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Salem Numbers and Pisot Numbers via Interlacing

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Abstract. We present a general construction of Salem numbers via rational functions whose zeros and poles mostly lie on the unit circle and satisfy an interlacing condition. This extends and unifies earlier work. We then consider the "obvious" limit points of the set of Salem numbers produced by our theorems and show that these are all Pisot numbers, in support of a conjecture of Boyd. We then show that all Pisot numbers arise in this way. Combining this with a theorem of Boyd, we produce all Salem numbers via an interlacing construction.

1 Introduction

A *Pisot number* is a real algebraic integer $\theta > 1$ all of whose other (algebraic) conjugates have modulus strictly less than 1. A *Salem number* is a real algebraic integer $\tau > 1$, whose other conjugates all have modulus at most 1, with at least one having modulus exactly 1. It follows that the minimal polynomial $P(z)$ of τ is *reciprocal* (*i.e.*, $z^{\deg P}P(1/z) = P(z)$), that τ^{-1} is a conjugate of τ, that all conjugates of τ other than τ and τ^{-1} have modulus exactly 1, and that $P(z)$ has even degree. The set of all Pisot numbers is traditionally denoted S, with T being used for the set of all Salem numbers.

In [17], we constructed Salem numbers via rational functions associated with certain rooted trees (the *quotients* of rooted *Salem trees*). In this paper we abstract the essential properties of these rational functions and give a much more general construction of Salem numbers (Theorems 3.1, 5.1, and 5.2) via rational functions whose zeros and poles mostly lie on the unit circle and satisfy an interlacing condition. In addition to extending the work of [17], this construction also extends the interlacing construction of [16]. We then consider the "obvious" limit points of the set of Salem numbers produced by our theorems and show that these are all Pisot numbers (Theorems 4.2 and 5.3). This supports a conjecture of Boyd [4, p. 327]. We then show that all Pisot numbers arise in this way (Theorem 6.4). Combining this with a theorem of Boyd, we show that all Salem numbers can be produced via interlacing. We conclude the paper with some applications to the study of small Salem numbers and negative-trace elements of S or T.

It is our hope that these ideas will lead to further improvements in our understanding of the set of Salem numbers, and may give a way to attack some outstanding

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problems: (i) Is there a least Salem number, and, if so, what is it? (ii) Is the set of Salem numbers below (say) 1.3 finite, and, if so, what are its members? (iii) What is the derived set of the set of Salem numbers?

For dealing with Pisot numbers one has the trivial but extremely useful observation that if $f(z)$ is a monic polynomial with integer coefficients having a simple real root $\theta > 1$ such that all roots other than θ have modulus strictly less than 1 and the constant term of $f(z)$ is not 0, then $f(z)$ is irreducible and is therefore the minimal polynomial of θ. (If $f(z)$ split into two nontrivial factors, then the factor that does not have θ as a root would have as its constant term something that on the one hand is a nonzero integer and on the other hand is a product of numbers all with modulus strictly less than one, which is absurd.) For Salem numbers, the analogous statement is not as pleasant: if $g(z)$ is a monic polynomial with integer coefficients having a simple real root $\tau > 1$ such that all the other roots of $g(z)$ have modulus at most one, with at least one having modulus equal to 1, and if the constant term of $g(z)$ is not zero, then $g(z) = t(z)u(z)$, where $t(z)$ is the minimal polynomial of τ, and $u(z)$ is a *cyclotomic polynomial* (for us, following [4], this means simply that all its roots are roots of unity: it need not be irreducible). It is the possibility that $u(z)$ might not equal 1 that renders explicit constructions of the minimal polynomials of Salem numbers more difficult. For Pisot numbers it is enough to find a polynomial that has all its roots in the right place; for Salem numbers one also has to deal with the possibility of cyclotomic factors. A further annoyance is that $t(z)$ might have degree 2, in which case one has that τ is a reciprocal quadratic Pisot number rather than a Salem number.

With these thoughts in mind, it is convenient to define a *Pisot polynomial* to be a polynomial of the form $z^k f(z)$, where $k \geq 0$ and $f(z)$ is the minimal polynomial of a Pisot number. And we define a *Salem polynomial* to be a polynomial of the form $t(z)u(z)$, where $u(z)$ is a cyclotomic polynomial and $t(z)$ is either the minimal polynomial of a Salem number or is the minimal polynomial of a reciprocal quadratic Pisot number.

The plan for the remainder of the paper is as follows. In Section 2 we define the various interlacing conditions that will subsequently be exploited. Section 3 shows how Salem numbers can be produced from pairs of polynomials that satisfy a simple circular interlacing condition; then Section 4 considers the obvious limit points of the set of Salem numbers produced and shows that these are all Pisot numbers. In Section 5 we prove analogous results for other naturally-arising variants of interlacing. In Section 6 we show that all Pisot numbers are generated by one of these interlacing constructions, and in Section 7 we show that all Salem numbers are produced, and we put Salem numbers into four (overlapping) subsets according to the flavour of interlacing used to produce them.

Several other interlacing constructions appear in the literature. Most notably, Bertin and Boyd [1] classify all Salem numbers in a way that involves interlacing. In Section 8 we briefly compare their results with ours before giving some concluding applications and remarks in Section 9. Other interlacing constructions have appeared in [6, Proposition 4.1], [11, 16]. For an encyclopaedic account of real interlacing, see [8].

We use $\mathbb{T}$ to denote the unit circle, $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

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2 Flavours of Interlacing

Several variants of interlacing will arise naturally as we study Salem numbers. We are concerned with interlacing on the unit circle, but the different flavours of interlacing are perhaps most easily understood when one moves to the real line via a Tchebyshev transformation. In Subsection 2.1 we recall this transformation; in Subsections 2.2–2.4 we describe interlacing in the complex world, and in Subsection 2.5 we view it from the real, post-Tchebyshev, perspective.

2.1 Moving to the Real World

Our ultimate objective is to understand Salem numbers and Pisot numbers, and these are firmly rooted in the world of complex numbers. We shall give constructions that involve reciprocal polynomials. Moreover, most (perhaps all) of their roots will be in $\mathbb{T}$, and other roots will be real and positive. It will be extremely convenient for the proofs to transform such polynomials to totally real polynomials. The transformation that we shall use is

$$x = \sqrt{z} + 1/\sqrt{z}\,. \tag{2.1}$$

It is a matter of historical accident (growing out of [14], where this particular transformation was essential) that this variant of the Tchebyshev transformation is used rather than the more familiar $x = z + 1/z$, which would serve just as well, but with many small differences in detail. In applying (2.1), a fixed branch of the square-root is used throughout the right-hand side, but since there is a choice of branch, we generally find two possible values of x. If $z \in \mathbb{T}$ or if z is real, then the corresponding one or two values of x are real.

The transformation (2.1) is generally a 2-to-2 map, with a reciprocal pair z, $1/z$ mapping to a pair x, $-x$. The exceptions are important for us: the single point $z = -1$ corresponds to the single point $x = 0$, and the single point $z = 1$ corresponds to the pair $x = 2$, $x = -2$. The inverse correspondence involves solving a quadratic equation, but we shall never have need for it explicitly.

2.2 CC-interlacing

Suppose that $P(z)$ and $Q(z)$ are coprime polynomials with integer coefficients and positive top coefficients. We say that Q and P satisfy the *CC-interlacing condition*, or that Q/P is a *CC-interlacing quotient* if:

- P and Q have all their roots in $\mathbb{T}$;
- all their roots are simple;
- their roots interlace on the unit circle, in the sense that between every pair of roots of $P(z)$ there is a root of $Q(z)$ and between every pair of roots of $Q(z)$ there is a root of $P(z)$.

Extending to real coefficients, one recovers the *circular interlacing condition* of [16]. If P and Q satisfy the CC-interlacing condition, then they must have the same degree. Moreover, both 1 and -1 must appear among their roots. One of P and Q is a reciprocal polynomial; the other is antireciprocal ($z - 1$ times a reciprocal polynomial).

