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MIXED PERIODIC JACOBI CONTINUED FRACTIONS

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§1. Introduction

Let b_0 be a positive real number and

$$J = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \end{bmatrix}. \qquad \begin{array}{l} -\infty < a_i < +\infty\,, \\ 0 < b_i\,, \\ 1 \leqq i < +\infty\,, \end{array}$$

be a Jacobi matrix. We can associate with them a Jacobi continued fraction, which will be abbreviated to a J fraction from the next section, as follows

$$\begin{aligned} \varphi(z) &= \frac{b_0^2}{|\, z - a_1} \Big| - \frac{b_1^2}{|\, z - a_2} \Big| - \frac{b_2^2}{|\, z - a_3} \Big| - \cdots \\ &= \lim_{n \to \infty} \frac{A_n(z)}{B_n(z)} \end{aligned}$$

where $A_n(z)/B_n(z)$ is the n-th Padé approximant of $\varphi(z)$. Under a suitable condition, which is always satisfied if $\max_i \{|a_i|, |b_i|\} \leq {}^{\exists}M < +\infty$ holds, $\varphi(z)$ can be described in a Stieltjes transform

$$\varphi(z) = \int_{-\infty}^{+\infty} \frac{d\mu(x)}{z - x}$$

for some Stieltjes measure $d\mu(x)$ on the real axis. And the denominators $B_n(x)$, $0 \leq n < +\infty$, constitute a system of orthogonal polynomials with respect to $d\mu(x)$.

In the previous papers [2], [3], we investigate what kind of measures $d\mu(x)$ gives a purely periodic Jacobi continued fraction, that is, $a_{i+N} = a_i$, $1 \leq i < +\infty$, $b_{i+N} = b_i$, $0 \leq i < +\infty$, for some positive integer N. In fact we succeed in showing that such a function $\varphi(z)$ can be explicitly written by means of abelian integrals on special hyperelliptic curves $\mathscr{R}$ introduced by van Moerbeke P. and Mumford D. [7]. If $N = 1$, $B_n(x)$ is

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essentially the n-th Tschebyscheff polymonial of the second type. But in case $N > 1$, we find a new phenomenon related to discrete measures.

This paper is the continuation of the preceding papers and we devote ourselves to study mixed periodic Jacobi continued fractions. Namely we deal with the followings

$$M\alpha) \qquad a_{i+N} = a_i \quad \text{for} \quad i \geq M \quad \text{and} \quad b_{i+N} = b_i \quad \text{for} \quad i \geq M-1$$
$$\text{but} \quad a_{M-1+N} \neq a_{M-1}\,,$$

$$M\beta) \qquad a_{i+N} = a_i\,, \quad b_{i+N} = b_i \quad \text{for} \quad i \geq M \quad \text{but} \quad b_{M-1+N} \neq b_{M-1}\,,$$

where M and N are positive integers. We refer them to be of type $M\alpha$ and $M\beta$ respectively. Our main results are stated in Section three. See Theorems 3.4 and 3.7. We give there complete characterizations of these functions and concrete methods to construct them. And the Stieltjes measures $d\mu(x)$ are explicitly described. Our work is closely related to physical problems. Namely some diagonal elements of the Hamiltonian of a tight binding formalism in solid physics which treat semiconductors or alloys admit a Jacobi continued fraction. And the measure $d\mu(x)$ represents the density of energy states of electrons. The non periodic parts of $M\alpha$) and $M\beta$) correspond to the effects of impurities. See, for examples, Lambin, Ph. and Gaspard, J-P. [4] and Turchi, P., Ducastelle, F. and Tréglia, G. [11].

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§ 2. Purely periodic J fractions

First we recall some preliminary knowledge and notation from [2] about purely periodic J fractions. But we slightly change the terminology.

Let

$$(2.1) \qquad C = \begin{pmatrix} \ddots & & & & & \\ \cdot & a_0 & b_0 & & & \\ & b_0 & a_1 & b_1 & & \\ & & b_1 & \ddots & \ddots & \\ & & & \ddots & \ddots & b_{N-1} \\ & & & & b_{N-1} & a_N & \cdot \\ & & & & & \cdot & \ddots \end{pmatrix}, \qquad \begin{array}{l} -\infty < a_i < +\infty\,, \\ 0 < b_i\,, \\ a_{i+N} = a_i\,, \\ b_{i+N} = b_i\,, \\ -\infty < i < +\infty\,, \end{array}$$

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be a complete N periodic Jacobi matrix and $f = (\cdots, f_{-1}, f_0, f_1, \cdots)^t$ be an infinite column vector. The matrix C acts canonically on f. We denote by D the shift operator of degree $+1$, $(Df)_i = f_{i+1}$ and set $S = D^N$. The condition that C and S admit a common eigenvector is parametrized by the following open Riemann surface

$$\begin{aligned}\mathscr{R}_0 &= \{(z, h) \in \boldsymbol{C} \times \boldsymbol{C}^* \mid Cf = zf,\ Sf = hf \text{ for some } f \neq 0\} \\ &= \{(z, h) \in \boldsymbol{C} \times \boldsymbol{C}^* \mid \det|zI - C_h| = F(h, h^{-1}, z) = 0\}.\end{aligned}$$

We denote by C_h the $N \times N$ matrix

$$C_h = \begin{pmatrix} a_1 & b_1 & \cdot & \cdot & \cdot & b_0 h^{-1} \\ b_1 & a_2 & b_2 & & & \cdot \\ \cdot & b_2 & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & b_{N-1} \\ b_N h & \cdot & \cdot & \cdot & b_{N-1} & a_N \end{pmatrix}, \tag{2.2}$$

and hence

$$zI - C_h = \begin{pmatrix} z - a_1 & -b_1 & & & & -b_0 h^{-1} \\ -b_1 & z - a_2 & -b_2 & & & \cdot \\ \cdot & -b_2 & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & -b_{N-1} \\ -b_N h & \cdot & \cdot & \cdot & -b_{N-1} & z - a_N \end{pmatrix}. \tag{2.3}$$

An easy calculation provides

$$F(h, h^{-1}, z) = -A(h + h^{-1}) + P(z) \tag{2.4}$$

where $A = \prod_{i=1}^{N} b_i$ and $P(z)$ is a monic polynomial of degree N with real coefficients as follows

$$P(z) = \begin{vmatrix} z - a_1 & -b_1 & & & \\ -b_1 & z - a_2 & -b_2 & & \\ & -b_2 & \cdot & \cdot & \\ & & \cdot & \cdot & -b_{N-1} \\ & & & -b_{N-1} & z - a_N \end{vmatrix} - b_0^2 \begin{vmatrix} z - a_2 & -b_2 & & & \\ -b_2 & z - a_3 & -b_3 & & \\ & -b_3 & \cdot & \cdot & \\ & & \cdot & \cdot & -b_{N-2} \\ & & & -b_{N-2} & z - a_{N-1} \end{vmatrix}. \tag{2.5}$$

