Proceedings of the Edinburgh Mathematical Society (1993) 36, 289–297 ©

THE p-HUPPERT-SUBGROUP AND THE SET OF p-QUASI-SUPERFLUOUS ELEMENTS IN A FINITE GROUP

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(Received 31st July 1991)

Based on the theory of p-supersoluble and supersoluble groups, a prime-number parametrized family of canonical characteristic subgroups $\Gamma_p(G)$ and their intersection $\Gamma(G)$ is introduced in every finite group G and some of its properties are studied. Special interest is dedicated to an elementwise description of the largest p-nilpotent normal subgroup of $\Gamma_p(G)$ and of the Fitting subgroup of $\Gamma(G)$.

1991 *Mathematics subject classification*: 20D25, 20E28.

0. Introduction

Let p be a prime number. The p-supersoluble groups ([**4**, p. 713]) are characterized among the p-soluble groups by a famous theorem due to Huppert ([**4**, p. 717, Th. 9.2/9.3], [**5**]). *The p-soluble group G is p-supersoluble, if and only if, for every maximal subgroup V of G, the index $|G: V|$ is p or relatively prime to p.*

Huppert's theorem has a general significance in all finite groups G: For every prime p we introduce the characteristic subgroup $\mathbf{\Gamma}_p(G)$ (*the p-Huppert-subgroup* of G) as being the intersection of all maximal subgroups of G which have composite index divisible by p. Let $\mathbf{A}_p(G)$ be the largest normal p-soluble subgroup of G. Then $\mathbf{A}_p(\mathbf{\Gamma}_p(G)) = \mathbf{A}_p(G) \cap \mathbf{\Gamma}_p(G)$ is p-supersoluble for every G. Moreover, $\mathbf{\Gamma}(G) = \bigcap_p \mathbf{\Gamma}_p(G)$, the intersection of all maximal subgroups of G of composite indices, is supersoluble. These results can be deduced from recent literature [2]. Using Huppert's theorem and a natural generalization for p-soluble groups of Gaschütz' theory of saturated formations, we give independently a short proof of these facts.

Our main attention we direct to $\mathbf{F}_p(\mathbf{\Gamma}_p(G))$, the largest p-nilpotent normal subgroup of $\mathbf{\Gamma}_p(G)$ and $\mathbf{F}(\mathbf{\Gamma}(G))$, the Fitting subgroup of $\mathbf{\Gamma}(G)$. These subgroups merit special interest: The elements of the Frattini-subgroup $\mathbf{\Phi}(G)$, the intersection of all maximal subgroups of G, are known as the *superfluous* elements of G (see [**4**, p. 268]). We call an $x \in G$ a *quasi-superfluous* element of G, if the cyclic group $\langle x \rangle$ is permutable with every maximal subgroup of G. With respect to a prime number p, we call x a *p-quasi-superfluous* element of G if $\langle x \rangle V = V \langle x \rangle$ holds for the maximal subgroups V of G which have index divisible by p. Let $\mathbf{Qs}_p(G)$ denote the set of all p-quasi-superfluous elements, $\mathbf{Qs}(G) = \bigcap_p \mathbf{Qs}_p(G)$ the set of all quasi-superfluous elements of G. We show: For every group G, the set $\mathbf{Qs}(G)$ coincides with $\mathbf{F}(\mathbf{\Gamma}(G))$ and, for odd prime p, the set $\mathbf{Qs}_p(G) \cap \mathbf{A}_p(G)$ is

$\mathbf{F}_p(\mathbf{\Gamma}_p(G))$. In particular these sets are subgroups of G which from their definition is not immediate. We use conventional notions and notation.

1. The p-Huppert-subgroup

Definition. Let G be a group.

(a) For every prime number p, the *p-Huppert-subgroup* $\mathbf{\Gamma}_p(G)$ of G is the intersection of all maximal subgroups V of G such that $p \mid |G:V| \neq p$.

(b) The intersection $\mathbf{\Gamma}(G)=\bigcap_p \mathbf{\Gamma}_p(G)$ we call the *Huppert-subgroup* of G.