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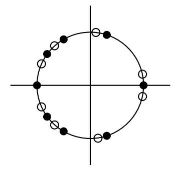

Figure 2.1: CC-interlacing. The roots of $(z-1)(z+1)(z^2+z+1)(z^4+z^3+z^2+z+1)$ [$\bullet$] interlace on $\mathbb{T}$ with those of $z^8+z^7-z^5-z^4-z^3+z+1$ [$\circ$].

The nomenclature is a shorthand for "cyclotomic-cyclotomic interlacing", which in turn is a slight abuse of terminology: the two polynomials have all their roots in $\mathbb{T}$, but need not be cyclotomic, since they need not be monic.

As an example (derived from the quotient attached to $\tilde{E}_8(8)$ in [17, p. 220]) to which we shall return later, take

$$\begin{aligned} P(z) &= (z-1)(z+1)(z^2+z+1)(z^4+z^3+z^2+z+1), \\ Q(z) &= z^8+z^7-z^5-z^4-z^3+z+1. \end{aligned} \tag{2.2}$$

Thus $Q(z)$ is the thirtieth cyclotomic polynomial, and $P(z)$ is the product of the first, second, third, and fifth cyclotomic polynomials. The roots of P and Q interlace on the unit circle as shown in Figure 2.1: Q/P is a CC-interlacing quotient.

Our definition is symmetric in P and Q: if Q/P is a CC-interlacing quotient, then so is P/Q.

Note that the definition of the CC-interlacing condition does not require either P or Q to be monic. When both are monic, then by a theorem of Kronecker [10] they are cyclotomic. In this case, all interlacing examples have essentially been classified by Beukers and Heckman [2].

2.3 CS-interlacing

Now we turn to another flavour of interlacing, where one polynomial has all its roots in $\mathbb{T}$, and the other has all but two roots in $\mathbb{T}$, with these two roots being θ and $1/\theta$ for some real $\theta > 1$. Here "CS" is short for "cyclotomic-Salem", with the same caveat as before that the polynomials need not be monic. One will be reciprocal, and the other will be antireciprocal.

Suppose that $P(z)$ and $Q(z)$ are coprime polynomials with integer coefficients and positive top coefficients. We say that P and Q satisfy the *CS-interlacing condition* and that Q/P is a *CS-interlacing quotient* if:

- P is reciprocal, and Q is antireciprocal;
- P and Q have the same degree;
- all the roots of P and Q are simple, except perhaps at $z=1$;
- $z^2-1 \mid Q$;

- Q has all its roots in $\mathbb{T}$;
- P has all but two roots in $\mathbb{T}$, with these two being real, positive and $\neq 1$;
- on the punctured unit circle $\mathbb{T}\backslash\{1\}$, the roots of Q and P interlace.

Notice the strange interlacing condition. On the unit circle, Q has two more roots than P, and necessarily $Q(1) = 0$. The interlacing condition implies that either Q has a triple root at 1, or it has a pair of simple roots that are closer to 1 on the unit circle than any of the roots of P.

A couple of pictures should clarify this; see Figures 2.2 and 2.3.

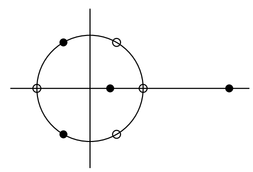

Figure 2.2: CS-interlacing with simple roots. The roots of $Q = (z^2 - 1)(z^2 - z + 1)$ [$\circ$] interlace on $\mathbb{T}\backslash\{1\}$ with those of $P = (z^2 + z + 1)(z^2 - 3z + 1)$ [$\bullet$].

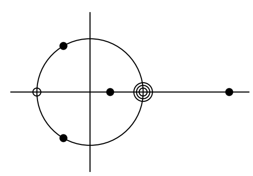

Figure 2.3: CS-interlacing with a triple root at 1. The roots of $Q = (z + 1)(z - 1)^3$ [$\circ$] interlace on $\mathbb{T}\backslash\{1\}$ with those of $P = (z^2 + z + 1)(z^2 - 3z + 1)$ [$\bullet$].

There is no symmetry in the CS-interlacing conditions: if Q/P is a CS-interlacing quotient, then P/Q is not.

2.4 SS-interlacing

For our third flavour of interlacing, "SS" suggests "Salem-Salem" with the usual caveats.

Suppose that $P(z)$ and $Q(z)$ are coprime polynomials with integer coefficients and positive top coefficients. We say that P and Q satisfy the *SS-interlacing condition* and that Q/P is an *SS-interlacing quotient* if:

- P and Q have the same degree;
- all the roots of P and Q are simple;
- P or Q is reciprocal, the other is antireciprocal;

- P and Q have all but two of their roots in $\mathbb{T}$, with these two being real, positive, and $\neq 1$;
- on the unit circle, the roots of $Q(z)$ and $P(z)$ interlace.

The behaviour of the real roots of P and Q gives us two possible types of SS-interlacing. If Q/P is an SS-interlacing quotient then we say that it is a *type* 1 *interlacing quotient* if the largest real root of PQ is a root of P, and it is a *type* 2 *interlacing quotient* if the largest real root of PQ is a root of Q. There is symmetry in the conditions for SS-interlacing, but between the two types: Q/P is a type 1 SS-interlacing quotient if and only if P/Q is a type 2 SS-interlacing quotient. Again it is helpful to see a picture; see Figure 2.4.

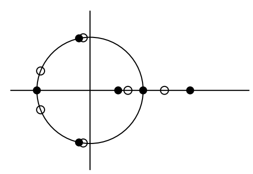

Figure 2.4: Type 1 SS-interlacing. The roots of $Q = z^6 - z^4 - z^3 - z^2 + 1$ [$\circ$] interlace on $\mathbb{T}$ with those of $P = z^6 - 2z^5 + 2z - 1$ [$\bullet$]. For type 2 SS-interlacing, interchange P and Q.

Swapping the roles of P and Q in the example in Figure 2.4 gives an example of type 2 SS-interlacing. Notice that we do not insist that the roots on the positive real axis interlace (although in this particular example they do).

2.5 Real Interlacing Quotients

From any of the above flavours and types of interlacing pairs, we shall consider transforming the pair to a rational function with only real zeros and poles. These zeros and poles will generally interlace (though the interlacing is not always perfect), and for convenience we shall refer to the rational function as a (real) interlacing quotient.

If P and Q satisfy the CC-interlacing condition, the CS-interlacing condition, or either type of SS-interlacing condition, then we transform the function $\sqrt{z}Q(z)/(z-1)P(z)$ via the map (2.1) to get a quotient $q(x)/p(x)$, with q and p coprime polynomials in $\mathbb{Z}[x]$ and $xq(x)/p(x)$ a rational function in x^2. Suppose P and Q have degree d. If $z-1 \mid Q(z)$ (which must be the case for CS-interlacing), then when considering $Q(z)/(z-1)P(z)$ we have pulled out a root of Q, and the remaining roots of P and Q transform in a 2-to-2 or 1-to-1 manner, so that q has degree $d-1$ and p has degree d. If $z-1 \mid P(z)$, then the factor $(z-1)^2$ in the denominator of $Q(z)/(z-1)P(z)$ transforms to x^2-4. We conclude that q has degree d and p has degree $d+1$. We call $q(x)/p(x)$ the *(real) interlacing quotient* corresponding to $Q(z)/P(z)$. The conditions on the roots of P and Q are sufficient to ensure that the roots of p and q are all real.

For CC-interlacing, CS-interlacing, and type 1 SS-interlacing, the roots of p and q interlace perfectly: the zeros and poles of the interlacing quotient interlace. The quotient $q(x)/p(x)$ is decreasing wherever it is defined and has partial fraction expansion

$$\sum_{i=1}^{\deg p} \frac{\lambda_i}{x-\alpha_i}, \tag{2.3}$$

where the α_i are the roots of p and the λ_i are all positive.