From (2.4), we have

$$h(z) = \frac{1}{2A}\{P(z) \pm \sqrt{P(z)^2 - 4A^2}\} = \frac{1}{\dfrac{1}{2A}\{P(z) \mp \sqrt{P(z)^2 - 4A^2}\}}. \tag{2.6}$$

The expression (2.6) implies that we can easily compactify $\mathscr{R}_0$ by adding two points P, whose $h = \infty$, and Q, whose $h = 0$, over $z = \infty$. The curve $\mathscr{R} = \mathscr{R}_0 \cup \{P, Q\}$ becomes a hyperelliptic curve of genus $g = N - 1$ branched at the $2N$ zeroes $\lambda_1, \lambda_2, \cdots, \lambda_{2N}$ of the polynomial $P(z)^2 - 4A^2$. As is explained in [2], they are all real and we can arrange them in increasing order

$$(2.7) \qquad \lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \cdots \leq \lambda_{2N-3} < \lambda_{2N-2} \leq \lambda_{2N-1} < \lambda_{2N} .$$

The interval $[\lambda_{2k-1}, \lambda_{2k}]$, $1 \leq k \leq N$, is called the k-th stable band and the interval $[\lambda_{2k}, \lambda_{2k+1}]$, $1 \leq k \leq N - 1$, called the k-th finite unstable band. The infinite interval $(-\infty, \lambda_1]$ (resp. $[\lambda_{2N}, +\infty)$) is the left (resp. right) exterior unstable band.

Remark 2.1. 1) Hereafter we denote by $\sqrt{P(z)^2 - 4A^2}$ the radical of $P(z)^2 - 4A^2$ which is approximately equal to $P(z)$ near the infinite point corresponding to Q.

2) Assume that we choose a real positive number A and a monic polynomial $P(z)$ of degree N with real coefficients such that the zeroes of $P(z)^2 - 4A^2$ are all real. Then conversely the hyperelliptic curve $\mathscr{R}$ being defined by (2.6) comes from some complete N periodic Jacobi matrix C, (2.1). This fact is derived from the general theory of van Moerbeke, P. and Mumford, D. [7].

We decompose $\mathscr{R}$ into three parts

$$\mathscr{R} = \mathscr{R}_+ \cup \mathscr{R}_R \cup \mathscr{R}_-$$

where

$$\begin{aligned} \mathscr{R}_+ &= \{p \in \mathscr{R} \mid |h(p)| > 1\} , \\ \mathscr{R}_R &= \{p \in \mathscr{R} \mid |h(p)| = 1\} , \\ \mathscr{R}_- &= \{p \in \mathscr{R} \mid |h(p)| < 1\} . \end{aligned}$$

The point P belongs to $\mathscr{R}_+$ and Q belongs to $\mathscr{R}_-$. If we project $\mathscr{R}_+$ and $\mathscr{R}_-$ onto the z-plane, both of them become biholomorphic to the domain $\{z\text{-plane}\} \cup \{\infty\} - \bigcup_{k=1}^{N} [\lambda_{2k-1}, \lambda_{2k}]$. Therefore for any element z_0 in this domain, we can lift it up to two points $z_0^+ \in \mathscr{R}_+$ and $z_0^- \in \mathscr{R}_-$.

Let $f = (\cdots, f_{-1}, f_0, f_1, \cdots)^t$ be a common eigenvector of C and S under the normalization $f_0 \equiv 1$. Then f_i, $-\infty < i < +\infty$, become meromorphic functions on $\mathscr{R}$ and it follows that $f_N = h$, $f_{i+N} = f_i \cdot f_N = f_i \cdot h$. If we put $\bar{f} = (f_1. \cdots . f_N)^t = (f_1, \cdots, h)^t$ then $(zI - C_h)\bar{f} = 0$. So we have

(2.8) $$f_i = \frac{\Delta_{k,i}}{\Delta_{k,j}} \cdot f_j \,, \qquad 1 \leq i, j, k \leq N \,,$$

where $\Delta_{i,j}$ denotes the (i, j) cofactor of $zI - C_h$. The cofactor $\Delta_{N,N}$ is a monic polynomial in z of degree $N - 1$ whose zeroes are all real. If we arrange them in increasing order $\mu_1 < \mu_2 < \cdots < \mu_{N-1}$, each μ_k lies on the k-th unstable band. Hence if we denote by $\sqrt{P(x)^2 - 4A^2}$ the limit

(2.9) $$\lim_{\varepsilon \downarrow 0} \sqrt{P(x + i\varepsilon)^2 - 4A^2} \,, \quad x \in \mathscr{R}_R \,, \quad x + i\varepsilon \in \mathscr{R}_- \,,$$

the form

(2.10) $$\frac{1}{2\pi i} \cdot \frac{\sqrt{P(x)^2 - 4A^2}}{\Delta_{N,N}(x)} dx$$

gives a positive measure on each stable band.

Under the same assumptions until now, we obtain the following fundamental lemmas. See the proof in [2].

LEMMA 2.2. *Let*

(2.11) $$\varphi(z) = \frac{b_0^2}{|\, z - a_1} - \frac{b_1^2}{|\, z - a_2} - \frac{b_2^2}{|\, z - a_3} - \cdots$$
$$a_{i+N} = a_i \,, \qquad 1 \leq i < +\infty \,,$$
$$b_{i+N} = b_i \,, \qquad 0 \leq i < +\infty \,,$$

be a purely N periodic J fraction. Then after an analytic prolongation, $\varphi(z)$ coincides with $b_0 \cdot f_1$. The analytic prolongation is possible from the neighborhood of Q.