Obviously $\mathbf{\Gamma}_p(G)$ and $\mathbf{\Gamma}(G)$ are characteristic subgroups of G which contain $\mathbf{\Phi}(G)$. We have $\mathbf{\Gamma}_p(G)=G$ if and only if every maximal subgroup of G is of index p or relatively prime to p. $\mathbf{\Gamma}(G)=G$ if and only if every maximal subgroup of G is of prime index in G.

By the definition it is clear that, for a normal subgroup N of G such that $N \leqq \mathbf{\Gamma}_p(G)$, one has $\mathbf{\Gamma}_p(G/N)=\mathbf{\Gamma}_p(G)/N$. Moreover, the largest normal p'-subgroup $\mathbf{O}_{p'}(G) \leqq \mathbf{\Gamma}_p(G)$.

One first observation is:

Proposition 1.1. (a) *The p-soluble group G is p-supersoluble if and only if* $\mathbf{F}_p(G) \leqq \mathbf{\Gamma}_p(G)$.

(b) *The soluble group G is supersoluble if and only if* $\mathbf{F}(G) \leqq \mathbf{\Gamma}(G)$.

Proof. Applying (a) for all p, we see that (b) is a consequence of (a). To prove (a) we mention that, by Huppert's theorem, the p-soluble group G is p-supersoluble if and only if $\mathbf{\Gamma}_p(G)=G$. So we only have to prove that $\mathbf{\Gamma}_p(G)=G$ if $\mathbf{F}_p(G) \leqq \mathbf{\Gamma}_p(G)$. By definition of $\mathbf{\Gamma}_p(G)$, it suffices to show that $G/\mathbf{\Gamma}_p(G)$ is abelian. We induct on $|G|$ to show that $G/\mathbf{F}_p(G)$ is abelian. If $N=\Phi(G)$ or $N=\mathbf{O}_{p'}(G)$, then $\mathbf{\Gamma}_p(G/N)=\mathbf{\Gamma}_p(G)/N$ and $\mathbf{F}_p(G/N)=\mathbf{F}_p(G)/N$. Therefore the result holds by induction in the case $N>1$. So we may assume that $\Phi(G)=\mathbf{O}_{p'}(G)=1$. Now $\mathbf{F}_p(G)$ is a p-group and $1=\Phi(G)=\mathbf{\Gamma}_p(G) \cap \mathbf{F}_p(G) \cap D = \mathbf{F}_p(G) \cap D$, where D is the intersection of all maximal subgroups U of G such that $\mathbf{F}_p(G) \nleqq U$. For every such U we have that $|G:U|=p$ and $\mathbf{F}_p(G)/\mathbf{F}_p(G) \cap U$ is a chief factor of G of order p. Therefore the commutator subgroup $G' \leqq \bigcap_U C_G(\mathbf{F}_p(G)/\mathbf{F}_p(G) \cap U) = C_G(\mathbf{F}_p(G)/\mathbf{F}_p(G) \cap D) = C_G(\mathbf{F}_p(G)) \leqq \mathbf{F}_p(G)$ ([4, p. 690, Th. 6.5]). □

The following lemmas are straightforward.

Lemma 1.2. *Let G be a group, V a subgroup of G of prime index* $|G:V|=p$ *and let* $V_G=\bigcap_{g\in G} V^g=1$. *Suppose there exists a normal p-soluble subgroup N of G, such that* $G=VN$. *Then G is metacyclic.*

Proof. If N is chosen minimal with $G=VN$, then $|N|=p$, N is self-centralizing in G and G/N is cyclic.

Corollary 1.3. *Let G be a group, V a subgroup of G of prime index p and let N be a p-soluble normal subgroup of G, such that $G=VN$. Then G/V_G is metacyclic.*

Proof. The group G/V_G fulfills the hypothesis of Lemma 1.2 with NV_G/V_G as p-soluble normal subgroup.

If G is a group, $\mathscr{F}$ a formation (see [4, p. 696]), we denote by $G^{\mathscr{F}}$ the smallest normal subgroup of G having $\mathscr{F}$-factor group ($G^{\mathscr{F}}$ is the so-called $\mathscr{F}$-residual of G). We remember that a subgroup $H \leqq G$ is called a *covering $\mathscr{F}$-subgroup* or *Sylow-$\mathscr{F}$-subgroup* of G, if $H \in \mathscr{F}$ and if for $H \leqq X \leqq G$ we have $HX^{\mathscr{F}}=X$.