For type 2 SS-interlacing, there is perfect interlacing of the zeros of q and p within the interval $[-2,2]$, but there is a blip to the right of $x=2$ (and to the left of $x=-2$) with the top (and bottom) zeros of p and q being in the wrong order for perfect interlacing. The derivative of the quotient $q(x)/p(x)$ changes sign twice, and the partial fraction expansion (2.3) has two of the λ_i negative.

Note that a real interlacing quotient $q(x)/p(x)$, as defined, is always an odd function: one of p and q is an even polynomial and the other is an odd polynomial. The degree of the denominator is one more than the degree of the numerator, and the top coefficients are positive. As $x \to \infty$, $q(x)/p(x) \to 0$ from above.

Proposition 3.3 of [16] extends to this setting.

Lemma 2.1 (a) *If Q_1/P_1 and Q_2/P_2 are CC-interlacing quotients, then so is their sum.*

(b) *Suppose that Q_1/P_1 is either a CS-interlacing quotient or an SS-interlacing quotient and that Q_2/P_2 is a CC-interlacing quotient. Then $Q_1/P_1 + Q_2/P_2$ is either a CS-interlacing quotient or an SS-interlacing quotient.*

Proof Part (a) is just [16, Proposition 3.3].

For (b), we transform to the real world, where it easy to see that everything is of the right shape. Let q_1/p_1 and q_2/p_2 be the corresponding real interlacing quotients. Then the partial fraction expansions of q_1/p_1 and q_2/p_2 as in (2.3) will have all the λ_i positive, except in the case of type 2 SS-interlacing, when the λ_i corresponding to the largest and smallest α_i are negative; these correspond to the roots of p_1 outside $[-2,2]$. The sum $q_1/p_1+q_2/p_2$ will be of the same form; either all the numerators in the partial fraction expansion will be positive, or there will be precisely two negative numerators corresponding to the roots of p_1 outside $[-2,2]$. This is the right shape for CS/SS-interlacing. For type-2 SS-interlacing we know that $q_1(x)/p_1(x) \to -\infty$ as x approaches the largest pole from above, so the same is true for the sum. Also, both $q_1(x)/p_1(x)$ and $q_2(x)/p_2(x)$ are positive for all sufficiently large x, so the sum has a zero to the right of this pole. ∎

3 Salem Numbers via CC-interlacing

We now show how to produce Salem numbers from CC-interlacing quotients. The first construction, which is essentially that of [16], uses a single quotient. We then consider a product construction combining two interlacing quotients in a multiplicative manner, inspired by (but greatly generalising) a formula for the quotients of certain Salem trees [17].

3.1 A Single Pair

Our first interlacing construction is a translation of [16, Proposition 3.2(a)]. This is also a special case of our second construction, Theorem 3.2.

Theorem 3.1 *Let Q/P be a CC-interlacing quotient, with the additional constraint that P is monic. Let q/p be the corresponding real interlacing quotient. If*

$$\lim_{x\to 2+}\frac{q(x)}{p(x)} > 2\,, \tag{3.1}$$

then the only solutions to the equation

$$\frac{Q(z)}{(z-1)P(z)} = 1+\frac{1}{z} \tag{3.2}$$

are a Salem number (or a reciprocal quadratic Pisot number), its conjugates, and possibly one or more roots of unity.

This is proved in [16] using the transformation $x = z + 1/z$. It also follows from Theorem 3.2, taking $P_1 = P$, $Q_1 = Q$, $P_2 = z+1$, and $Q_2 = z-1$. Nevertheless we give a proof here, using the transformation $x = \sqrt{z} + 1/\sqrt{z}$, as this provides a model for later generalisations.

Proof Suppose P and Q have degree d. Since the real interlacing quotient q/p is decreasing (except for jumps at poles), the equation $q(x)/p(x) = x$ has exactly one (simple) root between each pair of consecutive roots of p (these all lie in the interval $[-2, 2]$). The condition (3.1) implies the existence of exactly one solution to $q(x)/p(x) = x$ in the interval $(2,\infty)$. We have now accounted for all the roots of $xp(x) - q(x)$, which is a monic polynomial (given that P is monic) of degree $d+1$ or $d+2$ (according as $z-1 \mid Q$ or $z-1 \mid P$). Transforming back to the complex world, we see that all but two of the solutions to (3.2) lie in $\mathbb{T}$, and these two are a reciprocal pair $\{\tau, 1/\tau\}$ with $\tau > 1$. Clearing denominators in (3.2) gives a monic polynomial with integer coefficients and degree $d+1$ or $d+2$ as appropriate, so we are done. ■

The condition on q/p at $x = 2$ translates to $\lim_{z\to 1+} Q(z)/(z-1)P(z) > 2$, which amounts to either $P(1) = 0$ or ($Q(1) = 0$ and) $Q'(1) > 2P(1)$. Thus this condition can be checked readily without computing q and p.

As an example, take P and Q as in (2.2). We have CC-interlacing, and also $P(1) = 0$, and P is monic. Solving (3.2) gives the famous Lehmer polynomial

$$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

To see that cyclotomic factors may appear, consider $P(z) = z^{10} + z^7 - z^3 - 1$ and $Q(z) = 2z^{10} + z^8 + 2z^7 + z^6 + 2z^5 + z^4 + 2z^3 + z^2 + 2$. Again we have P monic and $P(1) = 0$. Now (3.2) gives the four primitive eighth roots of unity as solutions as well as the degree-8 Salem number with minimal polynomial $z^8 - 2z^7 - z^6 - 3z^4 - z^2 - 2z + 1$.

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3.2 A Product Construction

The following extension of Theorem 3.1 exploits two CC-interlacing pairs (P_1, Q_1) and (P_2, Q_2). Of course, after Lemma 2.1, one possible way of combining two such pairs is to write $P_1/Q_1 + P_2/Q_2 = P_3/Q_3$, giving a third pair (P_3, Q_3) that could be used in Theorem 3.1. Instead of the sum, we now consider the product. This will no longer give CC-interlacing, but we can still squeeze out Salem numbers.

Theorem 3.2 *Let Q_1/P_1 and Q_2/P_2 be two CC-interlacing quotients, with P_1 and P_2 both monic. Let q_1/p_1 and q_2/p_2 be the corresponding real interlacing quotients.*

(i) *Suppose that*

$$\lim_{x\to 2+}\Big(\frac{q_1(x)}{p_1(x)}-2\Big)\Big(\frac{q_2(x)}{p_2(x)}-2\Big) < 1.$$

Then the only solutions to the equation

$$\Big(\frac{Q_1(z)}{(z-1)P_1(z)}-1-\frac{1}{z}\Big)\Big(\frac{Q_2(z)}{(z-1)P_2(z)}-1-\frac{1}{z}\Big) = \frac{1}{z}$$

are a Salem number (or a reciprocal quadratic Pisot number), its conjugates, and possibly one or more roots of unity.

(ii) *Suppose that*

$$\lim_{x\to 2+}\frac{q_1(x)q_2(x)}{p_1(x)p_2(x)} > 1.$$

Then the only solutions to the equation

$$\frac{Q_1(z)Q_2(z)}{(z-1)^2P_1(z)P_2(z)} = \frac{1}{z}$$

are a Salem number (or a reciprocal quadratic Pisot number), its conjugates, and possibly one or more roots of unity.

Part (i) extends an explicit formula arising from a certain family of Salem trees [17, Lemma 7.1(ii)]. The proof makes use of the following lemma.

Lemma 3.3 *Let $\psi_1(x)$ and $\psi_2(x)$ be rational functions in $\mathbb{Z}(x)$, strictly decreasing on the real line (over intervals for which they are defined), with simple zeros and poles. Write $\psi_1(x)\psi_2(x) = f(x)/g(x)$, where $f(x)$ and $g(x)$ are coprime polynomials with integer coefficients. Suppose that $g(x)$ has real zeros at a and b (with $a < b$). Then, counted with multiplicity, the number of solutions to the equation $\psi_1(x)\psi_2(x) = c$ for $x \in (a, b)$ is independent of real $c \geq 0$.*

It will be evident from the proof that all relevant solutions to $\psi_1(x)\psi_2(x) = c$ are simple, except perhaps when $c = 0$. It is possible that $\psi_1\psi_2$ has one or more double zeros, but it cannot have zeros of higher order. The application of interest to us will use only that the number of solutions when $c = 1$ is the same as when $c = 0$.