We denote the residue of $\varphi(z)$ at $\mu_i^- \in \mathscr{R}_- \subset \mathscr{R}$ by

$$\gamma^i = \mathrm{Res}_{\mu_i^-} \varphi(z) = \mathrm{Res}_{\mu_i^-} b_0 \cdot f_1$$

and put

(2.12) $$S = S(\varphi) = \{i \,|\, \gamma^i \neq 0\} \subset \{1, 2, \cdots, N - 1\} \,.$$

And we associate with it a divisor

(2.13) $$\mathscr{D}_S = \sum_{i \in S} \mu_i^- + \sum_{j \in S^c} \mu_j^+ \,.$$

LEMMA 2.3. *We have*

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(2.14) 1) $\gamma^i = 0$ *or* $-\dfrac{\sqrt{P(\mu_i^-)^2 - 4A^2}}{\prod_{j \neq i} (\mu_i - \mu_j)} > 0\,.$

2) *From* (2.10), (2.14), *by putting*

$$d\mu(x) = d\mu_{S,d}(x) + d\mu_c(x) \tag{2.15}$$

where

$$d\mu_{S,d}(x) = \sum_{i \in S} \gamma^i \delta(x - \mu_i) dx, \qquad \textit{discrete measure}, \tag{2.16}$$

$$d\mu_c(x) = \sum_{j=1}^{N} \frac{1}{2\pi i} \chi_{[\lambda_{2j-1}, \lambda_{2j}]}(x) \frac{\sqrt{P(x)^2 - 4A^2}}{\Delta_{N,N}(x)} dx\,, \qquad \textit{continuous measure}\,, \tag{2.17}$$

we obtain a Stieltjes measure. Then $\varphi(z)$ *is the Stieltjes transform of it*

$$\varphi(z) = \int_{-\infty}^{+\infty} \frac{d\mu(x)}{z - x}\,. \tag{2.18}$$

LEMMA 2.4. *The following three conditions are all equivalent up to a nonzero constant factor.*

1) $\varphi(z)$ *admits a purely* N *periodic* J *fraction expansion.*

2) $\varphi(z)$ *is the first component* f_1 *of the common eigenvectors of* C *and* S.

3) $\varphi(z)$ *belongs to* $L(\mathscr{D}_S + P - Q)$.

For any other subset $S' \subset \{1, 2, \cdots, N-1\}$, there corresponds another purely N periodic J fraction. We can calculate it from $\varphi(z)$ by a standard method. See Appendix. Hereafter when we want to emphasize the dependence of $\varphi(z)$ on S, we denote it by $\varphi_S(z)$.

By use of Lemma 2.2 and (2.8), we can rewrite $\varphi(z) = \varphi_S(z)$ as follows

$$\begin{aligned}
\varphi(z) &= \frac{b_0^2}{|\, z - a_1} \Big| - \frac{b_1^2}{|\, z - a_2} \Big| - \frac{b_2^2}{|\, z - a_3} \Big| - \cdots \\
&= b_0 f_1 \\
&= \frac{Ah + b_0^2 \cdot \begin{vmatrix} z - a_2 & -b_2 & & & \\ -b_2 & z - a_3 & -b_3 & & \\ & -b_3 & \ddots & \ddots & \\ & & \ddots & \ddots & -b_{N-2} \\ & & & -b_{N-2} & z - a_{N-1} \end{vmatrix}}{\Delta_{N,N}(x)} \\
&= \frac{1}{2} \frac{\Lambda_0(z) - \sqrt{P(z)^2 - 4A^2}}{\Gamma_0(z)}
\end{aligned} \tag{2.19}$$

where we put

$$\Gamma_0(z) = \Delta_{N,N}(z)\,, \tag{2.20}$$

$$\Lambda_0(z) = \begin{vmatrix} z-a_1 & -b_1 & & & \\ -b_1 & z-a_2 & -b_2 & & \\ & -b_2 & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & -b_{N-1} \\ & & & -b_{N-1} & z-a_N \end{vmatrix} + b_0^2 \begin{vmatrix} z-a_2 & -b_2 & & & \\ -b_2 & z-a_3 & -b_3 & & \\ & -b_3 & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & -b_{N-2} \\ & & & -b_{N-2} & z-a_{N-1} \end{vmatrix} \tag{2.21}$$

$$= z^N + cz^{N-1} + \cdots .$$

Here $c = a_1 + a_2 + \cdots + a_N$. Notice that c coincides with the coefficient of z^{N-1} in $P(z)$, so it is fixed by the hyperelliptic curve $\mathscr{R}$ and is independent of μ_i, $1 \leq i \leq N-1$. If we compare Lemma 2.4. 3) and (2.19), we have

$$\Lambda_0(\mu_i) = \begin{cases} -\sqrt{P(\mu_i)^2 - 4A^2} & \text{if } i \in S \\ +\sqrt{P(\mu_i)^2 - 4A^2} & \text{if } i \in S^c\,. \end{cases} \tag{2.22}$$

From (2.21), (2.22), $\Lambda_0(z)$ is uniquely determined by use of Lagrange's interpolation. The importance of the following fact is pointed out by Prof. A. Magnus.

COROLLARY 2.5. *$\Gamma_0(z)$ divides the polynomial $\Lambda_0(z)^2 - P(z)^2 + 4A^2$.*

§ 3. Mixed periodic *J* fractions

In this section we study mixed periodic J fractions. We represent them as follows

$$\psi_M(z) = \frac{B_M^2}{|\,z - A_{M-1}} - \frac{B_{M-1}^2}{|\,z - A_{M-2}} - \cdots - \frac{B_1^2}{|\,z - A_0} - \frac{B_0^2}{|\,z - a_1} - \frac{b_1^2}{|\,z - a_2} - \frac{b_2^2}{|\,z - a_3} \cdots \tag{3.1}$$

where $a_{i+N} = a_i$, $b_{i+N} = b_i$, $1 \leq i < +\infty$. According to the number B_0, we distinguish them into two types

$$M\alpha) \quad B_0 = b_0 \quad \text{but} \quad A_0 \neq a_N\,, \tag{3.2}$$

$$M\beta) \quad B_0 \neq b_0\,. \tag{3.3}$$

If we denote the purely N periodic part by

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$$\varphi(z) = \varphi_S(z) = \frac{b_0^2}{|z - a_1} - \frac{b_1^2}{|z - a_2} - \frac{b_2^2}{|z - a_3} - \cdots, \tag{3.4}$$

we can write

$$\psi_M(z) = \frac{B_M^2}{|z - A_{M-1}} - \frac{B_{M-1}^2}{|z - A_{M-2}} - \cdots - \frac{B_1^2}{\left|z - A_0 - \left(\frac{B_0}{b_0}\right)^2 \varphi(z)\right.}. \tag{3.5}$$

For $1 \leq l \leq M - 1$, we use the notation

$$\psi_M(z) = \frac{B_M^2}{|z - A_{M-1}} - \frac{B_{M-1}^2}{|z - A_{M-2}} - \cdots - \frac{B_{M-l-1}^2}{|z - A_{M-l} - \psi_{M-l}(z)}. \tag{3.6}$$

From the expression (3.5), $\psi_M(z)$ is also a meromorphic function on the hyperelliptic curve $\mathscr{R}$ and the J fraction expansion is significant in the neighborhood of $Q \in \mathscr{R}$.

