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ADDITIVE FUNCTIONALS FOR DISCRETE-TIME MARKOV CHAINS WITH APPLICATIONS TO BIRTH–DEATH PROCESSES

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Abstract

In this paper we are interested in bounding or calculating the additive functionals of the first return time on a set for discrete-time Markov chains on a countable state space, which is motivated by investigating ergodic theory and central limit theorems. To do so, we introduce the theory of the minimal nonnegative solution. This theory combined with some other techniques is proved useful for investigating the additive functionals. This method is used to study the functionals for discrete-time birth–death processes, and the polynomial convergence and a central limit theorem are derived.

Keywords: Additive functional; birth–death process; ergodicity; central limit theorem

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1. Introduction

Let $\mathbb{Z}_+$ and $\mathbb{N}$ be the sets of all the nonnegative and positive integers, respectively. Let Φ_n be a discrete-time Markov chain (DTMC) on a countable state space $\mathbb{E} = \mathbb{Z}_+$ with the transition matrix $P = (p_{ij})$. Throughout this paper, we assume that the chain Φ_n is irreducible, aperiodic, and positive recurrent with a unique invariant distribution π. For a set A, the random variables

$$\tau_A := \inf\{n \in \mathbb{N}\colon \Phi_n \in A\}, \qquad \sigma_A := \inf\{n \in \mathbb{Z}_+\colon \Phi_n \in A\}$$

are said to be the first return and first hitting times, respectively. For any integer $p \in \mathbb{N}$ and any sequence $r(n)$ taking values in $\mathbb{N}$, we are interested in bounding or calculating functionals of the type $\mathrm{E}_i[(\sum_{k=0}^{\tau_A-1} r(n) f(\Phi_k))^p]$, which is very important in the context of ergodic theory and central limit theorems.

Let Λ be the class of subgeometric rate functions from [16], which includes, for example, the polynomial functions $r(n) = (n+1)^\ell$. It is given by Theorem 2.1 of [18] that if there exists a finite set A such that the functional $\mathrm{E}_i[\sum_{k=0}^{\tau_A-1} r(n) f(\Phi_k)]$ is finite for any $i \in A$, then we have the subgeometric convergence

$$r(n)\|P^n(i,\cdot) - \pi(\cdot)\|_f \to 0 \quad \text{as } n \to \infty$$

for any $i \in \mathbb{E}$, where $r \in \Lambda$, $f \geq 1$, and $\|\mu\|_f := \sup_{|g|\leq f} \mu(g)$ is the f-norm for a signed measure μ. An explicit expression of this functional is available only for some simple examples. It was shown in the same paper [18] that the functional is finite if and only if there exists a sequence of drift functions satisfying some conditions. Later, it was shown in [4], [5], and [9] that, for a large enough subclass (including polynomial case) of subgeometric rates, we can use

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a single drift function to replace a sequence of drift functions. Although these results have been applied successfully to many models, they seem to be difficult to apply to some models, such as discrete-time birth–death processes (or random walks) with general birth and death rates.

Another topic, which closely relates with the type of additive functionals, is the theory of central limit theorems (CLTs). Let f be a real-valued function $f\colon \mathbb{E} \to \mathbb{R}$. Define $\pi(f) = \sum_{i\in\mathbb{E}} \pi_i f_i$ and $\bar{f} = f - \pi(f)$. The sample mean is defined by

$$S_n(f) = \frac{1}{n}\sum_{k=0}^{n} f(\Phi_k).$$

When $\pi(|f|) < \infty$, the ergodic theorem guarantees that $S_n(f) \to \pi(f)$ with probability 1 as $n \to \infty$. We say that a CLT holds if there exists a constant $0 \le \sigma_f < +\infty$ such that

$$n^{1/2} S_n(\bar{f}) \Rightarrow \sigma_f N(0, 1) \quad \text{as } n \to \infty, \tag{1.1}$$

where $N(0, 1)$ denotes a standard normal random variable, and '$\Rightarrow$' denotes convergence in distribution. In the context of CLTs two important questions are:

(i) What conditions ensure a CLT?