Proof In intervals where $\psi_1\psi_2$ is positive, it is strictly monotonic: decreasing if both ψ_1 and ψ_2 are positive, and increasing if both are negative. As x passes through a zero $x = \alpha$ of $\psi_1\psi_2$, the function either decreases from ∞ to 0 as x approaches α from below, or $\psi_1\psi_2$ increases from 0 to ∞ as x increases from α (or both, in which case $\psi_1\psi_2$ has a double zero at α; note that if both ψ_1 and ψ_2 vanish at α, then necessarily both have the same sign in a punctured neighbourhood of α). For any $c \geq 0$ it follows that between any successive poles of $\psi_1\psi_2$ the number of solutions to $\psi_1(x)\psi_2(x) = c$ is independent of c. The result follows. ∎

The proof of Theorem 3.2 now follows. We take for ψ_1 and ψ_2 the rational functions $q_1/p_1 - ax$ and $q_2/p_2 - ax$, where $a = 1$ for part (i) and $a = 0$ for part (ii). These are decreasing where defined, since the q_i/p_i are real interlacing quotients corresponding to CC-interlacing quotients. Write $\psi_1(x)\psi_2(x) = f(x)/g(x)$, after cancelling any common factors, so that f and g are coprime polynomials with integer coefficients. Note that, from the remarks in Section 2.5, $f(x)/g(x)$ is an even function. The number of zeros of $\psi_1\psi_2$ between its extreme poles is equal to the degree of $f(x)$, since all roots are real. By Lemma 3.3, this equals the number of solutions to $f(x)/g(x) = 1$; all of these lie in the interval $[-2, 2]$. For part (i), $f/g \sim x^2 \to \infty$ as $x \to \infty$; for part (ii), f/g tends to a finite non-positive number as $x \to \infty$. The condition at $x = 2$ ensures a solution to $f(x)/g(x) = 1$ in the interval $(2, \infty)$, and by evenness also in $(-\infty, -2)$. Since $g(x) - f(x)$ is monic, and we have accounted for all its roots, we are done when we transform back to the complex world. As before, the condition at $x = 2$ transforms to an easily-checked condition at $z = 1$.

4 Pisot Numbers via CC-interlacing

We now construct Pisot numbers by taking limits of convergent sequences of Salem numbers. There is a conjecture of Boyd [4, p. 327] that, if true, would imply that this process will always yield either a Salem number or a Pisot number. Our results in this paper confirm this conjecture for all the cases considered. In this section we consider CC-interlacing. In Section 5 we will briefly treat the other flavours of interlacing.

4.1 CC-limit Functions

We define a *CC-limit function* to be a rational function $h(z)$ such that there is a sequence of CC-interlacing quotients $(h_n(z))$ for which $h_n(z)/(z-1)$ converges to $h(z)$ uniformly in any compact subset of the exterior of the unit disc. For example, $1/z$ is a CC-limit function, as we could take

$$h_n(z) = \frac{(z^n - 1)(z-1)}{z^{n+1} - 1};$$

indeed, in this case we have uniform convergence in the set $|z| \geq 1 + \varepsilon$, for any $\varepsilon > 0$.

Lemma 4.1 *Take any non-negative integers A, r_1, r_2, r_3, r_4, not all zero, and positive integers A_i, a_i $(1 \leq i \leq r_1)$, B_i, b_i $(1 \leq i \leq r_2)$, C_i, c_i $(1 \leq i \leq r_3)$, D_i, d_i $(1 \leq i \leq r_4)$.*

Then the rational function

$$\frac{A}{z-1}+\sum_{i=1}^{r_1}\frac{A_i(z^{a_i}-1)}{(z-1)z^{a_i}}+\sum_{i=1}^{r_2}\frac{B_i z^{b_i}}{(z-1)(z^{b_i}-1)}+\sum_{i=1}^{r_3}\frac{C_i(z^{c_i}+1)}{(z-1)z^{c_i}} + \sum_{i=1}^{r_4}\frac{D_i z^{d_i}}{(z-1)(z^{d_i}+1)} \tag{4.1}$$

is a CC-limit function.

Proof Using the Beukers–Heckman classification [2] (see also [17], where all these terms (or their reciprocals) appear as quotients of graphs (multiplied by $z-1$)) for interlacing cyclotomic polynomials and Lemma 2.1(a), we can define for each natural number n a CC-interlacing quotient Q_n/P_n by

$$\frac{Q_n(z)}{P_n(z)}=\frac{A(z^n+1)}{z^n-1}+\sum_{i=1}^{r_1}\frac{A_i(z^{a_i}-1)(z^n-1)}{z^{n+a_i}-1}+\sum_{i=1}^{r_2}\frac{B_i(z^{n+b_i}-1)}{(z^{b_i}-1)(z^n-1)} + \sum_{i=1}^{r_3}\frac{C_i(z^{c_i}+1)(z^n-1)}{z^{n+c_i}+1}+\sum_{i=1}^{r_4}\frac{D_i(z^{n+d_i}+1)}{(z^{d_i}+1)(z^n-1)}\,. \tag{4.2}$$

An easy estimate shows that for any $\varepsilon>0$ the sequence of functions

$$\frac{Q_n(z)}{(z-1)P_n(z)}$$

converges to the advertised limit function, uniformly in $|z|\geq 1+\varepsilon$. ■

We shall call a rational function of the shape (4.1) a *special CC-limit function*. For these we can exploit their explicit form to prove that certain limit points of the set of Salem numbers are in fact Pisot numbers.

4.2 A Single Interlacing Quotient

Given a single CC-interlacing quotient, we can take the limiting form of our Salem number construction and attempt to prove that the limit is a Pisot number.

Theorem 4.2 *Let Q/P be either a CC-interlacing quotient or zero ($Q=0$, $P=1$), with P monic, and put $g(z)=Q(z)/\big((z-1)P(z)\big)$. Let $h(z)$ be a special CC-limit function as in* (4.1)*. Let $f(z)=g(z)+h(z)-1-1/z$ (if this has a removable singularity at $z=0$, then remove it). If*

$$\lim_{z\to 1+}\big(g(z)+h(z)\big)>2\,, \tag{4.3}$$

then the only non-zero solutions to $f(z)=0$ are a Pisot number θ, the conjugates of θ, and possibly some roots of unity.

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Before proving this, let us make some remarks. One possible choice for $h(z)$ is $1/z$, giving simply $f(z) = g(z) - 1$. The construction of Pisot numbers in [14] is essentially that of Theorem 4.2 with $h(z) = k/z$ for some positive integer k; the construction in [16] uses $h(z) = 1/(z-1)$, which ensures that the condition (4.3) is satisfied. An application of Theorem 4.2 with the more interesting CC-limit function $z^7/(z-1)(z^7-1)$ is given in Subsection 9.1, where it is used to produce a Pisot number that has negative trace and degree only 16, a new record (for old records, see [12, 14, 16]).

Proof For the special CC-limit function $h(z)$ as in (4.1), let $Q_n(z)/P_n(z)$ be as in (4.2) and define $f_n(z) = g(z) + Q_n(z)/(z-1)P_n(z) - 1 - 1/z$. We have that for $|z| > 1$ the function $f(z)$ is the limit of the sequence $(f_n(z))$, with convergence uniform in compact subsets of that region. Moreover, from Theorem 3.1 each $f_n(z)$ has a unique root τ_n in the exterior of the unit disc, at least for all sufficiently large n, say $n \geq n_0$ (so that (3.1) holds).

Note that $f(z)$ has no poles outside the unit disc and has finitely many zeros there (it cannot be identically zero, as the condition (4.3) would then fail). The condition near $z = 1$ gives $\lim_{z\to 1+} f(z) > 0$, and we plainly have $\lim_{z\to +\infty} f(z) = -1$. So there is at least one θ in the real interval $(1, \infty)$ such that $f(\theta) = 0$.