Let $\Gamma(z)$ be a monic polynomial with real coefficients and decompose it into

$$\Gamma(z) = \prod_{i=1}^{r} (z - \nu_i) \prod_{j=1}^{s} (z - \zeta_j) \prod_{j=1}^{s} (z - \bar{\zeta}_j), \qquad r + 2s = \deg \Gamma(z). \tag{3.7}$$

Here $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_r$ denote real zeroes and ζ_j, $\operatorname{Im} \zeta_j > 0$, $\bar{\zeta}_j$, $1 \leq j \leq s$, denote the others. Since the case of multiple zeroes can be considered as limit case, for the sake of simplicity, we take the assumption that they are all distinct.

DEFINITION 3.1. For a polynomial $\Gamma(z)$, (3.7), we put the set of zeroes as follows

$$Z(\Gamma) = Z_{\mathrm{Re}}(\Gamma) \cup Z_{\mathrm{Im}}(\Gamma) \cup \overline{Z_{\mathrm{Im}}(\Gamma)}$$

where

$$Z_{\mathrm{Re}}(\Gamma) = \{\nu_i \mid \text{real},\ 1 \leq i \leq r\},$$
$$Z_{\mathrm{Im}}(\Gamma) = \{\zeta_j \mid \operatorname{Im} \zeta_j > 0,\ 1 \leq j \leq s\},$$
$$\overline{Z_{\mathrm{Im}}(\Gamma)} = \{\bar{\zeta}_j \mid \operatorname{Im} \bar{\zeta}_j < 0,\ 1 \leq j \leq s\}.$$

LEMMA 3.2. *Let us take a polynomial $\Gamma(z)$, (3.7), and a non zero real constant c. For any $x \in \bigcup_{k=1}^{N} [\lambda_{2k-1}, \lambda_{2k}]$, we denote by $\sqrt{P(x)^2 - 4A^2}$ the value*

$$\lim_{\varepsilon \downarrow 0} \sqrt{P(x + i\varepsilon)^2 - 4A^2}, \qquad x + i\varepsilon \in \mathscr{R}_-.$$

Then if

$$0 \leq \frac{c}{2\pi i} \frac{\sqrt{P(x)^2 - 4A^2}}{\Gamma(x)} \leq +\infty, \qquad x \in \bigcup_{k=1}^{N} [\lambda_{2k-1}, \lambda_{2k}]$$

holds, we have

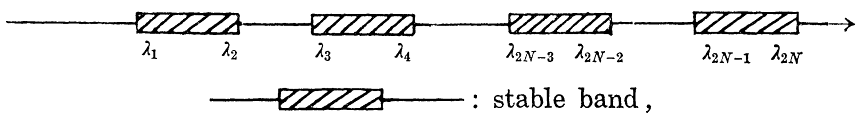
1) $\nu_i \in Z_{\mathrm{Re}}(\Gamma)$ *cannot belong to the interior of any stable band.*

2) *At least one of the elements* $\nu_i \in Z_{\mathrm{Re}}(\Gamma)$ *must belong to each finite unstable band. Especially* $r \geq N-1$.

Proof. If we note that the sign of $\sqrt{P(x)^2 - 4A^2}$ changes on the real axis as follows

N: even $+$ $\quad -i$ $\quad -$ $\quad +i$ $\quad \cdots +$ $\quad -i$ $\quad -$ $\quad +i$ $\quad +$

N: odd $-$ $\quad +i$ $\quad +$ $\quad -i$



these statements are bvoious.

Remark 3.3. Under the same assumptions as in Lemma 3.2, if the integrals are finite

$$\sum_{k=1}^{N} \lim_{\varepsilon \downarrow 0} \frac{c}{2\pi i} \int_{\lambda_{2k-1}+\varepsilon}^{\lambda_{2k}-\varepsilon} \frac{\sqrt{P(x)^2 - 4A^2}}{\Gamma(x)} dx < +\infty ,$$

the form

$$\omega = \frac{c}{2\pi i} \frac{\sqrt{P(x)^2 - 4A^2}}{\Gamma(x)} dx \tag{3.8}$$

gives a positive Radon measure on each stable band $[\lambda_{2k-1}, \lambda_{2k}]$, $1 \leq k \leq N$.

DEFINITION 3.3. For a pair c, $\Gamma(z)$, we put

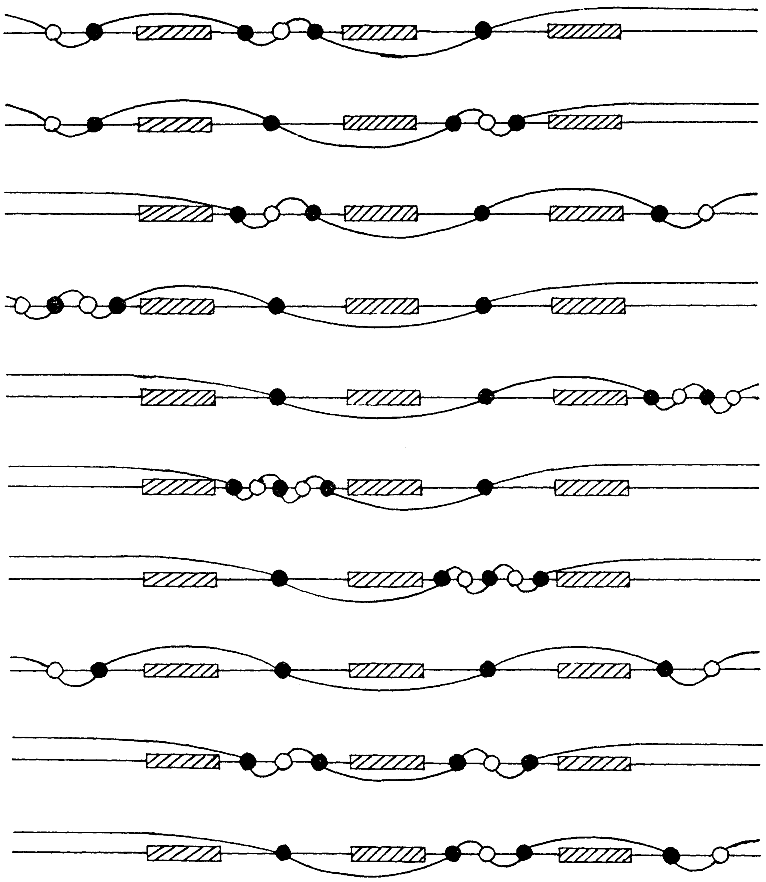
(3.9) 1) $$\gamma_i(c, \Gamma) = -c \frac{\sqrt{P(\nu_i^-)^2 - 4A^2}}{\prod_{k \neq i} (\nu_i - \nu_k) \prod_{j=1}^{s} (\nu_i - \zeta_j) \prod_{j=1}^{s} (\nu_i - \bar{\zeta}_j)},$$
$$\nu_i \in Z_{\mathrm{Re}}(\Gamma),$$

