed segments $N_1 \cap \mathscr{R}_1^*$ and $N_1 \cap \mathscr{R}_2^*$ lie in $N_1 \cap \mathscr{Q}$ and are disjoint. Hence there is a point $b \in (N_1 \cap \mathscr{Q}) \backslash (\mathscr{R}_1^* \cup \mathscr{R}_2^*)$ which separates them in $N_1 \cap \mathscr{Q}$ and there is a line N through b, close to N_1, which does not meet $\mathscr{R}_1^* \cup \mathscr{R}_2^*$. Since

$$N_1 \cap \mathscr{Q}' \cap (\mathscr{R}_1 \cup \mathscr{R}_2) = \emptyset,$$

N can be chosen so that it does not meet $(\mathscr{R}_2 \cup \mathscr{R}_2) \cap \mathscr{Q}'$. Thus

$$N \cap (\mathscr{R}_1 \cup \mathscr{R}_2) = \emptyset.$$

14. LEMMA. *Let* (x, y) *and* (u, v) *be subarcs of order two with the property that* $L = \langle \Gamma(x), \Gamma(y) \rangle$ *is a line and* $\Gamma(x, y) \cap \mathscr{R}_1 = \emptyset$ *where*

$$\mathscr{R}_1 = H(\Gamma[u, v]) \quad \text{and} \quad \mathscr{R}_2 = H(\Gamma[x, y]).$$

Then

1. *there is a line* N' *such that* $N' \cap (\mathscr{R}_1 \cup \mathscr{R}_2) = \emptyset$ *or*
2. $\{\Gamma(x), \Gamma(y)\} \subset \mathscr{R}_1$ *and there is a line* N *such that* $N \cap \mathscr{R}_1 = \emptyset$ *and* N *meets, and cuts,* Γ *at exactly one point of* (x, y).

Proof. If not 14.1 then $\mathscr{R}_1 \cap \mathscr{R}_2$ is not connected by 13. Since $\mathscr{R}_1$ and $\mathscr{R}_2$ are convex, the same applies to

$$\mathscr{R}_1 \cap \text{bd}(\mathscr{R}_2) = \mathscr{R}_1 \cap (\Gamma(x, y) \cup (L \cap \mathscr{R}_2)) = \mathscr{R}_1 \cap L \cap \mathscr{R}_2.$$

As $L \cap \mathscr{R}_1$ and $L \cap \mathscr{R}_2$ are closed segments of L and $L \cap \mathscr{R}_2$ has the end points $\Gamma(x)$ and $\Gamma(y)$, this yields that

$$\{\Gamma(x), \Gamma(y)\} \subset \mathscr{R}_1$$

and there is a point $p \in (L \cap \mathscr{R}_2) \backslash \mathscr{R}_1$. As $\mathscr{R}_1$ is convex, there is a line N through p disjoint from $\mathscr{R}_1$. Since

$$\{\Gamma(x), \Gamma(y)\} \subset \text{bd}(\mathscr{R}_1),$$

$N \neq L$ and 14.2 follows.

15. LEMMA. *Let* $m(\Gamma) = s(\Gamma)$ *and let* $p = \Gamma(t_1) = \Gamma(t_2)$ *be a double point of* Γ *such that* (t_1, t_2) *is of order two,* t_1 *or* t_2 *is ordinary,*

$$\text{ind}(\Gamma[t_2, t_1]) > 0 \quad \text{and} \quad s(\Gamma[t_1, t_2]) = 1.$$

Then there is a differentiable curve Γ^* *of even order such that* $\text{ind}(\Gamma^*) > 0$, $m(\Gamma^*) = s(\Gamma^*) = s(\Gamma) - 1$, $n_j(\Gamma^*) = n_j(\Gamma)$ *for* $j = 1, 3$ *and* $n_2(\Gamma^*)$ *equals* $n_2(\Gamma)$ *or* $n_2(\Gamma) + 1$.

Proof. Case 1. $\Gamma_1(t_1) = \Gamma_1(t_2)$.

Let T^* be the closed segment $[t_2, t_1]$ with t_2 and t_1 identified, say $\bar{t} \equiv t_1 \equiv t_2$. Let $\Gamma^*: T^* \to P^2$ be the curve defined by $\Gamma^*(t) = \Gamma(t)$

for $t \in T^*$. Since Γ is differentiable and $m(\Gamma) = s(\Gamma)$, we obtain that Γ^* is differentiable and

$$m(\Gamma^*) = s(\Gamma^*) = s(\Gamma) - 1.$$

It is easy to check that if t_1 $[t_2]$ is ordinary then $\bar{t}$ is the same type of point as t_2 $[t_1]$. Thus $n_i(\Gamma^*) = n_i(\Gamma)$ for $i = 1, 2, 3$, $\bar{n}(\Gamma^*) \equiv 0 (\text{mod } 2)$ and Γ^* is of even order. Finally

$$\text{ind}(\Gamma[t_2, t_1]) > 0$$

implies that $\text{ind}(\Gamma^*) > 0$.

Case 2. $\Gamma_1(t_1) \neq \Gamma_1(t_2)$ and t_1, t_2 are ordinary.

Since $[t_1, t_2]$ is ordinary with

$$\text{ind}(\Gamma[t_1, t_2]) = 0 \quad \text{and} \quad s(\Gamma[t_1, t_2]) = 1,$$

there exists an ordinary subarc (u, v) such that $t_1 < t_2$ in (u, v),

$$\text{ind}(\Gamma[u, v]) = 0 \quad \text{and} \quad s(\Gamma[u, v]) = 1.$$

Let $t^* \in (t_1, t_2)$. Since (t_1, t_2) is of order two and $\Gamma_1(t_1)$ $[\Gamma_1(t_2)]$ cuts Γ at t_2 $[t_1]$, there exist $t_1^* \in (u, t_1)$ and $t_2^* \in (t_2, v)$ such that

(1) any line meets Γ in at most three points of (t_1^*, t_2^*),

(2) (t_1^*, t^*) and (t^*, t_2^*) are both of order two,

(3) each line through two points of $\Gamma(t_1^*, t_1]$ $[\Gamma[t_2, t_2^*)]$ cuts Γ in $[t_2, t_2^*)$ $[(t_1^*, t_1]]$ and

(4) $\Gamma(t_1^*, t_2^*)$ has no double tangents, and a tangent of $\Gamma(t_1^*, t_2^*)$ meets this arc at no more than one other point.

Finally, let $\mathscr{P}$ be the closed triangle in P^2 with the vertices $\Gamma(t_1^*)$, $\Gamma(t_2^*)$ and $\Gamma(t^*)$ which contains the point p.

Let $\Gamma^*: T \to P^2$ be a curve with the property that $\Gamma^*(t) = \Gamma(t)$ for $t \in [t_2^*, t_1^*]$, t_i^* is an ordinary point of Γ^* with $\Gamma_1^*(t_i^*) = \Gamma_1(t_i^*)$ $(i = 1, 2)$, $[t_1^*, t_2^*]$ is a simple subarc of Γ^* such that $\Gamma^*(t_1^*, t_2^*) \subset \mathscr{P}$ and (t_1^*, t^*) and (t^*, t_2^*) are of order two and finally t^* is a cusp of Γ^* with

$$\Gamma^*(t^*) = \Gamma(t^*) \quad \text{and} \quad \Gamma_1^*(t^*) = \langle \Gamma(t^*), p \rangle;$$

cf. Figure 2.

