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Cancellative medial groupoids and arithmetic means

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It is shown that a homomorphism from a commutative, idempotent and medial groupoid with a reducible set of generators into another medial groupoid may be characterized by certain simultaneous equations. This result is used to characterize the arithmetic mean without introducing either continuity or order.

A characterization of the arithmetic mean on the real line was first given by Kolmogoroff [3] and independently by Nagumo [4]. Later Aczél [1] showed that if a compact interval is equipped with a jointly continuous binary operation which is commutative, cancellative, idempotent and medial, then the resulting groupoid is isomorphic and homeomorphic to the closed interval $[0, 1]$ under arithmetic mean. Fuchs [2] showed that Aczél's result holds for certain totally ordered groupoids. Sigmon [5] extended Aczél's result to the n-dimensional case.

A *groupoid* is a set together with a binary operation here denoted by multiplication. A groupoid is *medial* if the equation $(xy)(uv) = (xu)(yv)$ is an identity; *cancellative* if the maps $x \to xu$ and $x \to ux$ are one-to-one for each element u ; and *idempotent* if each element is idempotent $(uu = u)$. A subset A of a groupoid is *reducible* if given a, b and c in A there exists d in A such that $ab = cd$ or $bc = ad$ or $ca = bd$. If A has two elements then A is reducible, if A is a three element subset of a quasigroup G then a fourth element can be added to A to produce a reducible set in G .

If G is a groupoid and A, B are subsets of G we write

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$AB = \{ab : a \in A$ and $b \in B\}$ and $A^{n+1} = AA^n$, $(n = 1, 2, \ldots)$. We denote the closed interval $[0, 1]$ together with the arithmetic mean by Q, and the binary fractions in $[0, 1]$ with the arithmetic mean by Q_0. Both Q and Q_0 are commutative, idempotent, cancellative and medial groupoids.

LEMMA. *Let G be a commutative, idempotent and medial groupoid. If A generates G and A is reducible then* $G = AG = \bigcup_1^\infty A^n$.

Proof. Obviously $A \subset A^2$ and by induction $A^n \subset A^{n+1}$ for all n. We now prove that $A^nA^n = A^{n+1}$, whence $A^nA^{n+m} \subset A^{n+m+1}$ and $\bigcup_1^\infty A^n$ is a subgroupoid of G which contains A and so equals G. Now $A \subset A^n$ implies $A^{n+1} \subset A^nA^n$. Trivially $AA \subset A^2$. If $A^nA^n \subset A^{n+1}$ then, since A is reducible,

$$\begin{aligned} A^{n+1}A^{n+1} &= (AA^n)(AA^n) = (AA)(A^nA^n) \\ &\subset (AA)(AA^n) \subset A(AA^n) = A^{n+2}, \end{aligned}$$

and the result follows by induction. Finally

$$AG = A \bigcup_2^\infty A^n = \bigcup_1^\infty A^n = G .$$

THEOREM 1. *Let G be a commutative, idempotent and medial groupoid. Let A generate G and be reducible. Let H be a medial groupoid. Then $\phi : G \to H$ is a homomorphism if and only if*

(1) $\phi(\alpha x) = \phi(\alpha)\phi(x)$ *for all* α *in* A *and* x *in* G.

Proof. The "only if" part is trivial. Let us suppose (1) is true. We first prove that

(2) $\phi(\xi\eta) = \phi(\xi)\phi(\eta)$ for all ξ and η in A^2.

Let $\xi = \alpha\beta$ and $\eta = \gamma\delta$ where α, β, γ and δ are in A. Then

$$\phi\big((\alpha\beta)(\gamma\delta)\big) = \phi\big((\alpha\gamma)(\beta\delta)\big) = \phi\big((\beta\gamma)(\alpha\delta)\big)$$

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and

$$\begin{aligned}\phi(\alpha\beta)\phi(\gamma\delta) &= \big(\phi(\alpha)\phi(\beta)\big)\big(\phi(\gamma)\phi(\delta)\big)\\ &= \big(\phi(\alpha)\phi(\gamma)\big)\big(\phi(\beta)\phi(\delta)\big)\\ &= \phi(\alpha\gamma)\phi(\beta\delta) = \phi(\beta\gamma)\phi(\alpha\delta)\ .\end{aligned}$$

Hence when using the reducibility of A we may assume there exists ν in A such that $\alpha\beta = \gamma\nu$, so

$$\begin{aligned}\phi\big((\alpha\beta)(\gamma\delta)\big) &= \phi\big(\gamma(\nu\delta)\big) = \phi(\gamma)\big(\phi(\nu)\phi(\delta)\big)\\ &= \big(\phi(\gamma)\phi(\gamma)\big)\big(\phi(\nu)\phi(\delta)\big)\\ &= \big(\phi(\gamma)\phi(\nu)\big)\big(\phi(\gamma)\phi(\delta)\big)\\ &= \phi(\gamma\nu)\phi(\gamma\delta)\\ &= \phi(\alpha\beta)\phi(\gamma\delta)\ .\end{aligned}$$