Let $Syl_{\mathscr{F}}(G)$ denote the set of Sylow-$\mathscr{F}$-subgroups of G.

Lemma 1.4 ([4, p. 701, Th. 7.11]). *Let G be a group and $\mathscr{F}$ a formation. Suppose $H \in Syl_{\mathscr{F}}(G)$. If $\mathscr{F}$ contains all groups of prime orders, then $\mathbf{N}_G(H)=H$.*

We are interested in formations which satisfy the following:

Condition (*). Let p be a prime number, $\mathscr{F}$ a non-empty formation of p-soluble groups such that for any group G:

(a) If $G/\mathbf{\Phi}(G) \in \mathscr{F}$, then $G \in \mathscr{F}$ ($\mathscr{F}$ is a so-called *saturated* formation).

(b) If $G/\mathbf{O}_{p'}(G) \in \mathscr{F}$, then $G \in \mathscr{F}$.

Examples of formations which satisfy the condition (*) are: The class of p-nilpotent groups and the class of all p-soluble groups whose p-length does not exceed a given upper limit. It is clear that the essential contents of Huppert's theorem, cited in the introduction, can be stated as:

Lemma 1.5. *For any prime number p, the class of p-supersoluble groups is a formation satisfying* (*) *and which contains all metacyclic groups.*

It is not hard to show (see [4, p. 700, Th. 7.10]):

Lemma 1.6. *Let G be a p-soluble group, $\mathscr{F}$ a formation which satisfies the condition* (*). *Then*

(a) *G has a Sylow-$\mathscr{F}$-subgroup.*

(b) *Any two Sylow-$\mathscr{F}$-subgroups of G are conjugate.*

Lemma 1.7 (The Frattini Argument). *Let G be a group, $\mathscr{F}$ a formation which satisfies the condition* (*) *and let U be a normal p-soluble subgroup of G. If $H \in Syl_{\mathscr{F}}(U)$, then $G=\mathbf{N}_G(H)U$.*

Proof. If $g \in G$, then $H, H^g \in Syl_{\mathscr{F}}(U)$. By Lemma 1.6 there is a $u \in U$ such that $H^u=H^g$. Now, $g=(gu^{-1})u$ with $gu^{-1} \in N_G(H)$.

With these preparations we are now able to prove:

Proposition 1.8. *Let p be a prime number, G a group, U a p-soluble normal subgroup of G and let $\mathbf{\Gamma}_p(G)$ be the p-Huppert-subgroup of G. Let $\mathcal{F}$ be a formation which satisfies the condition (*) and which contains all metacyclic groups. Then the following holds:*

If $U\mathbf{\Gamma}_p(G)/\mathbf{\Gamma}_p(G)\in\mathcal{F}$, then $U\in\mathcal{F}$.

Proof. Suppose that the proposition fails, and let G be a minimal counterexample to the statement. We will prove a series of consequences of this assumption.

We put $D=\mathbf{\Gamma}_p(G)\cap U$.

(i) *We have $D>1$:*
If $D=1$, then $U\cong U\mathbf{\Gamma}_p(G)/\mathbf{\Gamma}_p(G)\in\mathcal{F}$ by hypothesis, which contradicts the choice of G.
Let N be a minimal normal subgroup of G, such that $N\leqq D$.

(ii) *$U/N\in\mathcal{F}$ and N is a p-group:*
Since

$$\frac{(U/N)\mathbf{\Gamma}_p(G/N)}{\mathbf{\Gamma}_p(G/N)}=\frac{(U/N)(\mathbf{\Gamma}_p(G)/N)}{\mathbf{\Gamma}_p(G)/N}\cong U\mathbf{\Gamma}_p(G)/\mathbf{\Gamma}_p(G)\in\mathcal{F},$$

by the minimality of $|G|$, we have $U/N\in\mathcal{F}$. By the p-solubility of U, certainly N is a p-group or a p'-group. If N is a p'-group, then $U/\mathbf{O}_{p'}(U)\in\mathcal{F}$ and by the condition (*) $U\in\mathcal{F}$, a contradiction.