Clearly Γ^* is a differentiable curve with

$$n_j(\Gamma^*) = n_j(\Gamma) \quad \text{for } j = 1, 3,$$

$$n_2(\Gamma^*) = n_2(\Gamma) + 1 \quad \text{and}$$

$$s(\Gamma^*) = m(\Gamma^*) = m(\Gamma^*[t_2^*, t_1^*]) = m(\Gamma[t_2^*, t_1^*]) = m(\Gamma) - 1 = s(\Gamma) - 1.$$

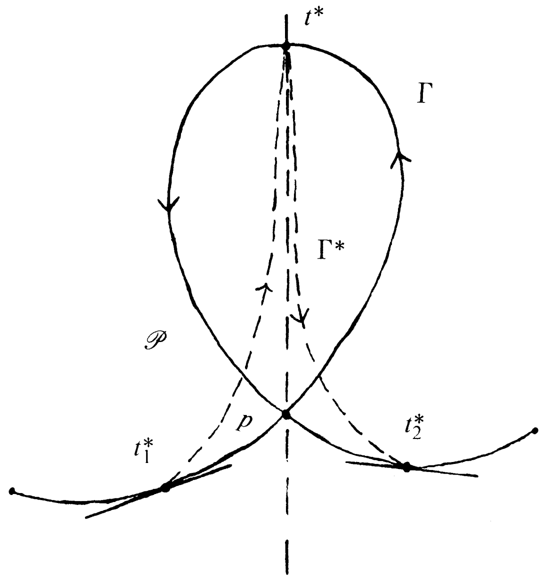

Figure 2.

It remains to show that $\mathrm{ind}(\Gamma^*) > 0$.

Let $L \subset P^2$. If $L \cap \Gamma[t_2^*, t_1^*] \neq \emptyset$ then

$$|L \cap \Gamma^*| \geqq |L \cap \Gamma^*[t_2^*, t_1^*]| = |L \cap \Gamma[t_2^*, t_1^*]| > 0.$$

Thus we may assume that

(5) $\quad L \cap \Gamma[t_2^*, t_1^*] = \emptyset.$

Suppose that L supports Γ at some $t \in (t_1^*, t_2^*)$ and thus $L = \Gamma_1(t)$. Since

$$\mathrm{ind}(\Gamma[t_2, t_1]) > 0$$

by assumption, L necessarily cuts $\Gamma[t_2, t_1]$ in at least one point. By (5), such a point lies in $\Gamma(t_1^*, t_2^*)$ and by (4), there is not more than one such point. Thus L supports Γ and L cuts Γ at one point each. Since Γ is of even order, this is a contradiction and hence L cuts Γ at every point of intersection.

We again note that $L \cap \Gamma[t_2, t_1] = \emptyset$ and by (5),

$$L \cap \Gamma[t_2, t_1] \subset \Gamma(t_1^*, t_1] \cup \Gamma[t_2, t_2^*).$$

As Γ is of even order, the preceding result implies that

$$|L \cap \Gamma| = |L \cap \Gamma(t_1^*, t_2^*)|$$

is even. Hence by (1),

$$L \cap \Gamma(t_1^*, t_2^*) = \{\Gamma(t'), \Gamma(t'')\}$$

where $\{t', t''\} \subset (t_1^*, t_1] \cup [t_2, t_2^*)$ and $t' \neq t''$.

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If $\{t', t''\} \subset (t_1^*, t_1]$ say, then $L \cap \Gamma[t_2, t_2^*) \neq \emptyset$ by (3). Since this is impossible, each of $(t_1^*, t_1]$ and $[t_2, t_2^*)$ contains exactly one of t' and t''. Hence $L \cap \mathscr{P}$ separates $\Gamma(t^*)$ from both $\Gamma(t_1^*)$ and $\Gamma(t_2^*)$ in $\mathscr{P}$ and L necessarily meets both $\Gamma^*(t_1^*, t^*)$ and $\Gamma^*(t^*, t_2^*)$.

Finally, we observe that the preceding yields that any line meets both $\Gamma^*(t_1^*, t_2^*)$ and $\Gamma(t_1^*, t_2^*)$ with the same parity. Thus Γ^* is also of even order.

Case 3. $\Gamma_1(t_1) \neq \Gamma_1(t_2)$ and t_1 or t_2 is singular.

Let t_1 be singular, say.

We choose $t_1^* \in (t_1, t_2)$ and $t_2^* \in (t_2, t_1)$ so close to t_2 that $\Gamma(t_1^*, t_2^*)$ is an arc of order two and that $\mathscr{P}'$, one of the closed triangles bounded by $\Gamma_1(t_1^*)$, $\Gamma_1(t_2^*)$ and $\langle \Gamma(t_1^*), \Gamma(t_2^*) \rangle$, contains $\Gamma[t_1^*, t_2^*]$. We may clearly assume that p is the only double point and $\Gamma(t_1)$ is the only singular point in $\mathscr{P}'$; cf. Figure 3.

The arc $\Gamma[t_1^*, t_2^*]$ decomposes $\mathscr{P}'$ into two subsets. If t_1^* and t_2^* are sufficiently close to t_2, one of these subsets, say $\mathscr{P}_0'$, does not meet $\Gamma(t_2^*, t_1)$. Let $\Gamma':T \rightarrow P^2$ be a curve with the property that $\Gamma'(t) = \Gamma(t)$ for $t \in [t_2^*, t_1^*]$ and $\Gamma'[t_1^*, t_2^*]$ is a convex curve in $\mathscr{P}_0'$ with

$$\Gamma_1'(t_i^*) = \Gamma_1(t_i^*); \quad i = 1, 2.$$

Then Γ' is a curve of even order with

$$\operatorname{ind}(\Gamma') > 0 \quad \text{and} \quad n_j(\Gamma') = n_j(\Gamma); j = 1, 2, 3.$$

If t_1 is a cusp or a beak (case (a)) then

$$s(\Gamma') = m(\Gamma') = S(\Gamma) - 1$$

and $\Gamma^* = \Gamma'$ has the required property. If t_1 is an inflection (case (b)) then

$$s(\Gamma') = m(\Gamma') = s(\Gamma)$$

and Γ' has a double point in $\mathscr{P}_0'$ which satisfies the assumptions of case 2.

16. *Remarks*. Since

$$\Gamma[t_2^*, t_1^*] = \Gamma^*[t_2^*, t_1^*] \quad \text{and}$$

$$\Gamma(t_1^*, t_2^*) \subset \operatorname{int}(H(\Gamma[t_1^*, t_2^*]),$$

$\operatorname{ind}(\Gamma) > 0$ implies that the construction in Figure 3(a) results in a curve Γ^* satisfying 15 even if $\operatorname{ind}(\Gamma[t_2, t_1]) = 0$.

The construction in Figure 3(b) performed when t_1 is an inflection point results in a differentiable curve Γ' of even order with $\operatorname{ind}(\Gamma') > 0$, $m(\Gamma') = s(\Gamma') = s(\Gamma)$ and $n_i(\Gamma') = n_i(\Gamma)$ for $i = 1, 2, 3$. Furthermore, Γ' contains a double point $p' = \Gamma'(t_1') = \Gamma'(t_2')$ such that neither (t_1', t_2') nor (t_2', t_1') is ordinary.