(ii) When a CLT holds, how is the asymptotic variance σ_f^2 calculated?

To answer the first question, we need to bound the additive functional $\mathrm{E}_i[(\sum_{k=0}^{\tau_i} |\bar{f}(\Phi_k)|)^2]$ for some $i \in \mathbb{E}$; see, e.g. [15, Chapter 17]. Many useful results have been obtained for the bound of this functional; see [10] and [17]. To answer the second question, we need to obtain an explicit expression for the functional $\mathrm{E}_i[\sum_{k=0}^{\tau_j} \bar{f}(\Phi_k)]$ for some j and any $i \in \mathbb{E}$ which is connected with the solution of a Poisson equation; see, e.g. [1], [6], and [21].

In this paper we aim to bound or calculate the additive functionals by using the theory of the minimal nonnegative solution and some other techniques. This theory, which is presented in Theorem 2.1 and is essentially from Chapter 6 of [8], makes it feasible to deal with the functionals for some specific models. Some remarks and one corollary are given in Section 2 to show the power of this theory.

Discrete-time birth–death (DTBD) processes are a class of important Markov chains, which has caused much research interest. For the ergodicity of DTBD processes, Mao [13] and Van Doorn and Schrijner [20] obtained criteria for the geometric ergodicity and uniform ergodicity in terms of the weaker total variation norm. For CLTs, explicit expressions for the asymptotic variance have been obtained for continuous-time birth–death processes on a finite state space; see, e.g. [7] and [21]. In Section 3, for a DTBD process, we derive a CLT and the polynomial convergence in terms of the stronger f-norm, by investigating their additive functionals through Theorem 2.1.

2. Additive functionals for DTMC processes

The following proposition about moments of additive functionals, which is copied from Theorem 6.7.4 of [8], is crucial for the subsequent analysis.

Proposition 2.1. ([8, Theorem 6.7.4].) *Let V be a nonnegative function on $\mathbb{E}$, and let $A \subset \mathbb{E}$ be a nonempty set. Define $\xi_A = \sum_{k=0}^{\tau_A - 1} V(\Phi_k)$ and $T_{iA}^{(p)} = \mathrm{E}_i[\xi_A^p]$, $p \ge 1$. The (p) superscript will be deleted when $p = 1$. Then $\{T_{iA}^{(p)}, i \in \mathbb{E}\}$ is the minimal nonnegative solution of the*

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equation

$$x_i = \sum_{k \notin A} p_{ik} x_k + \sum_{l=1}^{p} \binom{p}{\ell} (-1)^{l-1} V(i)^l T_{iA}^{(p-l)}, \qquad i \in \mathbb{E}, \tag{2.1}$$

where $\binom{p}{\ell} = p!/\ell!\,(p-\ell)!$.

Using Proposition 2.1, the localization theorem (see Theorem A.1 in Appendix A), and the comparison theorem (see Theorem A.2 in Appendix A), we obtain the following result.

Theorem 2.1. *Let $V \geq 0$, and let $A \subset \mathbb{E}$ be a nonempty set.*

(i) *For any $i \in A$,*

$$\sum_{k \notin A} p_{ik} T_{kA}^{(p)} = T_{iA}^{(p)} - \sum_{l=1}^{p} \binom{p}{\ell} (-1)^{l-1} V(i)^l T_{iA}^{(p-l)}.$$

(ii) *The sequence $\{x_i^*,\ i \in \mathbb{E}\}$ given by ${}^*_i = T_{iA}^{(p)}$, $i \notin A$, $x_i^* = 0, i \in A$, is the minimal nonnegative solution of the equations*

$$x_i \geq \sum_{k \in \mathbb{E}} p_{ik} x_k + \sum_{l=1}^{p} \binom{p}{\ell} (-1)^{l-1} V(i)^l T_{iA}^{(p-l)}, \quad i \notin A, \qquad x_i = 0, \quad i \in A,$$

and the sequence $\{x_i^*,\ i \in \mathbb{E}\}$ satisfies the system with equality.