(iii) *Let $H\in Syl_{\mathcal{F}}(U)$. Then H is not normal in G and $|N|=p$:*
Certainly $\mathcal{F}$ contains all groups of prime orders. So $\mathbf{N}_U(H)=H$ by Lemma 1.4. If $H\trianglelefteqq G$, then $U=G\cap U=\mathbf{N}_G(H)\cap U=\mathbf{N}_U(H)=H\in\mathcal{F}$, a contradiction. Since $U/N\in\mathcal{F}$, clearly $HN=U$ and by the Frattini argument $G=\mathbf{N}_G(H)U=\mathbf{N}_G(H)N$. Since $\mathbf{N}_G(H)\neq G$, we may choose V a maximal subgroup of G such that $\mathbf{N}_G(H)\leqq V$. Since $N\nleqq V$, the index $|G\colon V|=|N\colon N\cap V|$ is a p-power. Since $\mathbf{\Gamma}_p(G)\nleqq V$, we have $|N|=|G\colon V|=p$.

(iv) *The contradiction*
Since N is a normal p-soluble subgroup of G and $NV=G$, we have that G/V_G is metacyclic by Corollary 1.3. Also $U/U\cap V_G\cong UV_G/V_G$ is metacyclic and therefore an $\mathcal{F}$-group. So $N=U^{\mathcal{F}}\leqq V_G\cap U\leqq V_G\leqq V$, contradicting the choice of V. □

Our Proposition 1.8 contains:

Theorem 1.9. *For any prime number p and any group G we have that $\mathbf{A}_p(\mathbf{\Gamma}_p(G))=\mathbf{\Gamma}_p(G)\cap\mathbf{A}_p(G)$ is p-supersoluble.*

Proof. In our proposition we choose $U=\mathbf{A}_p(\mathbf{\Gamma}_p(G))$. By Lemma 1.5 we take for $\mathcal{F}$ the formation of the p-supersoluble groups. Since the unit group $\mathbf{\Gamma}_p(G)/\mathbf{\Gamma}_p(G)$ surely is p-supersoluble, the p-supersolubility of $\mathbf{A}_p(\mathbf{\Gamma}_p(G))$ follows. □

https://doi.org/10.1017/S0013091500018393 Published online by Cambridge University Press

Remark. The p-supersolubility of $\mathbf{A}_p(\mathbf{\Gamma}_p(G))$ can also be deduced from [2]: $\mathbf{A}_p(\mathbf{\Gamma}_p(G))$ is obtained as $\mathbf{\Phi}_f(G)$ in [2], choosing there the formation function f to be:

$$f(q)=\begin{cases}\text{the class of the abelian groups whose exponent divides } p-1 \text{ if } q=p\\ \text{the class of all finite groups if } q\neq p.\end{cases}$$

Since this f defines locally the formation of all p-supersoluble groups, the result of [2] shows the p-supersolubility of $\mathbf{A}_p(\mathbf{\Gamma}_p(G))$.

Corollary 1.10. *Let G be a group and p a prime number. Suppose that for all maximal subgroups V of G we have*

$$p=|G\colon V| \quad \text{or} \quad p\,\big|\,|G\colon V|.$$

Then $\mathbf{A}_p(G)$ is p-supersoluble.

Proof. By hypothesis, $\mathbf{\Gamma}_p(G)=G$. □

Corollary 1.11 (see [1] and [2]). *For every group G, the Huppert subgroup $\mathbf{\Gamma}(G)$ is supersoluble.*

Proof. By definition $\mathbf{\Gamma}(G)=\bigcap_p \mathbf{\Gamma}_p(G)$. Let $M=\bigcap_p \mathbf{A}_p(\mathbf{\Gamma}_p(G))$. By Theorem 1.9, M is supersoluble and $M\leqq\mathbf{\Gamma}(G)$. If we know that $\mathbf{\Gamma}(G)$ is soluble, then $\mathbf{\Gamma}(G)\leqq\bigcap_p \mathbf{A}_p(G)$ and $\mathbf{\Gamma}(G)=M$.

So we have to prove that $\mathbf{\Gamma}(G)$ is soluble. We proceed by induction on $|G|$: Clearly we may assume $\mathbf{\Gamma}(G)\neq 1$. Let p be the largest prime divisor of $|\mathbf{\Gamma}(G)|$ and consider a Sylow-p-subgroup P of $\mathbf{\Gamma}(G)$. The (ordinary) Frattini argument yields that $G=\mathbf{N}_G(P)\mathbf{\Gamma}(G)$. If P is not normal in G, choose a maximal subgroup V of G, such that $\mathbf{N}_G(P)\leqq V$. Since $\mathbf{\Gamma}(G)\nleqq V$, $|G\colon V|=q$ for some prime number q.