(3.10) 2) $$\mathscr{S}(c, \Gamma) = \{i \mid \gamma_i(c, \Gamma) > 0\}.$$

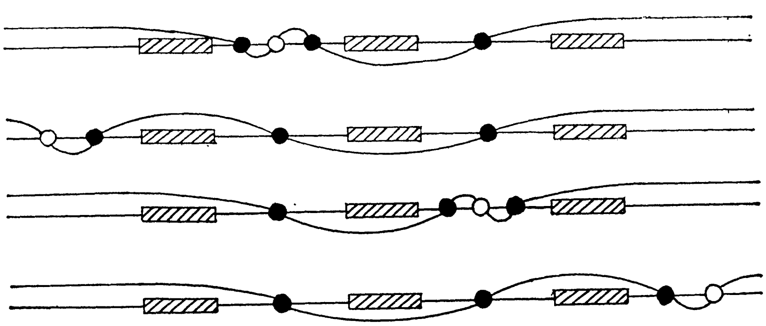
EXAMPLE 3.4. Let $N = 3$ and $\deg \Gamma(z) = 6$. And the pair $c \neq 0$, $\Gamma(z)$ satisfy the condition in Lemma 3.2. Then the following cases occur. Here the circles colored black represent the points ν_i, $i \in \mathscr{S}(c, \Gamma)$.

I) $c > 0$.

1) $r = 6$.



2) $r = 4$.



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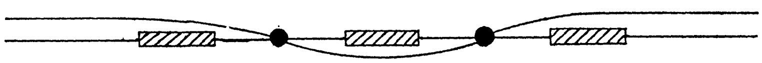
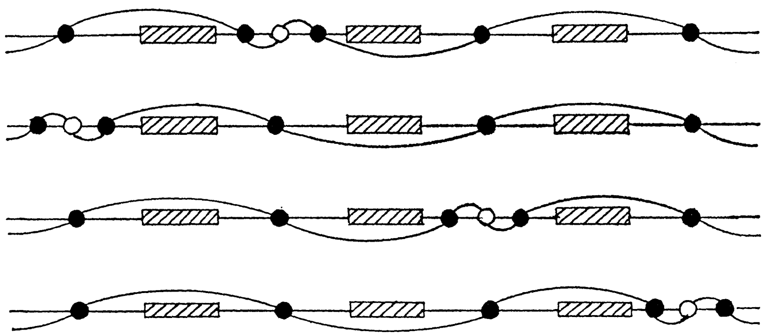
3) $r = 2$.

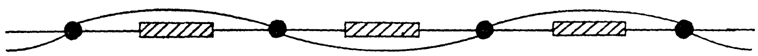


II) $c < 0$.

1) $r = 6$.



2) $r = 4$.



LEMMA 3.3. *Let $\Phi(z)$, $\Psi(z)$ be meromorphic functions on $\mathscr{R}$ which satisfy the relation*

$$\Psi(z) = \frac{B^2}{z - A - \Phi(z)}, \quad -\infty < A < +\infty, \quad 0 < B. \tag{3.11}$$

Then we have

1) *Let P and Q be zeroes of $\Psi(z)$ of first order. Then $\Psi(z)$ has $l + 2$ poles in $\mathscr{R} - \{P, Q\}$ if and only if $\Phi(z)$ has l poles in $\mathscr{R} - \{P, Q\}$.*

2) *Let Q be a zero of $\Psi(z)$ of first order and P be neither a zero nor a pole of it. Then $\Psi(z)$ has $l + 1$ poles in $\mathscr{R} - \{P, Q\}$ if and only if $\Phi(z)$ has l poles in $\mathscr{R} - \{P, Q\}$.*

Proof. First let us prove 1). From (3.11), besides at the two points P and Q, $\Psi(z)$ takes zeroes at the points in $\mathscr{R} - \{P, Q\}$ where $\Phi(z)$ takes poles and vice versa. Therefore if $\Phi(z)$ has l poles in $\mathscr{R} - \{P, Q\}$, $\Psi(z)$ has $l + 2$ zeroes. Hence $\Psi(z)$ must have $l + 2$ poles being contained in $\mathscr{R} - \{P, Q\}$. We obtain the proof of 1). We can prove 2) similarly.

THEOREM 3.4. *Up to a nonzero scalar factor, the conditions* I. 1.M) + 2.$M\alpha$), (*resp.* +2.$M\beta$)) *and* II. 1.M) + 3.M) + 4.M) + 2.$M\alpha$), (*resp.* +2.$M\beta$)), $1 \leq M < +\infty$, *are equivalent, where the conditions* I *and* II *are*

I. 1.M) $\psi_M(z)$ *can be expanded into a mixed N periodic J fraction as follows*

$$(3.12)_M \qquad \psi_M(z) = \frac{B_M^2 |}{|z - A_{M-1}} - \frac{B_{M-1}^2 |}{|z - A_{M-2}} - \cdots - \frac{B_1^2 |}{|z - A_0} - \frac{B_0^2 |}{|z - a_1} - \frac{b_1^2}{|z - a_2} - \frac{b_2^2 |}{|z - a_3} - \cdots$$

where $a_{i+N} = a_i$, $b_{i+N} = b_i$, $1 \leq i < +\infty$.

2.$M\alpha$) $B_0 = b_0$ *but* $A_0 \neq a_N$.

2.$M\beta$) $B_0 \neq b_0$.

II. 1.M) $\psi_M(z)$ *can be described as follows*

$$(3.13)_M \qquad \psi_M(z) = \frac{c_M}{2} \frac{\Lambda_M(z) - \sqrt{P(z)^2 - 4A^2}}{\Gamma_M(z)}$$

where c_M *is a non zero real number and both* $\Gamma_M(z)$ *and* $\Lambda_M(z)$ *are polynomials with real coeficients. Further* $\Gamma_M(z)$ *is monic.*

2.$M\alpha$) $\begin{cases} \deg \Gamma_1(z) = N, \\ \deg \Lambda_1(z) = N \text{ and } \Lambda_1(z) \text{ is monic.} \end{cases}$

$\begin{cases} \deg \Gamma_M(z) = 2M + N - 2, \\ \deg \Lambda_M(z) = 2M + N - 3, \ 2 \leq M < +\infty. \end{cases}$

2.$M\beta$) $\begin{cases} \deg \Gamma_M(z) = 2M + N - 1, \\ \deg \Lambda_M(z) = 2M + N - 2, \ 1 \leq M < +\infty. \end{cases}$

3.M) *The form*

$$(3.14)_M \qquad \omega = \frac{c_M}{2\pi i} \frac{\sqrt{P(x)^2 - 4A^2}}{\Gamma_M(x)} dx$$

gives a positive measure on each stable band $[\lambda_{2k-1}, \lambda_{2k}]$, $1 \leq k \leq N$.