Proof. Part (i) follows from Proposition 2.1 directly.

Now we prove (ii). From Proposition 2.1 we know that $\{T_{iA}^{(p)},\ i \in \mathbb{E}\}$ is the minimal nonnegative solution of (2.1). For any $i \in \mathbb{E}$, define $\tilde{P} = (\tilde{p}_{ij})$ by

$$\tilde{p}_{ij} = \begin{cases} p_{ij}, & j \notin A, \\ 0, & j \in A. \end{cases}$$

Then $\{T_{iA}^{(p)},\ i \in \mathbb{E}\}$ is the minimal nonnegative solution of the equations

$$x_i = \sum_{k \in \mathbb{E}} \tilde{p}_{ik} x_k + \sum_{l=1}^{p} \binom{p}{\ell} (-1)^{l-1} V(i)^l T_{iA}^{(p-l)}, \qquad i \in \mathbb{E}.$$

Let $G = \mathbb{E} \setminus A$ in Theorem A.1 (the localization theorem) in Appendix A. The sequence $\{T_{iA}^{(p)},\ i \in G\}$ is the minimal nonnegative solution of the equations

$$x_i = \sum_{k \in G} p_{ik} x_k + \sum_{l=1}^{p} \binom{p}{\ell} (-1)^{l-1} V(i)^l T_{iA}^{(p-l)}, \qquad i \in G.$$

By the comparison theorem we know that the sequence $\{T_{iA}^{(p)},\ i \notin A\}$ is the minimal nonnegative solution of the equations

$$x_i \geq \sum_{k \notin A} p_{ik} x_k + \sum_{l=1}^{p} \binom{p}{\ell} (-1)^{l-1} V(i)^l T_{iA}^{(p-l)}, \qquad i \notin A,$$

and satisfies the equation with equality, completing the proof of (ii).

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Remark 2.1. As will be shown later in this section and in the next section, Theorem 2.1 combined with (2.3) below is very useful to bound or calculate the additive functionals of the first hitting time. The key point of Theorem 2.1 is that the additive functional is the minimal nonnegative solution of a systems of equations.

The following definition is based on the definition and remark given on pages 778 and 779, respectively, of [18].

Definition 2.1. Let $f \geq 1$ and $r \in \Lambda$. The Markov chain Φ_n is said to be (f, r)-ergodic if, for some (then for all) finite nonempty A, $\mathrm{E}_i[\sum_{k=0}^{\tau_A-1} r(k) f(\Phi_k)] < \infty$ for all $i \in A$, or, equivalently, for some (then for all) $k \in \mathbb{E}$, $\mathrm{E}_k[\sum_{k=0}^{\tau_k-1} r(k) f(\Phi_k)] < \infty$. In particular, it is called (f, ℓ)-ergodic if $r(k) = (k+1)^\ell$, $\ell \in \mathbb{N}$, and f-ergodic if $r(k) \equiv 1$.

Corollary 2.1. *Let $r \in \Lambda$. Define $r(-1) = 0$ and $\Delta(r(k)) = r(k) - r(k-1)$, $k \in \mathbb{Z}_+$. The following statements are equivalent.*

(i) *For some finite nonempty set A, $\mathrm{E}_i[\sum_{k=0}^{\tau_A-1} r(k) f(\Phi_k)] < \infty$ for all $i \in A$.*

(ii) *For some finite nonempty set A, there exist a finite constant b and finite nonnegative functions x and h such that, for all $i \in \mathbb{E}$, $h(i) \geq \mathrm{E}_i[\sum_{k=0}^{\tau_A-1} \Delta(r(k)) f(\Phi_k)]$ and*

$$\sum_{k \in \mathbb{E}} p_{ik} x_k \leq x_i - h(i) + b\mathbf{1}_A(i). \tag{2.2}$$