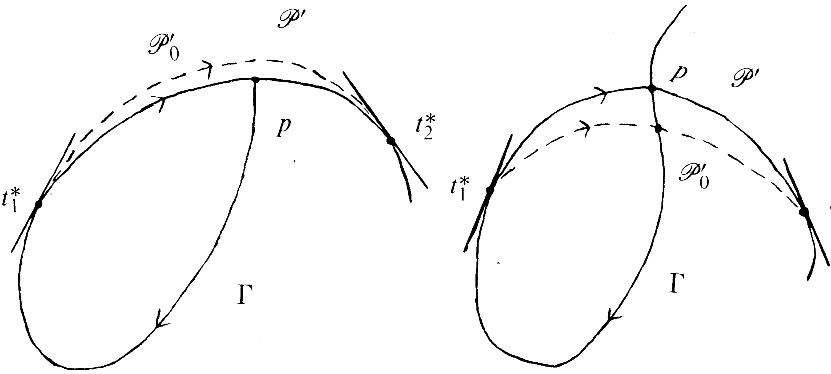

Figure 3.

As a final comment, we note that a similar construction allows us to replace any simple cusp [beak] of Γ by a pair of inflections [one inflection] in such a manner that the resultant curve $\tilde{\Gamma}$ has the property that $s(\tilde{\Gamma}) = S(\Gamma)$, $m(\tilde{\Gamma}) = m(\Gamma)$, $\mathrm{ind}(\tilde{\Gamma}) > 0$ and $\tilde{\Gamma}$ is of even order; cf. **[2]**, p. 147.

Proof of Theorem 3. Let $p = \Gamma(t_1) = \Gamma(t_2)$, $t_1 \neq t_2$, be the only strong point of Γ with

$$\mathrm{ind}(\Gamma[t_1, t_2]) = 0.$$

We assume that (t_1, t_2) and (t_2, t_1) are ordinary.

Since $\mathrm{ind}(\Gamma[t_1, t_2]) = 0$, every multiple point of (t_1, t_2) is then the common end-point of a subarc of index 0 in $\Gamma[t_1, t_2]$ and is therefore strong. Thus $s(\Gamma) = 1$ implies that (t_1, t_2) is simple. Then by 7, there exist $s_1 < s_2$ in $[t_1, t_2]$ such that (s_1, s_2) is of order two and

$$\Gamma[t_1, t_2] \subset \mathscr{R} = H(\Gamma[s_1, s_2]).$$

Let $\Gamma[v_1, v_2]$ be the maximal subarc of Γ contained in $\mathscr{R}$; $v_2 < v_1$ in $[t_2, t_1]$. If $t_1 \neq s_1$ then $v_1 \neq s_1$, $\Gamma(v_1) \notin \Gamma[s_1, s_2)$ and $\Gamma(v_1, s_1)$ has index 0. As in the preceding, $\mathrm{ind}(\Gamma(v_1, s_1)) = 0$ implies that every multiple point of (v_1, s_1) is strong. Thus $t_2 \notin (v_1, s_1)$ and $s(\Gamma) = 1$ imply that (v_1, s_1) is simple. By 12, (v_1, s_1) is not ordinary and $n(\Gamma) \geqq 1$. Hence we assume that $t_1 = s_1$ and by symmetry, $t_2 = s_2$.

Suppose that $\mathrm{ind}(\Gamma[t_2, t_1]) > 0$. Since (t_2, t_1) is ordinary, (t_2, t_1) is not simple by 7. Thus $m(\Gamma[t_2, t_1]) < \infty$ implies that there exist $u < u'$ in (t_2, t_1) such that $q = \Gamma(u) = \Gamma(u')$ and (u, u') is simple. But then $\mathrm{ind}(\Gamma[u, u']) = 0$ by 7 and $q \neq p$ is strong; a contradiction. Since $\mathrm{ind}(\Gamma[t_2, t_1]) = 0$, we assume as in the preceding that (t_2, t_1) is of order two. Let

$$\mathscr{R}' = H(\Gamma[t_2, t_1]).$$

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Since $\Gamma_1(t_i)$ meets Γ at exactly t_1 and t_2 and Γ is of even order, $\Gamma_1(t_i)$ supports Γ at both t_1 and t_2 or $\Gamma_1(t_i)$ cuts Γ at both t_1 and t_2; $i = 1, 2$. If $\Gamma_1(t_1)$ and $\Gamma_1(t_2)$ both support Γ at t_1 and t_2 then $\Gamma_1(t_1) = \Gamma_1(t_2)$ and it is easy to check that $\mathcal{R} \cap \mathcal{R}'$ is connected. Hence by 13, there is a line not meeting $\mathcal{R} \cup \mathcal{R}'$. Since $\Gamma \subset \mathcal{R} \cup \mathcal{R}'$, we obtain that $\text{ind}(\Gamma) = 0$. Thus $\Gamma_1(t_i)$ cuts Γ at t_i and t_i is singular.

In Figure 4, we present a Γ of even order with $\text{ind}(\Gamma) > 0$, $s(\Gamma) = 1 \neq m(\Gamma)$ and $n(\Gamma) = 1$.

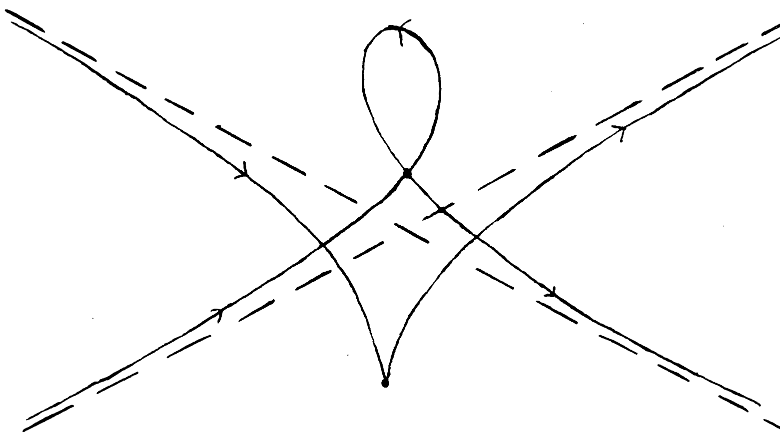

Figure 4.

Proof of Theorem 4. Let $p = \Gamma(t_1) = \Gamma(t_2)$, $t_1 \neq t_2$. Since $\bar{n}(\Gamma)$ is even, we need only to show that $n(\Gamma) \geqq 3$ or $n_2(\Gamma) \geqq 2$.

Case 1. Neither (t_1, t_2) nor (t_2, t_1) is ordinary.

If t_1 or t_2 is singular or if (t_1, t_2) or (t_2, t_1) contains more than one singular point then $n(\Gamma) \geqq 3$. Hence we assume that t_1 and t_2 are ordinary and say u_1 $[u_2]$ is the only singular point of (t_1, t_2) $[(t_2, t_1)]$. If u_1 or u_2 is a cusp then $\bar{n}(\Gamma) \geqq 4$ and hence we assume that they are not cusps. From 16, we may then assume that u_1 and u_2 are inflections. Finally as p is strong, we may assume that say

$$\text{ind}(\Gamma[t_2, t_1]) = 0.$$

Let $\mathcal{R} = H(\Gamma[t_2, t_1])$. Since no line through $\Gamma(u_2)$ supports Γ at u_2, $\Gamma(u_2) \in \text{int}(\mathcal{R})$. Hence some line L supports $\mathcal{R}$ in at least two distinct points of $\Gamma[t_2, t_1]$ and there is a segment $[r_2, r_1] \subset [t_2, t_1]$ such that

$$L = \langle \Gamma(r_1), \Gamma(r_2) \rangle, \quad L \cap \Gamma(r_2, r_1) = \emptyset \quad \text{and}$$

$$L \neq \Gamma_1(t) \quad \text{for all } t \in (r_2, r_1).$$

As $\{\Gamma(r_1), \Gamma(r_2)\} \subset \text{bd}(\mathcal{R})$, we have that r_1 and r_2 are ordinary. Then by 11,

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$$L = \Gamma_1(r_2) \text{ and either } r_1 = t_1 \text{ or } L = \Gamma_1(r_1).$$

Finally, let $\mathscr{R}'$ be the closed region in $\mathscr{R}$ bounded by $\Gamma[r_2, r_1]$ and L.

a) $r_1 \neq t_1$ and (r_2, r_1) is ordinary.