4.M) *If we decompose*

$$(3.15)_M \qquad \Gamma_M(z) = \prod_{i=1}^{r_M} (z - \nu_{M,i}) \prod_{j=1}^{s_M} (z - \zeta_{M,j}) \prod_{j=1}^{s_M} (z - \bar{\zeta}_{M,j})$$

then for some subset $S \subset \mathscr{S}(c_M, \Gamma_M) \subset \{1, 2, \cdots, r_M\}$, *we have*

$$(3.16)_M \qquad \Lambda_M(\nu_{M,i}) = \begin{cases} -\sqrt{P(\nu_{M,i})^2 - 4A^2} & \textit{if } i \in S \\ +\sqrt{P(\nu_{M,i})^2 - 4A^2} & \textit{if } i \in S^c = \{1, 2, \cdots, r_M\} - S \end{cases}$$

and

$$(3.17)_M \qquad \begin{aligned} \Lambda_M(\zeta_{M,j}) &= \sqrt{P(\zeta_{M,j})^2 - 4A^2} && \textit{for } \zeta_{M,j} \in Z_{\mathrm{Im}}(\Gamma_M)\,, \\ \Lambda_M(\bar\zeta_{M,j}) &= \overline{\Lambda_M(\zeta_{M,j})} = \sqrt{P(\bar\zeta_{M,j})^2 - 4A^2} && \textit{for } \bar\zeta_{M,j} \in Z_{\mathrm{Im}}(\Gamma_M)\,. \end{aligned}$$

Remark 3.5. From II. 2M), $(3.16)_M$ and $(3.17)_M$, $\Lambda_M(z)$ is uniquely determined by Lagrange's interpolation. Therefore if we fix the pair c_M, $\Gamma_M(z)$, there still exist 2^χ, where χ is the number of the elements in $\mathscr{S}(c_M, \Gamma_M)$, ambiguities to fix $\Gamma_M(z)$. Indeed if we know the J fraction expansion of $\psi_M(z)$, $(3.12)_M$, which corresponds to S, we can calculate all the other expansions. See Appendix.

Proof. Let us divide the proof from I to II into two steps.

Step 1. We prove the following assertions.

$(*)'_M$: I. 1.M), 2.M) $\Longrightarrow$ II. 1.M), 2.M),

$(*)''_M$: I. 1.M), 2.M) $\Longrightarrow$ $\Gamma_M(z)$ divides the polynomial
$$\Lambda_M(z)^2 - P(z)^2 + 4A^2\,,$$

by induction on M. Let $M = 1$. Then we have

$$\psi_1(z) = \cfrac{B_1^2}{z - A_0 - \left(\cfrac{B_0}{b_0}\right)^2 \varphi(z)}$$

$$(3.18) \qquad = \cfrac{B_1^2}{z - A_0 - \cfrac{1}{2}\left(\cfrac{B_0}{b_0}\right)^2 \cfrac{\Lambda_0(z) - \sqrt{P(z)^2 - 4A^2}}{\Gamma_0(z)}}$$

$$= \frac{B_1^2\Gamma_0(z)\{(z - A_0)\Gamma_0(z) - \frac{1}{2}(B_0/b_0)^2\Lambda_0(z) - \frac{1}{2}(B_0/b_0)^2\sqrt{P(z)^2 - 4A^2}\}}{(z - A_0)^2\Gamma_0(z)^2 - (B_0/b_0)^2(z - A_0)\Gamma_0(z)\Lambda_0(z) + \frac{1}{4}(B_0/b_0)^2(\Lambda_0(z)^2 - P(z)^2 + 4A^2)}\,.$$

By virtue of Corollary 2.5, (2.5) and (2.21), we can decompose as

$$\Lambda_0(z)^2 - P(z)^2 + 4A^2 = \Gamma_0(z)\Gamma_0^c(z)\,,$$
$$\deg \Gamma_0^c(z) \le N - 1\,.$$

Therefore we obtain the expression

$$\psi_1(z) = \frac{c_1'}{2}\,\frac{\Lambda_1(z) - \sqrt{P(z)^2 - 4A^2}}{\Gamma_1'(z)}$$

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where we put

$$c_1' = \left(\frac{B_0 B_1}{b_0}\right)^2,$$

$$\Gamma_1'(z) = (z - A_0)^2 \Gamma_0(z) - \left(\frac{B_0}{b_0}\right)^2 (z - A_0) \Lambda_0(z) + \frac{1}{4}\left(\frac{B_0}{b_0}\right)^2 \Gamma_0^c(z)$$

$$= \frac{b_0^2 - B_0^2}{b_0^2} z^{N+1} + \{\text{terms of lower degree}\},$$

$$\Lambda_1(z) = 2\left(\frac{b_0}{B_0}\right)^2 (z - A_0) \Gamma_0(z) - \Lambda_0(z)$$

$$= \frac{2b_0^2 - B_0^2}{B_0^2} z^N + \{\text{terms of lower degree}\}.$$

According to $B_0 = b_0$ or $B_0 \neq b_0$ we put

$$\begin{cases} c_1 = \dfrac{c_1'}{d_1} \\ \Gamma_1(z) = \dfrac{1}{d_1} \Gamma_1'(z), \end{cases}$$

where d_1 is the coefficient of z^N in $\Gamma_1'(z)$, and

$$\begin{cases} c_1 = c_1' \dfrac{b_0^2}{b_0^2 - B_0^2} = \dfrac{B_0^2 B_1^2}{b_0^2 - B_0^2} \\ \Gamma_1(z) = \dfrac{b_0^2}{b_0^2 - B_0^2} \Gamma_1'(z), \end{cases}$$

we get the assertion $(*)_1'$. Notice that if $B_0 = b_0$ and $d_1 = 0$ then it is the special case $A_0 = a_N$ being excluded. Next let us prove $(*)_1''$. Let $B_0 \neq b_0$. Then if we use Lemma 3.3. 1), we can show that $\psi_1(z)$ must have $N + 1$ poles in $\mathscr{R} - \{P, Q\}$. But from the expression $(3.13)_1$, for any zero $z_0 \in Z(\Gamma_1)$, at least one of the two lifted points z_0^+, z_0^- has to become a pole of $\psi_1(z)$. Since the degree of $\Gamma_1(z)$ is equal to $N + 1$, this means that

$$\Lambda_1(z_0) = \sqrt{P(z_0^-)^2 - 4A^2},$$

or

$$\Lambda_1(z_0) = \sqrt{P(z_0^+)^2 - 4A^2} = -\sqrt{P(z_0^-)^2 - 4A^2}$$

holds. Hence $(*)_1''$ follows. In case $B_0 = b_0$ but $A_0 \neq a_N$, Lemma 3.3. 2) is applicable and we can prove $(*)_1''$ similarly.