Take any circle, centred on θ, with radius sufficiently small that it lies outside the unit disc and such that no zeros of f other than θ lie in or on the circle. For all sufficiently large n, the function f dominates $f_n - f$ on this circle; hence by Rouché's Theorem (assuming also that $n \geq n_0$), there is exactly one root of f_n in this circle (for all sufficiently large n), and this root must be τ_n. We conclude that $\tau_n \to \theta$ as $n \to \infty$. Since this Rouché argument could be applied to any root of f outside the unit disc, but the sequence (τ_n) has at most one limit, we conclude that θ is the only root of f outside the unit disc.

We deduce that θ is either a Pisot number or a Salem number, and the theorem will follow if we show that θ is not a Salem number. Suppose, for a contradiction, that θ is a Salem number, and let z_0 be a conjugate of θ that lies in $\mathbb{T}$. For a rational function $k(z)$, write $\tilde{k}(z) = k(1/z)/z$. Since $f(z)$ has all coefficients real, $\overline{z_0} = 1/z_0$ is also a zero of $f(z)$. Thus z_0 is a zero of both $f(z) = g(z) + h(z) - 1 - 1/z$ and $\tilde{f}(z) = g(z) + \tilde{h}(z) - 1 - 1/z$ (using here that $(z-1)g(z)$ is a CC-interlacing quotient, so that $g(z)$ is a quotient of reciprocal polynomials). Thus z_0 is a zero of $h(z) - \tilde{h}(z)$, and by Galois conjugation so is θ.

For the five special cases $h(z) = 1/(z-1)$, $(z^a - 1)/(z-1)z^a$, $z^b/(z-1)(z^b-1)$, $(z^c+1)/(z-1)z^c$, and $z^d/(z-1)(z^d+1)$ one checks explicitly that $h(z) - \tilde{h}(z)$ has no roots outside the unit disc, giving the desired contradiction. For the general case, we appeal to Salem's theorem [18] that the set of Pisot numbers is closed. Write $h(z) = h_0(z) + h_1(z)$, where $h_0(z)$ is a single term in (4.1), and $h_1(z)$ is the rest. Then take Q_n/P_n as in the proof of Lemma 4.1 for the limit function h_1 (rather than h). Now for each sufficiently large n, we can apply our special result to conclude that the unique root θ_n of $g(z) + Q_n(z)/(z-1)P_n(z) + h_0(z) - 1 - 1/z$ outside the unit disc is a Pisot number. (Here we use Lemma 2.1(a) again to show that $(z-1)g(z) + Q_n(z)/P_n(z)$ is a CC-interlacing quotient.) Now another Rouché argument shows that θ is the limit of the θ_n, so that Salem's theorem gives that θ is a Pisot number. ∎

4.3 A Product of Two Quotients

Theorem 4.3 *Let Q_2/P_2 and Q_1/P_1 each be either a CC-interlacing quotient or zero, with P_1 and P_2 both monic, and define (for $i = 1, 2$) $g_i(z) = Q_i(z)/(z-1)P_i(z)$. Let h_1 be a special CC-limit function, and let h_2 be either a special CC-limit function or zero.*

(i) *Suppose that*

$$\lim_{z\to 1+} \big(g_1(z) + h_1(z) - 1 - 1/z\big)\big(g_2(z) + h_2(z) - 1 - 1/z\big) < 1.$$

Then the only non-zero roots of the rational function

$$f(z) = \big(g_1(z) + h_1(z) - 1 - 1/z\big)\big(g_2(z) + h_2(z) - 1 - 1/z\big) - 1/z$$

are a certain Pisot number θ, its conjugates, and perhaps some roots of unity.

(ii) *Suppose that*

$$\lim_{z\to 1+} \big(g_1(z) + h_1(z)\big)\big(g_2(z) + h_2(z)\big) > 1.$$

Then the only non-zero roots of the rational function

$$f(z) = \big(g_1(z) + h_1(z)\big)\big(g_2(z) + h_2(z)\big) - 1/z$$

are a certain Pisot number θ, its conjugates, and perhaps some roots of unity.

Proof The proof is very similar to that of Theorem 4.2, so we merely spell out the differences. We again use closure of S to reduce to the special case where $h_2(z) = 0$ and $h_1(z)$ is one of the five special functions $1/(z-1)$, $(z^a - 1)/(z-1)z^a$, $z^b/(z-1)(z^b-1)$, $(z^c+1)/(z-1)z^c$, and $z^d/(z-1)(z^d+1)$. Any Salem number that is a root of $f(z)$ is also a root of $f(1/z)/z^2$, so is a common root of

$$\big(g_1(z) + h_1(z) - a(1 + 1/z)\big)\big(g_2(z) - a(1+1/z)\big) - 1/z$$
$$\text{and}\quad \big(g_1(z) + \tilde{h}_1(z) - a(1 + 1/z)\big)\big(g_2(z) - a(1+1/z)\big) - 1/z,$$

so is a root of $\big(h_1(z) - \tilde{h}_1(z)\big)\big(g_2(z) - a(1+1/z)\big)$, where $a = 1$ or 0 as relevant. As before, $h_1 - \tilde{h}_1$ has no zeros outside the unit disc. Here $g_2(z) - a(1 + 1/z)$ may have a single zero outside the unit disc, but we see from the definition of f that this cannot be a zero of f. ∎

5 Salem and Pisot Numbers via CS/SS-interlacing

Several of the results of the previous section extend to obvious analogues for CS- and SS-interlacing quotients. We record these here briefly.

5.1 Salem Numbers

The analogue of Theorem 3.1 for CS-interlacing quotients is obvious from a sketch of the graph of $q(x)/p(x)$. Indeed, one necessarily has $q(2) \geq 0$ and $p(2) < 0$, so that $q(2)/p(2) \leq 0$, making a single root of $q(x)/p(x) = x$ in $(2, \infty)$ automatic.

Theorem 5.1 *Let Q/P be a CS-interlacing quotient, with the additional constraint that P is monic. Then the only solutions to equation* (3.2) *are a Salem number (or a reciprocal quadratic Pisot number), its conjugates, and possibly one or more roots of unity.*

For SS-interlacing quotients, a sufficient condition that there should be a unique solution to $q(x)/p(x) = x$ in the interval $(2, \infty)$ is that $q(2)/p(2) \leq 2$ (type 1) or $q(2)/p(2) < 2$ (type 2). For type 2 SS-interlacing the stated condition is not always necessary: it might be possible to have $q(2)/p(2) = 2$, depending on the derivative of q/p at $x = 2$. But it is simpler to restrict to a strong inequality.

Theorem 5.2 *Let Q/P be an SS-interlacing quotient (of either type), with the additional constraint that P is monic. Suppose further that*

$$\lim_{z \to 1+} \frac{Q(z)}{(z-1)P(z)} < 2. \tag{5.1}$$

Then the only solutions to equation (3.2) *are a Salem number (or a reciprocal quadratic Pisot number), its conjugates, and possibly one or more roots of unity. For type* 1 *SS-interlacing, a weak inequality in* (5.1) *would suffice.*

There is no analogue of Theorem 3.2, as the construction would give two roots outside the unit disc.

5.2 Pisot Numbers

Certain limiting cases of Theorems 5.1 and 5.2 yield Pisot numbers. Armed with Lemma 2.1(b), we can give the analogue of Theorem 4.2 in this setting.

Theorem 5.3 *Let Q/P be either a CS-interlacing quotient or an SS-interlacing quotient, with P monic, and put $g(z) = Q(z)/\big((z-1)P(z)\big)$. Let $h(z)$ be a special CC-limit function, as in* (4.1). *Let $f(z) = g(z) + h(z) - 1 - 1/z$ (if this has a removable singularity at $z = 0$, then remove it). If*

$$\lim_{z \to 1+} \big(g(z) + h(z)\big) < 2,$$

then the only non-zero solutions to $f(z) = 0$ are a Pisot number θ, the conjugates of θ, and possibly some roots of unity.