By 8, (r_2, r_1) is of order two and hence $\mathscr{R}' = H(\Gamma[r_2, r_1])$. Since L is a supporting line of both $\mathscr{R}$ and $\mathscr{R}'$ and $\Gamma[t_2, t_1]$ is simple, it follows that $\Gamma[t_2, t_1] \subset \mathscr{R}'$ and thus $\mathscr{R} = \mathscr{R}'$.

Since $\text{ind}(\Gamma) > 0$, there is a maximal subarc $\Gamma[v_2, v_1] \neq \Gamma$ in $\mathscr{R}$ with

$$[r_2, r_1] \subset (t_2, t_1) \subseteq (v_2, v_1).$$

As $m(\Gamma) = 1$, (v_2, r_2) and (r_1, v_1) are both simple and thus by 12, (v_2, r_2) and (r_1, v_1) are both singular. As $\{u_1, u_2\} \subset (v_2, r_2) \cup (r_1, v_1)$, we may assume that say

$$u_1 \in (t_1, t_2) \cap (r_1, v_1) = (t_1, v_1)$$

and

$$u_2 \in (t_2, t_1) \cap (v_2, r_2) = (t_2, r_2).$$

Finally, we note that since $\Gamma[v_2, v_1]$ is the maximal subarc contained in $\mathscr{R}$ and

$$\Gamma(v_i) \notin \text{bd}(\mathscr{R})\backslash\Gamma[r_2, r_1],$$

L cuts Γ at v_1 and v_2.

Suppose that (v_1, v_2) is ordinary and let $t \in (v_1, v_2) \subset (u_1, t_2)$. By 9,

$$\Gamma_1(t) \cap \Gamma[u_1, t) = \emptyset \quad \text{or} \quad \Gamma_1(t) \cap \Gamma(t, t_2) = \emptyset.$$

Hence $\Gamma(v_1)$ or $\Gamma(v_2)$ does not lie on $\Gamma_1(t)$, $\Gamma_1(t) \neq L$ and L cuts Γ at every point of $L \cap \Gamma[v_1, v_2]$. Since $\Gamma[v_1, v_2]$ is simple with index 0, L cuts Γ at only v_1 and v_2 by 10. Altogether then

$$L \cap \Gamma(v_1, v_2) = \emptyset$$

and thus (v_1, v_2) is of order two by 8. Let

$$\mathscr{R}'' = H(\Gamma[v_1, v_2]).$$

Since

$$\Gamma(v_1, v_2) \cap \Gamma(r_2, r_1) = \emptyset \quad \text{and} \quad L \cap \Gamma(v_1, v_2) = \emptyset,$$

we have that

$$\Gamma(v_1, v_2) \cap \text{bd}(\mathscr{R}) = \emptyset.$$

As L cuts Γ at v_1 and $\Gamma[v_2, v_1] \subset \mathscr{R}$, it follows that

$$\Gamma(v_1, v_2) \cap \mathscr{R} = \emptyset.$$

Thus by 14, either $\mathscr{R} \cup \mathscr{R}''$ is bounded in P^2 or there is a line not meeting $\mathscr{R}$ which meets and cuts Γ at exactly one point. Since $\Gamma \subset \mathscr{R} \cup \mathscr{R}''$, the latter yields that $\mathrm{ind}(\Gamma) = 1$; a contradiction. Thus (v_1, v_2) is not ordinary and $n(\Gamma) \geqq 3$ or (r_2, r_1) is not ordinary.

b) $r_1 \neq t_1$ and (r_2, r_1) is not ordinary.

Then $u_2 \in (r_2, r_1) \subset (t_2, t_1)$ and we may assume that $[t_2, r_2]$ and $[r_1, t_1]$ are ordinary. Since $\Gamma(t_2) \in \mathscr{R}\backslash\Gamma[r_2, r_1]$, either

$$\Gamma[t_2, r_2] \subset \mathrm{int}(\mathscr{R}') \quad \text{or} \quad \Gamma[t_2, r_2) \subset \mathscr{R}\backslash\mathscr{R}'.$$

Similarly, either

$$\Gamma(r_1, t_1] \subset \mathrm{int}(\mathscr{R}') \quad \text{or} \quad \Gamma(r_1, t_1] \subset \mathscr{R}\backslash\mathscr{R}'.$$

Since $\Gamma(t_1) = \Gamma(t_2)$, it follows that $\Gamma[t_2, r_2) \cup \Gamma(r_1, t_1]$ is in either $\mathrm{int}(\mathscr{R}')$ or $\mathscr{R}\backslash\mathscr{R}'$.

We recall that $L = \Gamma_1(r_1) = \Gamma_1(r_2)$ supports both Γ and the convex set $\mathscr{R}$ at $\Gamma(r_1)$ and $\Gamma(r_2)$ and $\mathscr{R}' \subset \mathscr{R}$. Let $\Gamma^*: T \to P^2$ be a curve with the property that $\Gamma^*(t) = \Gamma(t)$ for $t \in [r_2, r_1]$, $\Gamma_1^*(r_1) = \Gamma_1^*(r_2) = L$, $\Gamma^*(r_1, r_2)$ is of order two,

$$\Gamma^*(r_1, r_2) \cap \mathscr{R} = \emptyset \quad \text{and} \quad H(\Gamma^*[r_1, r_2]) \cap \mathscr{R} \subset L.$$

Clearly, Γ^* is a simple curve of even order with the three singular points $\Gamma^*(u_2)$, $\Gamma^*(r_1)$ and $\Gamma^*(r_2)$. We note that $\Gamma^*(u_2)$ is an inflection and since L cuts Γ^* at both r_1 and r_2, each of $\Gamma^*(r_1)$ and $\Gamma^*(r_2)$ is an inflection or a cusp. It is easy to check that $\Gamma[t_2, r_2) \cup \Gamma(r_1, t_1]$ in either $\mathrm{int}(\mathscr{R}')$ or $\mathscr{R}\backslash\mathscr{R}'$ yields that $\Gamma^*(r_1)$ and $\Gamma^*(r_2)$ are both cusps or both inflections. In either case, we then have that $\bar{n}(\Gamma^*)$ is odd and thus Γ^* is of odd order; a contradiction. Hence u_2 is not the only singular point of (r_2, r_1) and $n(\Gamma) \geqq 3$ or $r_1 = t_1$.

c) $r_1 = t_1$ and $u_2 \in (t_2, r_2)$.