By use of induction, since we can rewrite

$$\psi_{M+1}(z) = \frac{B_{M+1}^2}{z - A_M - \psi_M(z)}$$

(3.19)
$$= \frac{B_{M+1}^2}{z - A_M - \dfrac{c_M}{2}\dfrac{\Lambda_M(z) - \sqrt{P(z)^2 - 4A^2}}{\Gamma_M(z)}}$$

$$= \frac{B_{M+1}^2 \Gamma_M(z)\left\{(z - A_M)\Gamma_M(z) - \dfrac{c_M}{2}\Lambda_M(z) - \dfrac{c_M}{2}\sqrt{P(z)^2 - 4A^2}\right\}}{(z - A_M)^2\Gamma_M(z)^2 - c_M(z - A_M)\Gamma_M(z)\Lambda_M(z) + \left(\dfrac{c_M}{2}\right)^2(\Lambda_M(z)^2 - P(z)^2 + 4A^2)},$$

the rest we must do is to show that $(*)''_M$ follows from $(*)'_M$ and $(*)''_{M-1}$. For $M \geq 2$, we can write

(3.20)
$$\psi_M(z) = \frac{B_M^2}{z - A_M - \psi_{M-1}(z)} = \frac{c_M}{2}\frac{\Lambda_M(z) - \sqrt{P(z)^2 - 4A^2}}{\Gamma_M(z)},$$

and use Lemma 3.3. 1). The number of poles in $\mathscr{R} - \{P, Q\}$ of $\psi_M(z)$ is equal to

(3.21)
$$2M + N - 1 \quad \text{if} \quad B_0 = b_0,$$
$$2M + N - 2 \quad \text{if} \quad B_0 = b_0 \quad \text{but} \quad A_0 \neq a_N,$$

and it coincides with the degree of $\Gamma_M(z)$. But (3.20) implies that for any zero $z_0 \in Z(\Gamma_M)$, at least one of the two lifted points z_0^+, z_0^- must be a pole of $\psi_M(z)$. Therefore

(3.22)
$$\Lambda_M(z_0) = \sqrt{P(z_0^-)^2 - 4A^2}$$

or

$$= \sqrt{P(z_0^+)^2 - 4A^2} = -\sqrt{P(z_0^-)^2 - 4A^2}$$

holds and we obtain the assertion $(*)''_M$.

Step 2. Let us prove 3.M) and 4.M). Since $\psi_M(z)$ admits the J fraction expansion, $(3.12)_M$, it is expressible in a Stieltjes transform

(3.23)
$$\psi_M(z) = \int_{-\infty}^{+\infty} \frac{d\mu(x)}{z - x}$$

for some Stieltjes measure $d\mu(x)$. In particular, $\psi_M(z)$ is holomorphic in the domain $\mathscr{R}_- - \{\text{real axis}\}$, so $(3.17)_M$ must be satisfied.

LEMMA 3.6. *Any real zero* $\nu_{M,i} \in Z_{\mathrm{Re}}(\Gamma_M)$ *cannot belong to the interior of stable bands* $[\lambda_{2k-1}, \lambda_{2k}]$, $1 \leq k \leq N$.

Proof of Lemma 3.6. If $\nu_{M,i}$ lies there, it follows that

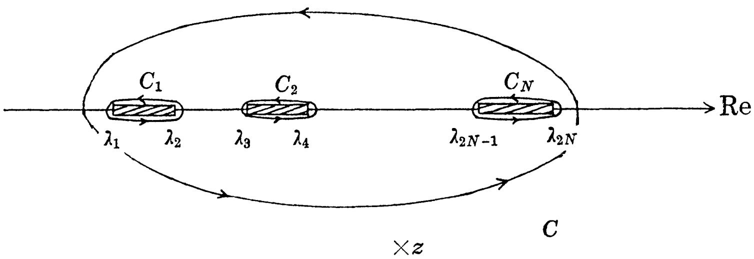
$$\Lambda_M(\nu_{M,i}) = \sqrt{P(\nu_{\bar{M},i})^2 - 4A^2}$$

or

$$-\sqrt{P(\nu_{\bar{M},i})^2 - 4A^2}\,.$$

Then the left hand side is real but the right hand sides are both pure imaginary This is a contradiction.

Let C and C_k, $1 \leq k \leq N$, be anti-clockwise contours in $\mathscr{R}_-$ as below



where the contour C contains all stable bands and all zeroes of $\Gamma_M(z)$ inside. Then for any element z in the outside of C, we can write

$$\begin{aligned}\psi_M(z) &= \frac{1}{2\pi i}\oint_C \frac{\psi_M(x)}{z-x}dx \\ &= \sum_{j=1}^{r_M}\frac{\gamma_{M,j}}{z-\nu_{M,j}} + \frac{1}{2\pi i}\sum_{j=1}^{N}\oint_{C_j}\frac{\psi_M(x)}{z-x}dx \\ &= \sum_{j=1}^{r_M}\frac{\gamma_{M,j}}{z-\nu_{M,j}} + \frac{c_M}{2\pi i}\sum_{j=1}^{N}\int_{\lambda_{2j-1}}^{\lambda_{2j}}\frac{\sqrt{P(x)^2-4A^2}}{(z-x)\Gamma_M(x)}dx\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}\gamma_{M,j} &= \mathrm{Res}_{\nu_{M,j}}\psi_M(z) \\ &= \begin{cases}\gamma_j(c_M, \Gamma_M) & \text{if } \Lambda_M(\nu_{M,j}) = -\sqrt{P(\nu_{\bar{M},j})^2-4A^2}\,, \\ 0 & \text{if } \Lambda_M(\nu_{M,j}) = +\sqrt{P(\nu_{\bar{M},j})^2-4A^2}\,.\end{cases}\end{aligned} \tag{3.25}$$

And we put

$$S = \{j \mid \Lambda_M(\nu_{M,j}) = -\sqrt{P(\nu_{\bar{M},j})^2-4A^2}\}\,. \tag{3.26}$$

Since the Stieltjes measure $d\mu(x)$ is uniquely determined, we have

$$d\mu(x) = d\mu_d(x) + d\mu_c(x) \tag{3.27}$$

where we put

$$d\mu_d(x) = \sum_{j\in S} \gamma_{M,j} \cdot \delta(x - \nu_{M,j})dx\,, \qquad \text{discrete measure,}$$

$$d\mu_c(x) = \sum_{j=1}^{N} \frac{c_M}{2\pi i} \chi_{[\lambda_{2j-1},\lambda_{2j}]}(x) \frac{\sqrt{P(x)^2 - 4A^2}}{\Gamma_M(x)} dx\,,$$

continuous measure.