Proof This is much as before, but now f, and each f_n in the sequence of functions converging to f, has a pole outside the unit disc (the same pole for each f_n and for f, corresponding to the Salem zero of P). When considering circles centred on roots of f outside the unit disc, the radii must be sufficiently small to avoid enclosing this pole. ∎

6 All Pisot Numbers via Interlacing

In this section we shall show that all Pisot numbers are produced by a special case of Theorem 5.3 (Theorem 6.4 below). We proceed in three steps: in §6.1 we define a sequence of polynomials $(P_k)_{k\geq 0}$, following Salem; in §6.2 we show that for all sufficiently large k the pair (P_k, P_{k+1}) is an SS-interlacing quotient (that these polynomials are Salem polynomials is contained in Salem's work—the novelty here is in establishing the interlacing property); in §6.3 we tie everything together to produce Theorem 6.4.

For any polynomial $A(z) \in \mathbb{Z}[z]$ of exact degree d, define $A^*(z) = z^d A(1/z)$.

6.1 The Polynomials P_k

Lemma 6.1 *Let $A(z)$ be any polynomial of degree d with integer coefficients. For $k \geq 0$, define $P_k(z) = \left(z^k A(z) - A^*(z)\right)/(z-1)$. Then for $k \geq 0$ we have*

$$z^k A(z) = P_{k+1}(z) - P_k(z). \tag{6.1}$$

If $k \geq 1$, then the polynomial P_k has degree $d + k - 1$. If $k \geq 1$, then P_k is a reciprocal polynomial; P_0 is a power of z times a reciprocal polynomial. The polynomials P_k satisfy the recurrence

$$P_{k+2} - (z+1)P_{k+1} + zP_k = 0 \tag{6.2}$$

for $k \geq 0$. For each $k \geq 1$, the pair of polynomials $(P, Q) = (P_{k+1}, P_k)$ is the unique pair of reciprocal polynomials such that the degrees of P and Q are $d + k$ and $d + k - 1$ and such that $z^k A(z) = P(z) - Q(z)$.

Proof This is a collection of simple assertions, each of which follows directly from the definitions. For the recurrence, its characteristic polynomial is $X^2 - (z+1)X + z = (X - z)(X - 1)$. ■

Suppose that $A(z)$ is monic. If $A(0) \neq 1$, then the degree of P_0 is $d - 1$, but if $A(0) = 1$, then this degree is at most $d - 2$. We record as a lemma the observation that no further cancellation in the degree of P_0 can occur if $A(z)$ is the minimal polynomial of a Pisot number θ, unless θ is a reciprocal quadratic Pisot number.

Lemma 6.2 *Let $A(z)$ be the minimal polynomial of a Pisot number θ. If $A(z)$ is not a reciprocal (and hence quadratic) polynomial, then $P_0(z) = (A(z) - A^*(z))/(z-1)$ has degree at least $d - 2$.*

Thus, writing $A(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_0$, this tells us that if $a_0 = 1$, then $a_{d-1} \neq a_1$. In this sense, Pisot polynomials are strongly non-palindromic.

Proof If $a_0 \neq 1$, then P_0 has degree $d - 1$. If $a_0 = 1$ and $a_1 = a_{d-1}$, then expanding $A(z)/A^*(z)$ about $z = 0$ gives

$$A(z)/A^*(z) = 1 + u_2 z^2 + u_3 z^3 + \cdots .$$

This contradicts [7, Théorème 1] (which asserts that the coefficient of z in such an expansion must be strictly positive), unless A is a reciprocal quadratic polynomial. ■

6.2 A Winding Argument

Theorem 6.3 *Suppose that $A(z)$ is the minimal polynomial of a Pisot number. For each $k \geq 0$, define $P_k(z)$ as in Lemma* 6.1.

For all large enough k, both P_{k+1} and P_k have all but two roots on the unit circle, with the other roots real and positive, and the roots of $P_{k+1}(z)$ and $(z-1)P_k(z)$ that lie on the unit circle interlace.

For all $k \geq 1$, the rational function $(z-1)P_k(z)/P_{k+1}(z)$ is an interlacing quotient (either CC, CS, or SS).

Proof Suppose that $A(z)$ has degree d. Note that $P_k(1) = kA(1) + A'(1) - (A^*)'(1)$, and this is negative for all large enough k, since $A(1) < 0$ ($A(z)$ is the minimal polynomial of a Pisot number). Hence $P_k(z)$ has at least one real root greater than 1, for all large enough k. Since P_{k+1} and P_k are both reciprocal, each has at least two positive real roots, for all large enough k. From now on, we assume that k is large enough (say $k \geq k_0 \geq 1$) for this to hold.

For z on the unit circle, $(z-1)P_k(z) = 0$ if and only if

$$z^k A(z) = A^*(z) = z^d A(\bar{z}) = z^d \overline{A(z)}\,,$$

which is equivalent to $z^{k-d}A(z)^2 = |A(z)|^2$, which is equivalent to $z^{k-d}A(z)^2$ being real and positive.

Now $z^{k-d}A(z)^2$ winds round the origin $k+d-2$ times as z winds round 0. Hence $(z-1)P_k(z)$ has at least $k+d-2$ zeros on the unit circle, and $P_k(z)$ has at least $k+d-3$ roots on the unit circle. Together with at least 2 roots not on the unit circle, we have accounted for all possible roots: $P_k(z)$ has exactly $k+d-3$ roots on the unit circle, and two other roots, real and positive (one of them being greater than 1 and the other between 0 and 1).

Similarly $P_{k+1}(z)$ has exactly $k+d-2$ roots on the unit circle, and two other roots, real and positive.

For the interlacing property, we look more closely at what happens as z winds round 0 in the positive sense (anticlockwise), on the unit circle, starting at $z = 1$. When $z = 1$, the argument of $z^{k-d}A^2(z)$ is 0. As z winds round the unit circle, the argument increases to $(k+d-2)2\pi$, not necessarily monotonically. The argument is an integer multiple of 2π precisely when $P_k(z) = 0$ (or when $z = 1$). The argument equals that of $1/z$ (modulo integer multiples of 2π) precisely when $P_{k+1}(z) = 0$. It is clear from Figure 6.1 that this must happen at least once (and hence, by counting, exactly once) between each two consecutive zeros of $(z-1)P_k(z)$, as claimed: the line running from bottom left to top right (which need not be a straight line!) must cross one of the short diagonals at least once between each pair of horizontal lines.

For the final assertion of the theorem, we need to consider what happens for smaller values of $k \geq 1$. The winding argument still accounts for all but two of the zeros of each P_k . We need to pin down the other two roots and establish the claimed interlacing property. Let $\theta > 1$ be the Pisot root of $A(z)$. If $A^*(\theta) > 0$, then $P_k(\theta) < 0$, in which case P_k always has a real root greater than θ. In this case there is nothing more to prove; we have SS-interlacing.

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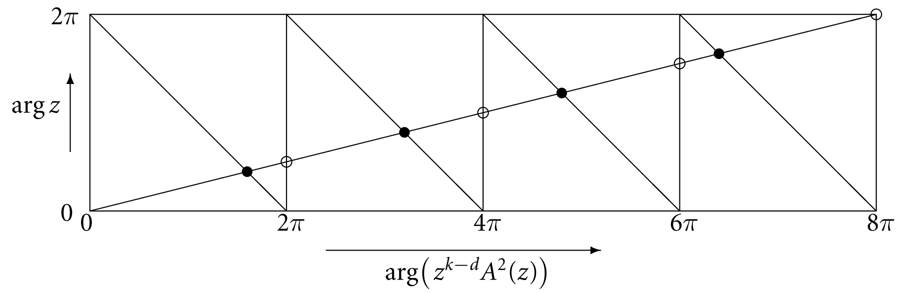

Figure 6.1: The case $d + k - 2 = 4$. The zeros of $(z-1)P_k(z)$ [○] and of $P_{k+1}(z)$ [●] interlace.