By the preceding cases, we may assume that L meets Γ at exactly t_2, r_2 and t_1 in $[t_2, t_1]$. Then $L \cap \Gamma[r_2, t_1) = \emptyset$ and (r_2, t_1) ordinary imply that (r_2, t_1) is of order two and

$$\mathscr{R} = H(\Gamma[r_2, t_1]).$$

We now consider (u_2, u_1) which is both simple and ordinary. Let $[s_2, s_1]$ be the convex cover of $[u_2, u_1]$. Thus (s_2, s_1) is of order two and

$$H(\Gamma[u_2, u_1]) = H(\Gamma[s_2, s_1]) = \mathscr{R}_u \quad \text{say.}$$

As $[r_2, t_1] \subset [u_2, u_1]$ we have $\mathscr{R} \subset \mathscr{R}_u$.

Let $\Gamma[z_2, z_1]$ be the maximal subarc of Γ contained in $\mathscr{R}_u$. Thus

$$[s_2, s_1] \subset [u_2, u_1] \subset [z_2, z_1].$$

Since $\mathscr{R} \subset \mathscr{R}_u$, we also have

$$[t_2, t_1] \subset [z_2, z_1] \quad \text{and} \quad [t_1, u_1] \subset [u_2, u_1].$$

Thus

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$$[t_2, u_1] \subset [z_2, z_1] \quad \text{and} \quad [z_1, z_2] \subset [u_1, t_2],$$

and (z_1, z_1) is ordinary and simple.

Let $[w_1, w_2]$ be the convex cover of $[z_1, z_2]$,

$$\mathscr{R}_z = H(\Gamma[w_1, w_2]).$$

We note that $\Gamma(w_1) \neq \Gamma(w_2)$, $L' = \langle \Gamma(s_2), \Gamma(s_1) \rangle$ is a line and

$$\text{bd}(\mathscr{R}_u) = \Gamma(s_2, s_1) \cup (L' \cap \mathscr{R}_u).$$

If $\{\Gamma(z_1), \Gamma(z_2)\} \subset L'$ then by arguing as in the preceding (with $L' = L$, $z_1 = v_1$ and $z_2 = v_2$), we obtain that $\text{ind}(\Gamma) \leqq 1$; a contradiction. Thus $\{\Gamma(z_1), \Gamma(z_2)\}$ is not contained in L' and in particular

$$s(\Gamma[s_2, s_1]) \neq 0.$$

As $p \in \Gamma[s_2, s_1]$, this implies that

$$t_2 < u_2 < s_2 < t_1 < s_1 \leqq u_1 \leqq z_1 < z_2 = t_2.$$

Hence $p = \Gamma(z_2) \notin L'$ and $\Gamma(z_1) \in L'$. Since L' is a supporting line of $\mathscr{R}_u$ and $\Gamma[z_2, z_1]$ is the maximal subarc in $\mathscr{R}_u$, L' cuts Γ at z_1. By 10, L' cuts Γ at exactly one point, say z, in (z_1, z_2) and by 9,

$$L' \cap \Gamma(z_1, z) = \emptyset$$

and L' supports Γ in at most one point of (z, z_2). Clearly (z_1, z) is of order two and

$$\Gamma(z_1, z) \cap \mathscr{R}_u = \emptyset.$$

It is now easy to check that $\Gamma(z) \notin \mathscr{R}_u$, $L' \cap \Gamma(z, z_2) = \emptyset$ and thus

$$\Gamma(z_1, z_2) \cap \mathscr{R}_u = \emptyset.$$

Since $w_1 < w_2$ in $[z_1, z_2]$,

$$\Gamma(w_1, w_2) \cap \mathscr{R}_u = \emptyset.$$

Since $\Gamma(w_1) \neq \Gamma(w_2)$, 14 implies that

$$N' \cap (\mathscr{R}_u \cup \mathscr{R}_z) = \emptyset$$

for some N' or

$$\{\Gamma(w_1), \Gamma(w_2)\} \subset \mathscr{R}_u$$

and there is an N such that $N \cap \mathscr{R}_u = \emptyset$ and N meets, and cuts, Γ at exactly one point of (w_1, w_2). Since

$$\Gamma = \Gamma[z_2, z_1] \cup \Gamma[z_1, z_2] \subset \mathscr{R}_u \cup \mathscr{R}_z,$$

the latter is true. But $\{\Gamma(w_1), \Gamma(w_2)\} \subset \mathscr{R}_u$ implies that $w_1 = z_1$ and $w_2 = z_2$ and thus (z_1, z_2) is of order two. Since $\Gamma[z_2, z_1] \subset \mathscr{R}_u$ and

$N \cap \mathscr{R}_u = \emptyset$, the intersection property of N implies that Γ is odd order; a contradiction. Thus (z_1, z_2) is not ordinary and $n(\Gamma) \geqq 3$.

d) $r_1 = t_1$ and $u_2 \in (r_2, t_1)$.

In this case, we consider the simple and ordinary arc (u_1, u_2) and argue as in c) with (u_2, u_1) to obtain a contradiction to $n(\Gamma) = 2$.

Case 2. (t_1, t_2) or (t_2, t_1) is ordinary.

Let (t_1, t_2) be ordinary and let $[s_1, s_2]$ be the convex cover of $[t_1, t_2]$ with $\mathscr{R} = H(\Gamma[s_1, s_2])$.

Suppose $s_1 \neq t_1$. Since $L = \langle \Gamma(s_1), \Gamma(s_2) \rangle$ supports both $\mathscr{R}$ and Γ at $\Gamma(s_1)$, we have $L = \Gamma_1(s_1)$. Hence $p \neq \Gamma(s_2)$ by 9 and either

$$[s_1, s_2] \subset (t_1, t_2)$$

or

$$(s_1, s_2) = (t_1, t_2).$$

Let $\Gamma[v_1, v_2]$ be the maximal subarc of Γ in $\mathscr{R}$ containing $\Gamma[t_1, t_2]$.

Assume $[s_1, s_1] \subset (t_1, t_2)$. Thus $L = \Gamma_1(s_1) = \Gamma_1(s_2)$. Since (t_1, t_2) is ordinary, we obtain that $p \in \text{int}(\mathscr{R})$, L cuts Γ at v_1 and v_2, and

$$t_1 < s_1 < s_2 < t_2 < v_2 < v_1 < t_1.$$

As both (v_1, s_1) and (s_2, v_2) are simple, 12 yields that each of them is singular. Arguing as in Case 1 a), we obtain that $\text{ind}(\Gamma) \leqq 1$ if (v_2, v_1) is ordinary. Hence (v_2, v_1) is not ordinary and $n(\Gamma) \geqq 3$. Thus we may assume that $s_1 = t_1$, $s_2 = t_2$ and (t_1, t_2) is of order two.

Since the preceding is symmetric in (t_1, t_2) and (t_2, t_1), we also have that (t_2, t_1) is of order two whenever (t_2, t_1) is ordinary. Since

$$\Gamma[t_1, t_2] \cap \Gamma[t_2, t_1] = \{p\},$$

the intersection of $H(\Gamma[t_1, t_2])$ and $H(\Gamma[t_2, t_1])$ is either $\{p\}$ or one of these two sets is contained in the other. In the latter case, $\text{ind}(\Gamma) = 0$ and in the former case we obtain that $\text{ind}(\Gamma) = 0$ by 13. Thus we may assume that (t_2, t_1) is not ordinary. We may also assume that say t_1 is ordinary, for otherwise $n(\Gamma) \geqq 3$.