We put $U=\mathbf{\Gamma}(G)\cap V$ and obtain

$$q=|G\colon V|=|\mathbf{\Gamma}(G)V\colon V|=|\mathbf{\Gamma}(G)\colon U|.$$

Applying Sylow's Theorem for P in $\mathbf{\Gamma}(G)$ and in U, we obtain

$$1\equiv|\mathbf{\Gamma}(G)\colon \mathbf{N}_{\Gamma(G)}(P)|=|\mathbf{\Gamma}(G)\colon U|\,|U\colon \mathbf{N}_U(P)|\equiv q\cdot 1 \bmod p,$$

whence $q\equiv 1 \bmod p$. Since q divides $|\mathbf{\Gamma}(G)|$, q is not bigger than p, a contradiction. So $P\trianglelefteq G$. By induction. $\mathbf{\Gamma}(G)/P=\mathbf{\Gamma}(G/P)$ is soluble. Therefore also $\mathbf{\Gamma}(G)$ is soluble. □

2. The p-quasi-superfluous elements

The purpose of this second section is to describe the elements of $\mathbf{F}(\mathbf{\Gamma}(G))$ and (for odd p) those of $\mathbf{F}_p(\mathbf{\Gamma}_p(G))$ by means of permutability properties.

Definition. Let G be a group and $x \in G$.

(a) If p is a prime number, we call x a *p-quasi-superfluous element* of G, if $\langle x \rangle V = V \langle x \rangle$ holds for the maximal subgroup V of G, whenever $p \mid |G\colon V|$.

(b) We call x a *quasi-superfluous element* of G, if x is p-quasi-superfluous for every p.

Let $\mathbf{Qs}_p(G)$ denote the set of p-quasi-superfluous elements, $\mathbf{Qs}(G) = \bigcap_p \mathbf{Qs}_p(G)$ the set of the quasi-superfluous elements of G.

Certainly, $\mathbf{Qs}_p(G)$ and $\mathbf{Qs}(G)$ are characteristic subsets of G. $\mathbf{Qs}(G)$ is exactly the set of elements x of G such that $\langle x \rangle$ is M-embedded (M-eingebettet) in G in the sense of [6].

Let $\Delta(G)$ denote the intersection of the non-normal maximal subgroups of G (see [3]). Clearly all elements of $\Delta(G)$, in particular the elements of the Frattini subgroup, as well as the elements of the hypercentre of G, are quasi-superfluous.

In the simple group $G = \mathbf{PSL}(2,7)$ we have $\mathbf{Qs}_3(G) = G$ whereas $\mathbf{Qs}_7(G) = \{x \in G \mid x^7 = 1\}$ is not a subgroup of G.

In the symmetric group $\mathbf{S}_4$ of degree four we get $\mathbf{Qs}_3(\mathbf{S}_4) = \mathbf{A}_4$, the alternating group, whereas $\mathbf{Qs}_2(\mathbf{S}_4)$ is not a subgroup of $\mathbf{S}_4$, because the four cycle $(iklm) \in \mathbf{Qs}_2(\mathbf{S}_4)$, but $(il)(km) = (iklm)^2 \notin \mathbf{Qs}_2(\mathbf{S}_4)$.

Our aim is to prove:

Theorem 2.1. *Let G be a group and p a prime number.*

(a) $\mathbf{F}_p(\mathbf{\Gamma}_p(G)) \subseteq \mathbf{Qs}_p(G) \cap \mathbf{A}_p(G)$.

(b) *If p is odd or if G has no factor group isomorphic to $\mathbf{S}_4$, then $\mathbf{Qs}_p(G) \cap \mathbf{A}_p(G) = \mathbf{F}_p(\mathbf{\Gamma}_p(G))$. In particular, $\mathbf{Qs}_p(G) \cap \mathbf{A}_p(G)$ is a p-nilpotent characteristic subgroup of G.*

(c) $\mathbf{Qs}(G) = \mathbf{F}(\mathbf{\Gamma}(G))$. *In particular, $\mathbf{Qs}(G)$ is always a nilpotent characteristic subgroup of G.*

Remark. Since $\mathbf{Qs}_2(S_4)$ is not a subgroup of $\mathbf{S}_4$, the hypothesis in (b) can not be omitted.