Proof. Let $V(i) = \mathrm{E}_i[\sum_{k=0}^{\tau_A-1} \Delta(r(k)) f(\Phi_k)]$ for any $i \in \mathbb{E}$. It follows from the proof of Theorem 3.5 of [18] that

$$\mathrm{E}_i\Bigg[\sum_{k=0}^{\tau_A-1} V(\Phi_k)\Bigg] = \mathrm{E}_i\Bigg[\sum_{k=0}^{\tau_A-1} r(k) f(\Phi_k)\Bigg], \qquad i \in \mathbb{E}. \tag{2.3}$$

First, we prove that (i) implies (ii). Suppose that $\mathrm{E}_i[\sum_{k=0}^{\tau_A-1} r(k) f(\Phi_k)] < \infty$ for all $i \in A$. Then it follows from (i) of Propositions 3.1 and 3.2 of [18] that, for any finite set B, $\mathrm{E}_i[\sum_{k=0}^{\tau_B-1} r(k) f(\Phi_k)] < \infty$ for all $i \in \mathbb{E}$. Define $\{x_i,\ i \in \mathbb{E}\}$ by $x_i = \mathrm{E}_i[\sum_{k=0}^{\tau_A-1} r(k) f(\Phi_k)]$, $i \notin A$, and $x_i = 0$, $i \in A$. From (2.3) and Theorem 2.1(ii), $\{x_i,\ i \in \mathbb{E}\}$ is a finite nonnegative solution to (2.2) with $b = \max_{i \in A}\{\mathrm{E}_i[\sum_{k=0}^{\tau_A-1} r(k) f(\Phi_k) + \mathrm{E}_i[\sum_{k=0}^{\tau_A-1} \Delta(r(k)) f(\Phi_k)]\} < \infty$.

Now we prove that (ii) implies (i). Suppose that $\{x_i,\ i \in \mathbb{E}\}$ is a finite nonnegative solution to (2.2). For $i \notin A$, we have $\sum_{k \notin A} p_{ik} x_k \leq x_i - h(i)$. From the comparison theorem, Theorem 2.1(ii), and (2.3), we see that $\mathrm{E}_i[\sum_{k=0}^{\tau_A-1} r(k) f(\Phi_k)] \leq x_i$ for all $i \notin A$. Hence, for all $i \notin A$, we have

$$\sum_{k \notin A} p_{ik}\, \mathrm{E}_k\Bigg[\sum_{k=0}^{\tau_A-1} r(k) f(\Phi_k)\Bigg] \leq \sum_{k \notin A} p_{ik} x_k \leq x_i + b < \infty,$$

from which, together with Theorem 2.1(i), we obtain

$$\mathrm{E}_i\Bigg[\sum_{k=0}^{\tau_A-1} r(k) f(\Phi_k)\Bigg] = \sum_{k \notin A} p_{ik}\, \mathrm{E}_k\Bigg[\sum_{k=0}^{\tau_A-1} r(k) f(\Phi_k)\Bigg] + \mathrm{E}_i\Bigg[\sum_{k=0}^{\tau_A-1} \Delta(r(k)) f(\Phi_k)\Bigg] < \infty.$$

Remark 2.2. From Corollary 2.1 we can obtain some known results. (i) When $f \equiv 1$, we easily deduce (iii) of Theorem 3 of [19]. (ii) When $\{A\} = \{0\}$ and $r(k) = k^\ell$, $\ell \in \mathbb{N}$, we immediately obtain Theorem 1.3 of [14].

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Remark 2.3. When $r(k) = (k+1)^\ell$, $\ell \in \mathbb{N}$, the drift condition (2.2) can be changed to finitely many drift functions. Using Theorem 2.1 and similar arguments to those in the proof of Corollary 2.1 of [12], we can show that the following statements are equivalent.

(i) For some finite nonempty set A, $\mathrm{E}_i[\sum_{k=0}^{\tau_A-1}(k+1)^\ell f(\Phi_k)] < \infty$ for all $i \in A$.