Then the positivity of $d\mu(x)$ implies II. 3.M) and 4.M).

Now let us prove the converse II to I. If we assume II then $\psi_M(z)$ can be expressed in the Stieltjes transform of the measure defined by (3.27) for some subset $S \subset \mathscr{S}(c_M, \Gamma_M)$. Therefore it admits a J fraction expansion

$$\begin{aligned} \psi_M(z) &= \frac{c_M}{2} \frac{A_M(z) - \sqrt{P(z)^2 - 4A^2}}{\Gamma_M(z)} \\ &= \frac{B_M^2|}{|z - A_{M-1}} - \frac{B_{M-1}^2|}{|z - A_{M-2}} - \cdots - \frac{B_1^2|}{|z - A_0} \\ &\quad - \frac{B_0^2|}{|z - a_1} - \frac{b_1^2|}{|z - a_2} - \cdots . \end{aligned} \tag{3.28}$$

At this stage, we do not know the periodicity. But since we can decompose as

$$\frac{c_M}{2}(A_M(z)^2 - P(z)^2 + 4A^2) = \frac{2}{c'_{M-1}} \Gamma_M(z) \cdot \Gamma_{M-1}(z)\,,$$

$\Gamma_{M-1}(z)$ is monic ,

we have

$$\psi_M(z) = \cfrac{1}{\cfrac{\Gamma_M(z)(A_M(z) + \sqrt{P(z)^2 - 4A^2})}{\frac{c_M}{2}(A_M(z)^2 - P(z)^2 + 4A^2)}} \tag{3.29}$$

$$= \cfrac{1}{az + b - \cfrac{c'_{M-1}}{2} \cfrac{A_{M-1}(z) - \sqrt{P(z)^2 - 4A^2}}{\Gamma_{M-1}(z)}}. \tag{3.30}$$

Here we draw the linear part $az + b$ so that deg $A_{M-1}(z)$ satisfies the condition 2.M-1). If we compare (3.28) and (3.30), we have

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$$B_M = \frac{1}{\sqrt{a}}, \quad A_{M-1} = -\frac{b}{a}, \quad c_{M-1} = \frac{c'_{M-1}}{a}$$

and

$$\frac{c_{M-1}}{2}\,\frac{A_{M-1}(z) - \sqrt{P(z)^2 - 4A^2}}{\Gamma_{M-1}(z)}$$

(3.31)
$$= \frac{B_{M-1}^2}{|z - A_{M-2}} - \frac{B_{M-2}^2}{|z - A_{M-3}} - \cdots - \frac{B_1^2}{|z - A_0} - \frac{B_0^2}{|z - a_1} - \frac{b_1^2}{|z - a_2} - \cdots .$$

If we use Lemma 3.3 and note the J fraction expansion (3.31), the condition 3. M–1) and 4. M–1) are also satisfied for this function. Since the conditions of the case $M = 0$ coincide with those of purely periodic J fractions, we can obtain the proof from II to I by induction on M.

We have already given the proof of the following theorem.

Theorem 3.7. *Let $\psi_M(z) = \psi_{M,S}(z)$ be a mixed N periodic J fraction $(3.12)_M$ in Theorem 3.4 corresponding to some subset $S \subset \mathscr{S}(c_M, \Gamma_M)$. Then*

1) *$\psi_{M,S}(z)$ can be described in the Stieltjes transform of the measure*

$$d\mu_S(x) = d\mu_{S,d}(x) + d\mu_c(x)$$

where

$$d\mu_{S,d}(x) = \sum_{j \in S} \gamma_{M,j}\delta(x - \nu_{M,j})dx\,,$$

$$d\mu_c(x) = \sum_{j=1}^{N} \frac{c_M}{2\pi i}\chi_{[\lambda_{2j-1},\lambda_{2j}]}(x)\frac{\sqrt{P(x)^2 - 4A^2}}{\Gamma_M(x)}dx\,.$$

2) *For any two subsets S, $S' \subset \mathscr{S}(c_M, \Gamma_M)$, we have the relation as follows*

(3.23)
$$\psi_{M,S'}(z) = \psi_{M,S}(z) + \sum_{j \in S'-S} \frac{\gamma_{M,j}}{z - \nu_{M,j}} - \sum_{k \in S-S'} \frac{\gamma_{M,k}}{z - \nu_{M,k}}\,.$$

§ 4. Appendix

We take an arbitrary J fraction $\varphi(z)$ and expand it at the infinity

$$\varphi(z) = \frac{b_0^2}{|z - a_1} - \frac{b_1^2}{|z - a_2} - \frac{b_2^2}{|z - a_3} - \cdots$$

$$= \frac{c_0}{z^1} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots .$$

Then the coefficients a_i, b_i are calculated from the moments c_j by the following formulae

$$a_n = \frac{1}{H'_{n-2}}\left(\frac{H_{n-2}H'_{n-1}}{H_{n-1}} + \frac{H_{n-1}H'_{n-3}}{H_{n-2}}\right),$$

$$b_n^2 = \frac{H_n H_{n-2}}{H_{n-1}^2},$$

with

$$H_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix}, \qquad H'_n = \begin{vmatrix} c_1 & c_2 & \cdots & c_{n+1} \\ c_2 & c_3 & \cdots & c_{n+2} \\ \vdots & \vdots & & \vdots \\ c_{n+1} & c_{n+2} & \cdots & c_{2n+1} \end{vmatrix}$$

and by definition $H'_{-2} = 0$, $H'_{-1} = H_{-1} = 1$.

If we expand the right hand side of (3.32) at the infinity and use these formulae, we can calculate the J fraction expansion of $\psi_{M,S'}(z)$ from that of $\psi_{M,S}(z)$ at finite steps.

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