We are left with the case that $A^*(\theta) < 0$. Then P_k has no real root greater than θ, for any k. In particular, for $k \geq k_0$ the Salem root τ_k of P_k satisfies $1 < \tau < \theta$. Then from $z^k A(z) = P_{k+1}(z) - P_k(z)$ we have $P_{k+1}(\tau_k) < 0$, and hence $\tau_{k+1} > \tau_k$. If follows that $(z-1)P_k(z)/P_{k+1}(z)$ is a type 1 SS-interlacing quotient for $k \geq k_0$, and the roots and poles of the corresponding real interlacing quotient $p_k(x)/p_{k+1}(x)$ interlace perfectly on the real line. The recurrence (6.2) translates to the real world as

$$p_{k+1}(x) = xp_k(x) - p_{k-1}(x)\,.$$

Since the zeros of p_{k_0} and p_{k_0+1} interlace, one deduces that those of p_{k_0-1} and p_{k_0} interlace, and then those of p_{k_0-2} and p_{k_0-1}, and so on. Thus $P_{k+1}(z)$ and $(z-1)P_k(z)$ interlace for all $k \geq 1$. If $p_k(2) = 0$, then P_k has a double zero at $z = 1$, and $(z-1)P_k$ has a triple zero there; in this case we have CS-interlacing. If P_{k+1} has all roots on the unit circle, then we have CC-interlacing (if $p_{k+1}(2) = 0$, then P_{k+1} has a double zero at $z = 1$, and $P_{k+1}(z)/(z-1)$ interlaces with $P_k(z)$). If P_{k+1} is Salem but P_k is cyclotomic, we have CS-interlacing. ■

6.3 Theorem 5.3 Gives all Pisot Numbers

Now we are in a position to show that the interlacing construction given in Theorem 5.3, with $h(z) = 1/z$, produces all Pisot numbers.

Theorem 6.4 *Given any Pisot number θ, there exists an SS-interlacing quotient $Q(z)/P(z)$ satisfying the conditions of Theorem 5.3 (with $h(z) = 1/z$) such that the only solutions to $Q(z)/P(z) = 1$ are θ, its conjugates, and 0.*

Proof Let $A(z)$ be the minimal polynomial of θ. We consider

$$P(z) = P_{k+1}(z) = \left(z^{k+1}A(z) - A^*(z)\right)/(z-1),$$

$$\text{and} \quad Q(z) = (z-1)P_k(z) = z^k A(z) - A^*(z).$$

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We have seen (Theorem 6.3) that for all large enough k the quotient Q/P is an SS-interlacing quotient. To apply Theorem 5.3 with $h(z) = 1/z$ we need the condition $\lim_{z\to 1+} Q(z)/(z-1)P(z) < 1$. But

$$\lim_{z\to 1+} \frac{Q(z)}{(z-1)P(z)} = \lim_{z\to 1+} \frac{P_k(z)}{P_{k+1}(z)} = \frac{kA(1) + A'(1) - (A^*)'(1)}{(k+1)A(1) + A'(1) - (A^*)'(1)},$$

and this is less than 1 if k is large enough, since $A(1) < 0$.

Finally we note that $Q(z)/(z-1)P(z) = 1$ is equivalent to $z^k A(z) = 0$, which has as its roots θ, all the conjugates of θ, and 0 (assuming $k > 0$). ■

For smaller values of k the quotient Q/P in the proof of Theorem 6.4 might be CS-interlacing or CC-interlacing. The case $k = 0$ and $P_0(0) = 0$ is exceptional, as ever.

The proof of Theorem 6.4 uses Salem's method to construct the P_k from A, and then shows, conversely, how A can be recovered from P_k via (6.1) of Lemma 6.1 for k sufficiently large.

7 All Salem Numbers via Interlacing

7.1 Boyd's Theorem

We recall the following fundamental result of Boyd [4].

Theorem 7.1 ([4, Theorem 4.1]) *Let τ be a Salem number with minimal polynomial $R(z)$. Define $S_1(z) = z^2 + 1$, $S_{-1}(z) = z - 1$. Then for each choice of $\varepsilon = \pm 1$, there exist infinitely many Pisot polynomials $A(z)$ such that (with $A^*(z)$ as before)*

(7.1) $$S_\varepsilon(z)R(z) = zA(z) + \varepsilon A^*(z).$$

7.2 All Salem Numbers via Interlacing

Armed with Theorem 7.1, we show first (Lemma 7.2) that we can produce all Salem numbers via SS-interlacing quotients, but with a "right-hand side" other than $1+1/z$, as used in Theorems 3.1, 5.1, and 5.2.

Lemma 7.2 *Let τ be any Salem number and choose $\varepsilon = \pm 1$. Then for all sufficiently large k there exists an SS-interlacing quotient $Q(z)/P(z)$ such that the only non-zero solutions to*

(7.2) $$\frac{Q(z)}{(z-1)P(z)} = \frac{z^{k-1} + \varepsilon}{z^k + \varepsilon}$$

are τ, its conjugates, and perhaps some roots of unity.

Proof Let $R(z)$ be the minimal polynomial of τ, and let $A(z)$ be a Pisot polynomial such that (7.1) holds with our choice of ε. As in the proof of Theorem 6.4, we put

$$P = P_{k+1}(z) = \left(z^{k+1}A(z) - A^*(z)\right)/(z-1) \quad \text{and}$$
$$Q = (z-1)P_k(z) = z^k A(z) - A^*(z),$$

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and repeat the observation that for all sufficiently large k the quotient $Q(z)/P(z)$ is an SS-interlacing quotient. With $z^kA(z) = P_{k+1}(z) - P_k(z)$ we have $A^*(z) = P_{k+1}(z) - zP_k(z)$, and hence (with $S_\varepsilon(z) = z^2+1$ or $z-1$ according as $\varepsilon = 1$ or -1) from (7.1) we have

$$\begin{aligned} z^{k-1}S_\varepsilon(z)R(z) &= z^kA(z) + \varepsilon z^{k-1}A^*(z) \\ &= (P_{k+1}(z) - P_k(z)) + \varepsilon z^{k-1}(P_{k+1}(z) - zP_k(z)) \\ &= (1+\varepsilon z^{k-1})P_{k+1}(z) - (1+\varepsilon z^k)P_k(z), \end{aligned}$$

from which the result follows. ■

Instead of taking large k in Lemma 7.2, we can consider choosing k of any size. The choice of $k = 1$ gives the following theorem.

Theorem 7.3 *Consider the equation*

$$\frac{Q(z)}{(z-1)P(z)} = \frac{2}{z+1}\,. \tag{7.3}$$

Define four types of Salem number I, II, III, IV as follows. A Salem number τ is of type I (respectively, II, III, IV) if there exist monic polynomials $P(z)$, $Q(z)$ such that $Q(z)/P(z)$ is a CC-interlacing quotient (respectively, CS-interlacing, type 1 SS-interlacing, type 2 SS-interlacing) and for which the only non-zero solutions to (7.3) are τ, its conjugates, and pehaps some roots of unity. Then every Salem number is of at least one of these four types.

Proof We take $k = 1$ and $\varepsilon = 1$ in the proof of Lemma 7.2. For the interlacing properties, we appeal to Theorem 6.3. ■

8 Comparison with the Bertin-Boyd Classification

Let τ be any Salem number, with minimal polynomial $R(z)$. Bertin and Boyd [1] showed that there exist reciprocal polynomials $K(z)$ and $L(z)$ such that $L(z)$ interlaces with $K(z)R(z)$ on the unit circle. Their Theorem B is most relevant here, as it relates to expressing $K(z)R(z)$ in the shape $zA(z) + \varepsilon A^*(z)$, where $A(z) = z^mA_0(z)$ is a Pisot polynomial, with $A_0(0) \neq 0$.