(ii) For some nonempty finite set A, there exist $\ell+1$ finite nonnegative functions $V^{(n)}(i)$, $0 \le n \le \ell+1$, and a constant b such that $V^{(0)}(i) \ge f(i)$, and, for all m, $0 \le m \le \ell$,

$$\sum_{k\in\mathbb{E}} p_{ik} V^{(m+1)}(k) \le V^{(m+1)}(i) - V^{(m)}(i) + b\,\mathbf{1}_A(i).$$

This result was first shown in Corollary 1 of [5] using different arguments.

3. Additive functionals for DTBD processes

Let Φ_n be a DTBD process with transition matrix $P = (p_{ij})$, where

$$p_{ij} = \begin{cases} b_i, & j = i+1,\ i \ge 0, \\ a_i, & j = i-1,\ i \ge 1, \\ 1-b_0, & j = i,\ i = 0, \\ 1-(a_i+b_i), & j = i,\ i \ge 1, \\ 0, & \text{otherwise.} \end{cases}$$

Define $\mu_0 = 1$ and $\mu_i = b_0 b_1 \cdots b_{i-1}/a_1 a_2 \cdots a_i$, $i \ge 1$. Suppose that Φ_n is irreducible and aperiodic. It is well known that Φ_n is recurrent if and only if $\sum_{i=0}^\infty 1/\mu_i b_i = \infty$, and that Φ_n is positive recurrent if and only if $\sum_{i=0}^\infty \mu_i < \infty$. If Φ_n is positive recurrent then the invariant probability measure π such that $\pi P = \pi$ can be computed as

$$\pi_i = \frac{\mu_i}{\mu}, \qquad \mu = \sum_{i=0}^\infty \mu_i. \tag{3.1}$$

Lemma 3.1. *Suppose that Φ_n is a recurrent DTBD process. Let $V \ge 0$. Then we have*

$$\mathrm{E}_i\Bigg[\sum_{k=0}^{\tau_j-1} V(\Phi_k)\Bigg] = \begin{cases} \displaystyle\sum_{n=i}^{j-1} \frac{1}{b_n\mu_n} \sum_{k=0}^{n} V(k)\mu_k & \text{if } i < j, \\ \displaystyle\sum_{n=j}^{i-1} \frac{1}{b_n\mu_n} \sum_{k=n+1}^{\infty} V(k)\mu_k & \text{if } i > j, \\ \displaystyle\frac{1}{\mu_i} \sum_{k=0}^{\infty} V(k)\mu_k & \text{if } i = j. \end{cases}$$

Proof. Let $A = \{j\}$ and $p = 1$. Then Theorem 2.1(ii) becomes

$$x_i = \sum_{k\in\mathbb{E}} p_{ik} x_k + V(i), \quad i \ne j, \qquad x_j = 0.$$

First, we consider the case when $i < j$. Substituting the value of p_{ij} and using induction on n gives

$$\begin{aligned} x_n - x_{n+1} &= \frac{a_n}{b_n}(x_{n-1} - x_n) + \frac{V(n)}{b_n} \\ &= \frac{a_n a_{n-1}}{b_n b_{n-1}}(x_{n-2} - x_{n-1}) + \frac{a_n}{b_n b_{n-1}} V(n-1) + \frac{V(n)}{b_n} \\ &= \cdots \\ &= \frac{1}{b_n \mu_n} \sum_{k=0}^{n} V(k)\mu_k \end{aligned}$$

for any n, $0 \le n \le j-1$, where $x_{-1} = 0$. Summing over n from i to $j-1$ yields

$$x_i = \sum_{n=i}^{j-1} \frac{1}{b_n \mu_n} \sum_{k=0}^{n} V(k)\mu_k, \qquad i < j.$$

Now we consider the case when $i > j$. Substituting the value of p_{ij} and using induction on n gives, for $n \ge j+1$,

$$\begin{aligned} x_{n+1} - x_n &= \frac{a_n}{b_n}(x_n - x_{n-1}) - \frac{V(n)}{b_n} \\ &= \frac{a_n a_{n-1}}{b_n b_{n-1}}(x_{n-1} - x_{n-2}) - \frac{a_n}{b_n b_{n-1}} V(n-1) - \frac{V(n)}{b_n} \\ &= \cdots \\ &= \frac{1}{b_n \mu_n}\left(\mu_j b_j x_{j+1} - \sum_{k=j+1}^{n} V(k)\mu_k\right). \end{aligned}$$