If $\text{ind}(\Gamma[t_2, t_1]) > 0$ we apply 15 and 1 to obtain that $n_2(\Gamma^*) = 0$ implies $n(\Gamma) = n(\Gamma^*) \geqq 3$ and $n_2(\Gamma^*) > 0$ implies that

$$\bar{n}(\Gamma) \geqq \bar{n}(\Gamma^*) - 2 \geqq 4.$$

Hence, let

$$\text{ind}(\Gamma[t_2, t_1]) = 0.$$

If t_2 is singular then the construction in 15, Case 3 yields a curve Γ' with all the properties of Γ except that either $m(\Gamma') = 0$ or Γ' has exactly one strong double point and Γ' is ordinary at that point. Since 1 is applicable when $m(\Gamma') = 0$, we may assume that t_2 is also ordinary. Thus

(t_2, t_1) contains two inflections or a cusp. From 16, we may assume that (t_2, t_1) contains two inflections. Let $\mathscr{R}^*$ denote the convex hull of the bounded arc (curve) $\Gamma[t_2, t_1]$. We claim that

(1) there exist $r < s$ in (t_2, t_1) such that

$$\{\Gamma(r), \Gamma(s)\} \subset \mathrm{bd}(\mathscr{R}^*) \quad \text{and} \quad \Gamma(t_2, r) \cup \Gamma(s, t_1) \subset \mathrm{int}(\mathscr{R}^*).$$

If $p \in \mathrm{int}(\mathscr{R}^*)$ then clearly (1). If $\Gamma_1(t_1) \neq \Gamma_1(t_2)$ then (cf. 15, Case 2) for any ordinary t arbitrarily close to t_1 or t_2 in (t_2, t_1), $\Gamma_1(t)$ cuts Γ in (t_2, t_1). Thus $\Gamma_1(t)$ is not a supporting line of $\mathscr{R}^*$ and $\Gamma(t) \notin \mathrm{bd}(\mathscr{R}^*)$. Since there exist ordinary $U^+(t_2)$ and $U^-(t_1)$ in (t_2, t_1), (1) follows. Let

$$p \in \mathrm{bd}(\mathscr{R}^*) \quad \text{and} \quad \Gamma_1(t_1) = \Gamma_1(t_2).$$

Since t_1 and t_2 are ordinary and (t_1, t_2) is of order two, it is immediate that $\Gamma_1(t_1)$ is a supporting line of both $\mathscr{R}$ and $\mathscr{R}^*$. Hence

$$\Gamma[t_1, t_2] \cap \Gamma[t_2, t_1] = \{p\}$$

yields that $\mathscr{R} \subseteq \mathscr{R}^*$ or $\mathscr{R}^* \subseteq \mathscr{R}$ and thus $\mathrm{ind}(\Gamma) = 0$. This is a contradiction and hence (1).

Since $\{\Gamma(r), \Gamma(s)\} \subset \mathrm{bd}(\mathscr{R}^*)$, there are lines through $\Gamma(r)$ and $\Gamma(s)$ which support $\mathscr{R}^*$. Since (t_2, t_1) contains at most inflections, it follows that $\Gamma(r)$ and $\Gamma(s)$ are ordinary. In particular; $\Gamma_1(r)$ is a supporting line of $\mathscr{R}^*$,

$$\Gamma_1(r) \cap \Gamma(t_2, r) = \emptyset$$

and there exists a $U(r)$ of order two in (t_2, t_1). Let t tend r in $(t_2, r) \cap U(r)$. Then $\Gamma(t) \in \mathrm{int}(\mathscr{R}^*)$ and 6 imply that $\Gamma_1(t)$ cuts Γ in $(t_2, t_1)\backslash U(r)$. Hence by the continuity of tangents, $\Gamma_1(r)$ meets Γ in $(r, t_1]$. Similarly, $\Gamma_1(s)$ meets Γ in $[t_2, s)$.

Let u_1 and u_2 be inflections, $u_1 < u_2$ in (t_2, t_1) and suppose that $n(\Gamma) = 2$. Then

$$\{\Gamma(u_1), \Gamma(u_2)\} \subset \mathrm{int}(\mathscr{R}^*)$$

and $\Gamma_1(r)$ meets Γ at some point $r' \in (r, t_1]$ such that

$$\Gamma_1(r) \cap \Gamma(r, r') = \emptyset.$$

Since $\Gamma(r') \in \mathrm{bd}(\mathscr{R}^*)$, we have that $u_1 \neq r' \neq u_2$ and r' is ordinary.

Let $r' \neq t_1$. If (r, r') is ordinary or (r, r') contains only u_1 or u_2, we argue as in Case 1 to obtain a contradiction. If $\Gamma_1(r)$ meets Γ at say $r < r' < r''$ in (t_2, t_1) then one of (r, r') or (r', r'') again contains at most one of u_1 and u_2, which is again a contradiction. Hence we may assume that $\Gamma_1(r)$ meets Γ at exactly r and r' in (t_2, t_1) and that $\{u_1, u_2\} \subset (r, r')$. If $p \in \Gamma_1(r)$ then $(t_2, r) \cup (r', t_1)$ ordinary yields that (t_2, r) and (r', t_1) are both of order two by 8. Since t_1 and t_2 are ordinary, it readily follows that either

$$\mathscr{R}^* = H(\Gamma[t_2, r]) \quad \text{or} \quad \mathscr{R}^* = H(\Gamma[r', t_1]);$$

a contradiction by (1). Hence

$$p \notin \Gamma_1(r) \quad \text{and} \quad |\Gamma_1(r) \cap \Gamma[t_1, t_2]| = 2.$$

Then $\Gamma(t_2, r) \subset \text{int}(\mathscr{R}^*)$ and 11.3 imply that

$$\Gamma(U^+(r)) \subset \text{bd}(\mathscr{R}^*) \quad \text{for } U^+(r) \subset (r, r').$$

Let $\mathscr{R}'$ be the closed region in $\mathscr{R}^*$ bounded by $\Gamma[r, r']$ and $\Gamma_1(r)$. Thus

$$\Gamma(U^+(r)) \subset \text{bd}(\mathscr{R}')$$

as well. But now

$$\Gamma(U^+(r)) \subset \text{bd}(\mathscr{R}^*) \cap \text{bd}(\mathscr{R}')$$

and

$$\Gamma(r, r') \cap (\Gamma[t_2, r) \cup \Gamma(r', t_1]) = \emptyset$$

clearly imply that

$$\Gamma[t_2, r) \cup \Gamma(r', t_1] \subset \mathscr{R}' \quad \text{and} \quad \Gamma[t_2, t_1] \subset \mathscr{R}' = \mathscr{R}^*.$$

Let $\Gamma[z_2, z_1]$ be the maximal subarc of Γ contained in $\mathscr{R}'$, $z_1 < z_2$ in $[t_1, t_2]$. Then

$$[r, r'] \subset (t_2, t_1) \subset (z_2, z_1).$$

As $t_1 \notin (z_2, r)$ we have that (z_2, r) is both simple and ordinary; a contradiction by 12.