In the case $\varepsilon = 1$, which they use only when $A_0(0) < 0$, their polynomial $L(z)$ is $A(z) + A^*(z)$; in the case $\varepsilon = -1$, which they use only when $A_0(0) > 0$, their $L(z)$ is our $P_0(z)$. Our proof of interlacing comes from a winding argument; theirs is via a characterisation of "entrances" and "exits" to and from the unit disc for the associated algebraic curve $zA(z) + \varepsilon tA^*(z) = 0$ ($t \geq 0$ real); see [4, Lemma 3.1].

9 Final Remarks

9.1 Pisot Numbers of Negative Trace

As one application of the construction in Theorem 4.2, we produce an example of a Pisot number that has trace -1 and degree only 16. Earlier examples of Pisot numbers that had negative trace had much larger degrees ([12, 14]). The algorithm in

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[16] for producing Pisot numbers of any desired trace gives an example with degree 38. The construction there was that in Theorem 4.2 with $h(z) = 1/(z-1)$. Instead, take $g(z) = (z-1)(z^8+z^7-z^5-z^4-z^3+z+1)/(z^2-1)(z^3-1)(z^5-1)$ and $h(z) = z^7/(z-1)(z^7-1)$ to give a Pisot number with degree 16 and trace -1; its minimal polynomial is

$$z^{16}+z^{15}-z^{14}-4z^{13}-6z^{12}-7z^{11}-7z^{10}-7z^9-6z^8-4z^7-2z^6-z^5+z^3+2z^2+2z+1\,.$$

The choice of $g(z)$ (see §2.2) produces a low-degree example of a Salem number with trace -1 via Theorem 3.1. The choice of the CC-limit function $h(z)$ is motivated by the desire to introduce a new, negative-trace, low-degree cyclotomic factor into the denominator.

We used the Dufresnoy–Pisot–Boyd algorithm [5] to search for small Pisot numbers of small degree and negative trace. For Pisot numbers below 2, we found 10 examples, of degrees between 22 and 48. The degree-16 example above is for a Pisot number slightly larger than 2. Finding the smallest degree (perhaps 16?) for a Pisot number of negative trace remains a challenge.

9.2 Salem Numbers of Large Negative Trace

For Salem numbers of trace -1, see [19]. Salem numbers of trace below -1 first appeared via a graph construction [15], which can now be seen as a special case of Theorem 3.1. In [16], Salem numbers of arbitrary trace were produced by interlacing, but the interlacing quotients were not optimal for producing minimal degrees. Starting with the interlacing quotient

$$g(z) = \frac{(z-1)(z^8+z^7-z^5-z^4-z^3+z+1)}{(z^2-1)(z^3-1)(z^5-1)}$$

from Subsection 2.2, add the CC-interlacing quotient

$$\frac{z^{18}-1}{(z-1)(z^7-1)(z^{11}-1)} + \frac{z^{30}-1}{(z-1)(z^{13}-1)(z^{17}-1)}$$

and apply Theorem 3.1 to produce a Salem number of degree 54 and trace -3, the smallest degree currently known for this trace:

$$z^{54}+3z^{53}+2z^{52}-11z^{51}-48z^{50}-122z^{49}-245z^{48}+\cdots\,.$$

(One needs to check that this polynomial has no cyclotomic factors. This can be done using the algorithm of Beukers and Smyth [3] or by checking irreducibility.)

A real transform of this polynomial can be found in [13, §5.4].

9.3 Small Salem Numbers

Proposition 9.1 *Let τ be any Salem number below the real root of z^3-z-1 (so that τ is smaller than any Pisot number). Let $R(z)$ be the minimal polynomial of τ, and let $A(z)$ be any Pisot polynomial such that* (7.1) *holds with $\varepsilon = 1$. Then $A(z)$ has at least three real roots, with at least one between $1/\tau$ and 1.*

Proof The conditions on τ imply that $A(\tau) < 0$. Putting $z = \tau$ in (7.1) we deduce that $A^*(\tau) > 0$, and hence A^* has a real root between 1 and τ. Thus A has at least two real roots: the Pisot number and another root between $1/\tau$ and 1. Since A has odd degree, it must have at least three real roots. ■

For example, taking τ to be Lehmer's number, one possibility for $A(z)$ is

$$z^{11} - 2z^9 - 4z^8 - 4z^7 - 3z^6 - z^5 + z^4 + 3z^3 + 4z^2 + 3z + 1.$$

Sure enough, this has real roots approximately equal to $-0.74616, 0.98390, 2.20974$.

It is not known whether or not there is a smallest Salem number. If there is one, then the next theorem gives some information about it.

Theorem 9.2 *If there is a smallest Salem number τ, then it is of type IV (as defined in Theorem* 7.3*) and not of any other type.*

Proof Suppose that there is a smallest Salem number τ. We take $A(z)$ and the sequence $P_k(z)$ as in the proof of Lemma 7.2, and claim that $Q(z)/P(z) = (z-1)P_1(z)/P_2(z)$ must be a type 2 SS-interlacing quotient, showing that τ is of type IV.

If $Q(z)/P(z)$ were either CS-interlacing or type 1 SS-interlacing, then the Salem root of $P(z)$ would be smaller than τ, giving a contradiction.

Finally we eliminate the possibility that $Q(z)/P(z)$ is CC-interlacing. In this case we increase k until P_{k+1} becomes Salem, with P_k still cyclotomic; then we get (using (7.2) with $Q/P = (z-1)P_k/P_{k+1}$, as in the proof of Lemma 7.2) that the root of P_{k+1} is smaller than our Salem number, again contradicting the minimality of τ. ■

Note that this is not saying that there are no examples of small Salem numbers that come from CC or CS interlacing, merely that if we go via (7.3) (as in the definition of types) we will not see small Salem numbers arising other than as type IV.

9.4 Further Consequences of CC-interlacing

We conclude with two amusing remarks concerning CC-interlacing quotients, which we record as a single proposition.

Proposition 9.3 *Let Q/P be a CC-interlacing quotient. Then*

(a) *$P^2 + Q^2$ has all its roots in $\mathbb{T}$;*
(b) *$P + Q$ has all its roots in the open unit disc $|z| < 1$.*

Proof For (a) we just apply Lemma 2.1(a). The sum $P/Q + Q/P$ is a CC-interlacing quotient, so its numerator has all roots in $\mathbb{T}$. We can say even further that these roots interlace with those of PQ.

For (b) we use another winding argument. Let d be the common degree of P and Q, and suppose that $z - 1 \mid Q$. We observe that it is enough to show that $f(z) = P(z^2) + Q(z^2)$ (a polynomial of degree $2d$) has all its roots in the open unit disc.

Write $f(z) = z^d g(z) = z^d\big(P(z^2)/z^d + Q(z^2)/z^d\big)$. Since P is reciprocal and Q is antireciprocal, $P(z^2)/z^d$ and $Q(z^2)/z^d$ give the real and imaginary parts of $g(z)$ when

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z is on the unit circle. As z goes round the unit circle, anticlockwise, the argument of z^d increases by $2d\pi$. The roots of P and Q interlace on the unit circle, so as z goes round the unit circle the pair $(\Re g(z), \Im g(z))$ cycles d times through one of the patterns $(+,+)$, $(+,-)$, $(-,-)$, $(-,+)$ or $(+,+)$, $(-,+)$, $(-,-)$, $(+,-)$. We do not (yet) know which of these patterns occurs, nor at what point in the pattern we start, but d complete cycles through one of these two patterns must be made. In either case, $g(z)$ winds d times round the origin. In the former case it winds clockwise, and in the latter case anticlockwise. We conclude that as z goes anticlockwise around the unit circle, the argument of $f(z)$ increases by either 0 or $4d\pi$. It follows that $P(z^2) + Q(z^2)$ has either all of its roots in the open unit disc or none of them, and the same holds for $P(z) + Q(z)$. But since P is reciprocal and Q is antireciprocal, we have $P(0) + Q(0) = 1 + (-1) = 0$, so that $P + Q$ has at least one root, namely 0, that has modulus strictly less than 1. Hence all d roots must have modulus strictly less than 1. ■

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