We mention the following interesting and immediate consequence, which is not at all evident from the definition of the sets $\mathbf{Qs}(G)$ and $\mathbf{Qs}_p(G)$.

Corollary 2.2. (a) *Let $x, y \in G$ be elements such that $\langle x \rangle$ and $\langle y \rangle$ are permutable with every maximal subgroup of G. Then for all $z \in \langle x, y \rangle$, the group $\langle z \rangle$ is also permutable with every maximal subgroup of G.*

(b) *Let $p > 2$ be a prime number and G a p-soluble group. If $x, y \in G$ are elemenets such that $\langle x \rangle$ and $\langle y \rangle$ are permutable with every maximal subgroup of G of index divisible by p, then, for all $z \in \langle x, y \rangle$, also $\langle z \rangle$ is permutable with the same maximal subgroups.*

As a consequence of 2.1 and 1.1 we have:

Corollary 2.3. (a) *Let G be a p-soluble group. Suppose $p>2$ or G has no factor group isomorphic to S_4. Then G is p-supersoluble if and only if $\mathbf{F}_p(G)=\mathbf{Qs}_p(G)$.*

(b) *The soluble group G is supersoluble if and only if $\mathbf{F}(G)=\mathbf{Qs}(G)$.*

Proof. (a) Since $\mathbf{A}_p(G)=G$, we conclude $\mathbf{Qs}_p(G)=\mathbf{F}_p(\mathbf{\Gamma}_p(G))$ by 2.1(b). Now $\mathbf{F}_p(G)=\mathbf{F}_p(\mathbf{\Gamma}_p(G))$ if and only if $\mathbf{F}_p(G)\leqq\mathbf{\Gamma}_p(G)$. We apply 1.1(a).

(b) The proof is similar to (a).

We prepare the proof of Theorem 2.1.

Lemma 2.4. *Let p be a prime number, G a group and V a maximal subgroup of G. Suppose there exists a normal p-soluble subgroup N of G and a p-element $x\in G$, such that $G=VN=V\langle x\rangle$. Then $|G:V|=p$ or $p=2$ and $|G:V|=4$.*

Proof. This is a generalization of a classical result due to Ritt [**8**]. See [**7**, Th. 2.5].

Lemma 2.5. *Let G be a group, V a maximal subgroup of G and let X be a subgroup of G such that $X\leqq V$. If $V^gX=XV^g$ for every $g\in G$, then $X^G\leqq\bigcap_{g\in G}V^g$.*

Proof. See [**6**, Th. 2.6].

Let G be a group and π a set of prime numbers. We recall that G is said to satisfy the Sylow-π-theorem, if there exists a Hall-π-subgroup H of G and if every π-subgroup of G is conjugate to a subgroup of H (see [**4**, p. 284]).

Lemma 2.6. *Let G be a group and π a set of prime numbers, π' its complementary set in the set of all prime numbers. If $x\in\bigcap_{q\in\pi}\mathbf{Qs}_q(G)$ and if the normal closure $Y=(\mathbf{O}_{\pi'}(\langle x\rangle))^G$ is a group which satisfies the Sylow-π'-theorem, then $\mathbf{O}_{\pi'}(\langle x\rangle)\leqq\mathbf{O}_{\pi'}(G)$.*

Proof. Let H be a Hall-π'-subgroup of Y, such that $\mathbf{O}_{\pi'}(\langle x\rangle)\leqq H$. By the Frattini argument $G=\mathbf{N}_G(H)Y$. If $H\trianglelefteq G$, then $\mathbf{O}_{\pi'}(\langle x\rangle)\leqq H\leqq\mathbf{O}_{\pi'}(G)$. Suppose $\mathbf{N}_G(H)\neq G$ and let V be a maximal subgroup of G such that $\mathbf{N}_G(H)\leqq V$. Then $Y\nleqq V$. Clearly the index $|G:V|$ is divisible (only) by primes in π. Therefore $\langle x\rangle$ is permutable with all conjugates of V. If $\langle x\rangle\leqq V$, then $Y\leqq\langle x\rangle^G\leqq\bigcap_{g\in G}V^g\leqq V$ by Lemma 2.5, a contradiction. If $\langle x\rangle\nleqq V$, we get that $G=\langle x\rangle V=V\langle x\rangle$, whence $Y=(\mathbf{O}_{\pi'}(\langle x\rangle))^G=(\mathbf{O}_{\pi'}(\langle x\rangle))^{\langle x\rangle V}=(\mathbf{O}_{\pi'}(\langle x\rangle))^V\leqq H^V\leqq V$, the same contradiction. □