Let $r' = t_1$. Then

$$\Gamma_1(r) \cap \Gamma[t_2, t_1] = \{p, \Gamma(r)\}.$$

Symmetrically, we obtain that

$$\Gamma_1(s) \cap \Gamma[t_2, t_1] = \{p, \Gamma(s)\}.$$

Thus $p \in \Gamma_1(r) \cap \Gamma_1(s)$ and in fact, $p \in \text{bd}(\mathscr{R}^*)$. From (1),

$$\text{bd}(\mathscr{R}^*) \subset \Gamma[r, s] \cup \Gamma_1(r) \cup \Gamma_1(s).$$

If (t_2, r) is ordinary then (t_2, r) is of order two. Since r is also ordinary,

$$\Gamma[t_2, r] \subset \text{bd}(H(\Gamma[t_2, r]))$$

implies that there is a $U^+(t) \subset (r, s)$ such that

$$\Gamma(U^+(r)) \subset \text{int}(H(\Gamma[t_2, r])).$$

Since (t_2, t_1) is simple, it follows that

$$\mathscr{R}^* \subseteq H(\Gamma[t_2, r])$$

and thus

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$$\Gamma(t_2, r) \subset \text{bd}(\mathscr{R}^*);$$

a contradiction by (1). Thus (t_2, r) is not ordinary and similarly, (s, t_1) is not ordinary. Let $u_2 \in (t_2, r)$, $u_1 \in (s, t_1)$ and thus

$$(r, s) \subset (u_2, u_1) \subset (t_2, t_1).$$

We recall that (r, s) is ordinary and simple. Since $\Gamma_1(r)[\Gamma_1(s)]$ meets $\Gamma[r, s]$ at only $\Gamma(r)[\Gamma(s)]$, we have that

$$\langle \Gamma(r), \Gamma(s) \rangle \cap \Gamma(r, s) = \emptyset$$

and thus (r, s) is of order two by 8. Next

$$\Gamma[r, s] \cap \Gamma[t_1, t_2] = \emptyset.$$

Since $p \in \Gamma_1(r)$ and (t_1, t_2) is of order two, it follows that

(2) $\Gamma_1(r) \cap \Gamma(t_1, t_2) = \emptyset$

or $\Gamma_1(r)$ cuts Γ at t_1, t_2 and exactly one point of (t_1, t_2). As $\Gamma_1(r)$ does not cut Γ in (t_2, t_1), we have that Γ is of odd order in the latter case; a contradiction. Thus (2) and symmetrically,

(3) $\Gamma_1(s) \cap \Gamma(t_1, t_2) = \emptyset.$

But then $\text{bd}(\mathscr{R}) = \Gamma[t_1, t_2]$ implies that $\mathscr{R} \cap \mathscr{R}^* = \{p\}$ and thus $\mathscr{R} \cup \mathscr{R}^*$ is bounded by 13. Since $\Gamma \subset \mathscr{R} \cup \mathscr{R}^*$, this is a contradiction and hence (r, s) is not ordinary and $n(\Gamma) \geqq 3$.

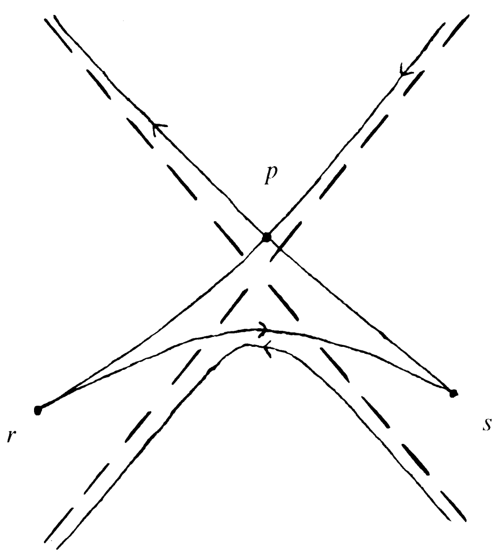

Figure 5.

We note the arguments in the proof of Theorem 4 not only show that $n(\Gamma) \geqq 2$ but also indicate how Γ may be constructed. For example, the

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curve Γ in Figure 5 is of even order with $\mathrm{ind}(\Gamma) > 0$, $m(\Gamma) = s(\Gamma) = 1$, $p = \Gamma(t_1) = \Gamma(t_2)$, (t_1, t_2) of order two, $\mathrm{ind}\,\Gamma[t_2, t_1]) = 0$ and r and s cusps in (t_2, t_1).

Proof of Theorem 5. Let Γ be an ordinary curve with the strong double points $p = \Gamma(t_1) = \Gamma(t_2)$, $t_1 \neq t_2$, and $q = \Gamma(u_1) = \Gamma(u_2)$; $u_1 \neq u_2$. By suitable labelling, either $t_1 < u_1 < t_2 < u_2 < t_1$ or $t_1 < t_2 < u_1 < u_2 < t_1$.

Case 1. $t_1 < u_1 < t_2 < u_2 < t_1$.

As (t_1, t_2) is ordinary and simple $[t_1, t_2]$ has a convex cover $[s_1, s_2]$. Let $\Gamma[v_1, v_2]$ be the maximal subarc of Γ contained in $H(\Gamma[s_1, s_2])$ and containing $\Gamma[s_1, s_2]$. Then either

$$[s_1, s_2] \subset (t_1, t_2) \quad \text{or} \quad [s_1, s_2] = [t_1, t_2];$$

cf. Case 2 of the proof of 4.

Suppose $[s_1, s_2] \subset (t_1, t_2)$. Then the quoted argument yields that $(t_1, t_2) \subset (v_1, v_2)$. We now apply 12 repeatedly. If $\Gamma(v_1) \in \Gamma[s_1, s_2]$ then $v_1 = u_2$ and $u_1 \in [s_1, s_2)$. Hence

$$v_2 \notin \{u_1, u_2\} \quad \text{and} \quad \Gamma(v_2) \notin \Gamma(s_1, s_2).$$

But then (s_2, v_2) is simple; a contradiction by 12. Hence $\Gamma(v_2) \notin \Gamma[s_1, s_2]$ and (v_1, s_1) is not simple. Symmetrically, (s_2, v_2) is not simple and thus

$$\{u_1, u_2\} \subset (v_1, s_1) \cap (s_2, v_2).$$

As this is impossible, we obtain that $[s_1, s_2] = [t_1, t_2]$.

By the preceding, (t_1, t_2) is of order two and symmetrically, (t_2, t_1) is of order two. The line $\Gamma_1(t_1)$ supports both $H(\Gamma[t_2, t_1])$ and $H(\Gamma[t_1, t_2])$ and thus

$$\Gamma_1(t_1) \cap \Gamma = \{p\}.$$

Suppose $\Gamma_1(t_1) \neq \Gamma_1(t_2)$. Then $\Gamma_1(t_1)$ supports Γ at t_1 and cuts Γ at t_2. Thus $\Gamma_1(t) \cap \Gamma = \{p\}$ yields that Γ is of odd order. This is a contradiction and hence $L = \Gamma_1(t_1) = \Gamma_1(t_2)$ supports Γ at both t_1 and t_2. Symmetrically, $L' = \Gamma_1(u_1) = \Gamma_1(u_2)$ meets Γ at only q and supports Γ at both u_1 and u_2. Clearly, $L \neq L'$ and $L \cap L' \notin \Gamma$. Thus Γ lies in one of the closed half-planes bounded by L and L' and $\mathrm{ind}(\Gamma) = 0$. This is a contradiction and therefore Γ is singular.

Case 2. $t_1 < t_2 < u_1 < u_2 < t_1$.