We now prove that if $\xi \in A^2$ and $x \in G$ then

$$\phi(\xi x) = \phi(\xi)\phi(x)\ .$$

If $\delta \in A$, $\xi, \eta \in A^2$, and $x \in G$ then

(3) $$\phi(\eta\delta)\big(\phi(\xi)\phi(x)\big) = \phi(\xi\eta)\phi(\delta x)$$

since

$$\begin{aligned}\text{LHS} &= \big(\phi(\eta)\phi(\delta)\big)\big(\phi(\xi)\phi(x)\big)\\ &= \big(\phi(\eta)\phi(\xi)\big)\big(\phi(\delta)\phi(x)\big) = \text{RHS}\ .\end{aligned}$$

Let $\xi \in A^2$ and $x \in G$. Then there exist α in A and s in G such that $x = \alpha s$; and since A is reducible there exist δ and μ in A such that $\xi(\alpha y) = \delta(\mu y)$ for all y in G . Hence writing $\eta = \alpha\mu$ we have $\xi\alpha = \delta\eta$ and $\delta\mu = \xi\eta$,

$$\begin{aligned}\phi(\xi x) &= \phi\big(\delta(\mu s)\big) = \phi(\delta)\big(\phi(\mu)\phi(s)\big)\\ &= \big(\phi(\delta)\phi(\mu)\big)\big(\phi(\delta)\phi(s)\big)\\ &= \phi(\delta\mu)\phi(\delta s) = \phi(\xi\eta)\phi(\delta s)\\ &= \phi(\eta\delta)\big(\phi(\xi)\phi(s)\big) \qquad \big(\text{by } (3)\big)\\ &= \phi(\xi\alpha)\big(\phi(\xi)\phi(s)\big)\\ &= \phi(\xi)\phi(\alpha s) \qquad \big(\text{by } (3)\big)\\ &= \phi(\xi)\phi(x)\ .\end{aligned}$$

Let n be a positive integer such that $\phi(xy) = \phi(x)\phi(y)$ for all x and y in A^n . If $\alpha, \beta \in A$ and $s, t \in A^n$ then

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$$\begin{aligned}\phi((\alpha s)(\beta t)) &= \phi((\alpha\beta)(st)) = \phi(\alpha\beta)\phi(st) \\ &= (\phi(\alpha)\phi(\beta))(\phi(s)\phi(t)) \\ &= (\phi(\alpha)\phi(s))(\phi(\beta)\phi(t)) \\ &= \phi(\alpha s)\phi(\beta t)\ .\end{aligned}$$

It follows by induction that ϕ is a homomorphism.

THEOREM 2. *Let G be a commutative, cancellative, idempotent and medial groupoid generated by the elements a and b so that $ax = by$ only if $x = b$ and $y = a$. Then G is isomorphic to Q_0.*

Proof. Let $A = \{a, b\}$. We define $\phi : G \to Q_0$ so that $\phi(a) = 0$, $\phi(b) = 1$, and inductively extend the domain of ϕ to G so that the equations

(4) $$\phi(ax) = \tfrac{1}{2}\phi(x)\ ,$$

(5) $$\phi(bx) = \tfrac{1}{2} + \tfrac{1}{2}\phi(x)$$

are valid for all $x \in A^n$ and each n. Let $S = \{0, 1\}$ and $nS = \{\tfrac{1}{2}x+\tfrac{1}{2}y : x \in S, y \in (n-1)S\}$, $(n = 2, 3, \ldots)$. Then (4) and (5) obviously well define ϕ as a one-to-one map of A^2 onto $2S$.

Suppose (4) and (5) well define ϕ as a one-to-one map of A^p onto pS. If x is in $A^p\backslash A^{p-1}$ then ax and bx are in $A^{p+1}\backslash A^p$, $\phi(x)$ is in $pS\backslash(p-1)S$ and the numbers $\tfrac{1}{2}\phi(x)$ and $\tfrac{1}{2} + \tfrac{1}{2}\phi(x)$ are distinct elements of $(p+1)S\backslash pS$. Hence (4) and (5) well define ϕ as a one-to-one map of A^{p+1} onto $(p+1)S$. But $G = \cup A^n$ and $Q_0 = \cup(nS)$. Hence ϕ is a well defined one-to-one map of G onto Q_0 and ϕ satisfies (4) and (5). These equations correspond to (1) in Theorem 1. Hence ϕ is an isomorphism.

Note 1. Let G be the compact interval $[a, b]$ together with a jointly continuous binary operation which is commutative, cancellative, idempotent and medial. Then aG and bG have only the element ab in common. The binary operation is intern $(x < xy < y)$, so $\cup\{a, b\}^n$ is dense in G. The isomorphism constructed in Theorem 2 may be shown to be strictly increasing and extended to all of G whence G is isomorphic to Q as in [1].

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Note 2. Soublin ([6], p. 106) has shown that a cancellative and distributive $\{x(uv) = (xu)(xv)\}$ groupoid with two or three generators is medial. It follows that Theorem 2 remains true if "medial" is replaced by "distributive" (*cf.* [6], p. 253); then of course it is well known that a cancellative and distributive groupoid is idempotent so that "idempotent" may be removed from Theorem 2.

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