Corollary 2.7. *Let G be a group.*

(a) *We have $\mathbf{Qs}(G)\subseteq\mathbf{F}(G)$.*

(b) *If $x\in\mathbf{Qs}_p(G)\cap\mathbf{A}_p(G)$ for some prime number p, then $\mathbf{O}_{p'}(\langle x\rangle)\leqq\mathbf{O}_{p'}(G)$.*

Proof. (a) Let p be a prime number and π the set of all primes $\neq p$. If $x\in\mathbf{Qs}(G)$, then also $x\in\bigcap_{q\in\pi}\mathbf{Qs}_q(G)$. Since $\pi'=\{p\}$, the group $Y=(\mathbf{O}_p(\langle x\rangle))^G$ certainly satisfies the

Sylow-π'-theorem. Therefore $\mathbf{O}_p(\langle x\rangle)\leqq\mathbf{O}_p(G)\leqq\mathbf{F}(G)$. Since this holds for all p, we conclude $x\in\langle x\rangle\leqq\mathbf{F}(G)$.

(b) Let $x\in\mathbf{Qs}_p(G)\cap\mathbf{A}_p(G)$. The group $Y=(\mathbf{O}_{p'}(\langle x\rangle))^G\leqq\mathbf{A}_p(G)$ certainly satisfies the Sylow-p'-theorem. Therefore $\mathbf{O}_{p'}(\langle x\rangle)\leqq\mathbf{O}_{p'}(G)$. □

Proof of Theorem 2.1. (a) Let $x\in\mathbf{F}_p(\mathbf{\Gamma}_p(G))$ be an arbitrary element. Certainly $x\in\mathbf{A}_p(G)$. To show that $x\in\mathbf{Qs}_p(G)$, let V be a maximal subgroup of G, such that $p\,|\,|G\colon V|$. We prove $\langle x\rangle V=V\langle x\rangle$ by induction on $|G|$. Clearly we may assume $|G|>1$ and $x\notin V$.

Case I: $\mathbf{O}_{p'}(G)>1$.

Let $L=\mathbf{O}_{p'}(G)$. We have $L\leqq\mathbf{\Gamma}_p(G)$ and $L\leqq\mathbf{F}_p(G)$. Since $\mathbf{F}_p(G/L)=\mathbf{F}_p(G)/L$ and $\mathbf{\Gamma}_p(G/L)=\mathbf{\Gamma}_p(G)/L$ we see that $xL\in\mathbf{F}_p(\mathbf{\Gamma}_p(G))/L=\mathbf{F}_p(\mathbf{\Gamma}_p(G/L))$. Since $L\leqq V$, we conclude by the inductive hypothesis $\langle xL\rangle(V/L)=(V/L)\langle xL\rangle$ and therefore $\langle x\rangle V=V\langle x\rangle$.

Case II: $\mathbf{O}_{p'}(G)=1$.

We have that $R=\mathbf{F}_p(\mathbf{\Gamma}_p(G))$ is now a p-group, $RV=G$ and since $\mathbf{\Gamma}_p(G)\nleqq V$ we get that $|R\colon V\cap R|=|G\colon V|=p$ by definition of $\mathbf{\Gamma}_p(G)$. So $V\cap R\trianglelefteq R$. We conclude $R=\langle x\rangle(V\cap R)=(V\cap R)\langle x\rangle$, whence $V\langle x\rangle=V(V\cap R)\langle x\rangle=VR=G=\langle x\rangle V$.