Then $\Gamma[t_1, t_2] \cap \Gamma[u_1, u_2] = \emptyset$ and (t_1, t_2) and (u_1, u_2) are both simple and ordinary. As in Case 1, we then obtain that (t_1, t_2) and (u_1, u_2) are both of order two. Let

$$\mathcal{R}_t = H(\Gamma[t_1, t_2]) \quad \text{and} \quad \mathcal{R}_u = H(\Gamma[u_1, u_2]).$$

If $\mathrm{ind}(\Gamma[t_2, t_1]) > 0$ then 15 implies that there is a differentiable curve Γ^* of even order with

$$\text{ind}(\Gamma^*) > 0,$$

$$m(\Gamma^*) = s(\Gamma^*) = s(\Gamma) - 1 = 1 \quad \text{and}$$

$$n(\Gamma) \geqq n(\Gamma^*) - 1.$$

Thus by 4, $n(\Gamma^*) \geqq 2$ and $n(\Gamma) \geqq 1$. Hence we may assume that

$$\text{ind}(\Gamma[t_2, t_1]) = \text{ind}(\Gamma[u_2, u_1]) = 0.$$

Let $\mathscr{R}_t^* = H(\Gamma[t_2, t_1])$.

As in the Case 2 of the proof of 4:

(1) there exist $r < s$ in (t_2, t_1) such that

$$\{\Gamma(r), \Gamma(s)\} \subset \text{bd}(\mathscr{R}_t^*),$$

$$\Gamma(t_2, r) \cup \Gamma(s, t_1) \subset \text{int}(\mathscr{R}_t^*)$$

and if $q \neq \Gamma(r)$ [$q \neq \Gamma(s)$] then $\Gamma_1(r)$ [$\Gamma_1(s)$] meets Γ in $(r, t_1]$ [$[t_2, s)$].

Next we observe that both

(2) $\Gamma[t_2, u_1] \subset \Gamma[u_2, t_1] \subset \mathscr{R}_u$

and

(3) $\Gamma[u_1, u_2] \subset H(\Gamma[t_2, u_1] \cup \Gamma[u_2, t_1])$

lead to a contradiction. Since (2) implies that

$$p \in \Gamma[t_2, t_1] \subset \mathscr{R}_u;$$

$\text{bd}(\mathscr{R}_u) = \Gamma[u_1, u_2]$ and $\Gamma[u_1, u_2] \cap \Gamma[t_1, t_2] = \emptyset$ yield that

$$p \in \text{int}(\mathscr{R}_u)$$

and in particular

$$\Gamma[t_2, t_2] = \Gamma \subset \text{int}(\mathscr{R}_u) \quad \text{and} \quad \text{ind}(\Gamma) = 0.$$

In case of (3),

$$H(\Gamma[t_2, u_1] \cup \Gamma[u_2, t_1]) \subseteq H(\Gamma[u_2, u_1])$$

implies that $\Gamma \subset H(\Gamma[u_2, u_1])$. Thus $\text{ind}(\Gamma[u_2, u_1]) = 0$ now yields that $\text{ind}(\Gamma) = 0$.

Since $\Gamma[t_2, u_1] \cup \Gamma[u_2, t_1]$ and $\Gamma[u_1, u_2]$ are curves which meet only at q, $\Gamma_1(u_1) = \Gamma_1(u_2)$ clearly implies either (2) or (3). Hence $\Gamma_1(u_1) \neq \Gamma_1(u_2)$, $\Gamma_1(u_1)$ cuts Γ at u_2 and $q \notin \text{bd}(\mathscr{R}_t^*)$. Since r is ordinary, we have that

$$\Gamma_1(r) \cap \Gamma(t_2, r) = \emptyset$$

and $\Gamma_1(r)$ meets Γ at a point $r' \in (r, t_1]$ such that

$$\Gamma_1(r) \cap \Gamma(r, r') = \emptyset.$$

Since (r, r') is ordinary, (r, r') is of order two whenever (r, r') is simple. Let $\mathscr{R}'$ be the closed region in $\mathscr{R}_t^*$ bounded by $\Gamma[r, r']$ and $\Gamma_1(r)$.

Let $r' \neq t_1$. If $q \notin \Gamma(r, r')$ then (r, r') is simple and

$$\mathscr{R}' = H(\Gamma[r, r']).$$

Clearly since r is ordinary, $\Gamma[t_2, r] \subset \mathscr{R}'$ and thus $\mathscr{R}' = \mathscr{R}_t^*$ and (3). If (r, r') contains u_1 and not u_2 then (r, r') is still simple,

$$\mathscr{R}' = H(\Gamma[r, r']) \quad \text{and} \quad p \in \Gamma[t_2, u_2] \subset \mathscr{R}'.$$

Since $q \in \text{bd}(\mathscr{R}')$ and $\Gamma_1(u_1)$ cuts Γ at t_2, it follows that

$$p \in \Gamma(u_2, t_1) \subset \mathscr{R}_t^* \backslash \mathscr{R}';$$

a contradiction. The preceding is symmetric in u_1 and u_2 and thus $u_1 < u_2$ in (r, r'). But then it is clear that either

$$\Gamma(r, r') \subset \text{int}(\mathscr{R}_t^*) \quad \text{or} \quad \Gamma[t_2, t_1] \subset \mathscr{R}'.$$

Since $\Gamma(r, r') \subset \text{int}(\mathscr{R}_t^*)$ implies (3), there exist $v_1 < v_2$ in $[t_2, t_1]$ such that $\Gamma[v_2, v_1]$ is the maximal subarc of Γ contained in $\mathscr{R}'$. Then $r < u_1 < u_2 < r'$ in (t_2, t_1) implies that

$$\{\Gamma(v_1), \Gamma(v_2)\} \cap \Gamma[r, r'] = \emptyset$$

and $m(\Gamma) = 2$ yields that (v_1, r) or (r', v_2) is simple. Hence $n(\Gamma) > 0$ by 12.

Let $r' = t_1$. Then $\Gamma_1(r)$ meets Γ at exactly t_2, r and t_1 in $[t_2, t_1]$ and (r, t_1) is ordinary. Since (t_2, r) is also ordinary, (as in the preceding) $r \notin (u_1, u_2)$ implies that (t_2, r) or (r, t_1) is of order two with $\mathscr{R}_t^*$ equal to its convex hull. This is a contradiction by (1) and thus $r \in (u_1, u_2)$ and (r, t_1) is of order two. Since the preceding arguments are symmetric in r and s; $\Gamma_1(s)$ meets Γ at exactly t_2, s and t_1 in $[t_2, t_1]$ and $s \in (u_1, u_2)$. Since $r < s$ in (t_2, t_1), $s \in (r, t_1)$. As (r, t_1) is of order two, 6 implies that $\Gamma(t_1) \notin \Gamma_1(s)$; a contradiction. Thus (t_2, r) cannot be ordinary and $n(\Gamma) > 0$.

From the curve represented in Figure 5, it is easy to deduce that there exists a differentiable curve Γ of even order with $\text{ind}(\Gamma) > 0$, $m(\Gamma) = s(\Gamma) = 2$ and $n(\Gamma) = n_2(\Gamma) = 1$.

References

1. T. Bisztriczky, *On the singularities of almost-simple plane curves*, Pac. J. Math. *109* (1983), 257-273.
2. ——— *On the singularities of simple plane curves*, Mich. Math. J. *32* (1985), 141-151.
3. H. Brunn, *Sätze über zwei getrennte Eikörper*, Math. Ann. *104* (1931), 300-324.
4. R. Park, *Topics in direct differential geometry*, Can. J. Math. *24* (1972), 98-148.

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