(b) By (a) we only have to show that $\mathbf{Qs}_p(G)\cap\mathbf{A}_p(G)\subseteq\mathbf{F}_p(\mathbf{\Gamma}_p(G))$. Suppose that this is false, and let G be a minimal counterexample. Let x be an element of $\mathbf{Qs}_p(G)\cap\mathbf{A}_p(G)$ such that $x\notin\mathbf{F}_p(\mathbf{\Gamma}_p(G))$.

We will prove a series of items under this assumption, which will lead to a contradiction. Certainly the hypothesis in (b) is inherited by factor groups.

(i) $\mathbf{O}_{p'}(G)=1$:

Suppose $L=\mathbf{O}_{p'}(G)>1$. Then $xL\in\mathbf{Qs}_p(G/L)\cap\mathbf{A}_p(G/L)$. Since $|G|$ is minimal, we get that $xL\in\mathbf{F}_p(\mathbf{\Gamma}_p(G/L))=\mathbf{F}_p(\mathbf{\Gamma}_p(G))/L$. So $x\in\mathbf{F}_p(\mathbf{\Gamma}_p(G))$, a contradiction.

(ii) *x is a p-element:*

$\mathbf{O}_{p'}(\langle x\rangle)\leqq\mathbf{O}_{p'}(G)=1$ by Corollary 2.7.

(iii) $x\notin\mathbf{\Gamma}_p(G)$:

If $x\in\mathbf{\Gamma}_p(G)$, then $x\in\mathbf{A}_p(\mathbf{\Gamma}_p(G))$. Since $\mathbf{A}_p(\mathbf{\Gamma}_p(G))$ is p-supersoluble by Theorem 1.9 and $\mathbf{O}_{p'}(G)=1$, we have that $\mathbf{A}_p(\mathbf{\Gamma}_p(G))$ has a normal Sylow-p-subgroup (see [4, p. 691, Th. 6.6]). We conclude that $x\in\mathbf{F}_p(\mathbf{\Gamma}_p(G))\in Syl_p(\mathbf{A}_p(\mathbf{\Gamma}_p(G)))$, contradicting the choice of x.

(iv) *The contradiction*

Let $N=\langle x\rangle^G$. Since $x\notin\mathbf{\Gamma}_p(G)$, there exists a maximal subgroup V of G such that $p\,|\,|G\colon V|\neq p$ and $N\nleqq V$. Then $G=\langle x\rangle V=NV$ and by Lemma 2.4 we have that $|G\colon V|=4$. So $G/V_G\cong\mathbf{S}_4$, a contradiction. □

(c) By (a) we have

$$\mathbf{F}(\mathbf{\Gamma}(G)) = \mathbf{F}(G) \cap \mathbf{\Gamma}(G) = \bigcap_p \mathbf{F}_p(G) \cap \bigcap_p \mathbf{\Gamma}_p(G)$$

$$= \bigcap_p \mathbf{F}_p(\mathbf{\Gamma}_p(G)) \subseteq \bigcap_p \mathbf{Qs}_p(G) \cap \bigcap_p \mathbf{A}_p(G) \subseteq \mathbf{Qs}(G).$$

Let $x \in \mathbf{Qs}(G)$ be an arbitrary element. By Corollary 2.7(a) we have $x \in \mathbf{F}(G)$. If V is a maximal subgroup of G such that $x \notin V$, then $\mathbf{F}(G) \nleq V$ and $G = V\mathbf{F}(G)$. Moreover $D = \mathbf{F}(G) \cap V \trianglelefteq G$ because of the maximality of V and the nilpotency of $\mathbf{F}(G)$. So $\mathbf{F}(G)/D$ is a chief factor of G of prime exponent. Also $\langle x \rangle V = V\langle x \rangle = G$, whence $\mathbf{F}(G) = G \cap \mathbf{F}(G) = V\langle x \rangle \cap \mathbf{F}(G) = (V \cap \mathbf{F}(G))\langle x \rangle = D\langle x \rangle$. So $\mathbf{F}(G)/D \cong \langle x \rangle / D \cap \langle x \rangle$ is also cyclic. We conclude that $|\mathbf{F}(G)/D| = |G:V|$ is a prime number. Therefore $x \in \mathbf{\Gamma}(G)$ by the definition of $\mathbf{\Gamma}(G)$ and finally $x \in \mathbf{\Gamma}(G) \cap \mathbf{F}(G) = \mathbf{F}(\mathbf{\Gamma}(G))$. □

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