Summing over n from $j+1$ to $i-1$ yields

$$\begin{aligned} x_i &= x_{j+1} + \sum_{n=j+1}^{i-1} \frac{1}{b_n \mu_n}\left(\mu_j b_j x_{j+1} - \sum_{k=j+1}^{n} V(k)\mu_k\right) \\ &= \sum_{n=j}^{i-1} \frac{1}{b_n \mu_n}\left(\mu_j b_j x_{j+1} - \sum_{k=j+1}^{n} V(k)\mu_k\right). \end{aligned}$$

Since $\sum_{i=1}^{\infty} 1/b_i\mu_i = \infty$, to ensure that x_i is nonnegative, we must have

$$\mu_j b_j x_{j+1} \ge \sum_{k=j+1}^{\infty} V(k)\mu_k.$$

Also, considering that x_i is the minimal solution, we further have $\mu_j b_j x_{j+1} = \sum_{k=j+1}^{\infty} V(k)\mu_k$. Thus, the second assertion is established.

Finally, we consider $i = j$. Applying Theorem 2.1(i), and using the just proved two assertions and the fact that $a_{i+1}\mu_{i+1} = b_i\mu_i$, we have

$$\mathrm{E}_i\left[\sum_{k=0}^{\tau_i - 1} V(\Phi_k)\right] = b_i\,\mathrm{E}_{i+1}\left[\sum_{k=0}^{\tau_i - 1} V(\Phi_k)\right] + a_i\,\mathrm{E}_{i-1}\left[\sum_{k=0}^{\tau_i - 1} V(\Phi_k)\right] + V(i) = \frac{1}{\mu_i}\sum_{k=0}^{\infty} V(k)\mu_k.$$

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Remark 3.1. (i) Since

$$\sum_{j=1}^{\infty} V_j \mu_j = \frac{\pi(V) - \pi_0 V_0}{\mu},$$

we know that, for any $i \geq j$, $E_i[\sum_{k=0}^{\tau_j - 1} V(\Phi_k)] < \infty$ if and only if $\mu(V) < \infty$, or, equivalently, $\pi(V) < \infty$.

(ii) Let $f \equiv 1$. Then we obtain the expression for $E_i[\tau_j]$, which is well known in the literature; see, e.g. [3, Part I, Section 12].

3.1. (f, ℓ)-ergodicity for DTBD processes

Theorem 3.1. *Let $f \geq 1$, and let Φ_n be an f-ergodic DTBD process. Then Φ_n is (f, ℓ)-ergodic if and only if $\sum_{j=1}^{\infty} g_j^{(\ell-1)}(f)\mu_j < \infty$, where*

$$g_n^{(p)}(f) = \sum_{i=0}^{n-1} \frac{1}{b_i \mu_i} \sum_{j=i+1}^{\infty} g_j^{(p-1)}(f)\mu_j, \qquad p, n \in \mathbb{N},$$

and

$$g_n^{(0)}(f) = E_n\Bigg[\sum_{k=0}^{\tau_0 - 1} f(\Phi_k)\Bigg] = \sum_{i=0}^{n-1} \frac{1}{b_i \mu_i} \sum_{j=i+1}^{\infty} f(j)\mu_j.$$

Proof. Define $V^{(p)}(i) = E_i[\sum_{k=0}^{\tau_0 - 1} \Delta((k+1)^p) f(\Phi_k)]$ for any $1 \leq p \leq \ell$. Using (2.3), we obtain

$$E_i\Bigg[\sum_{k=0}^{\tau_0 - 1} V^{(p)}(\Phi_k)\Bigg] = E_i\Bigg[\sum_{k=0}^{\tau_0 - 1} (k+1)^p f(\Phi_k)\Bigg], \qquad i \in \mathbb{E}. \tag{3.2}$$

Now we consider the lower bound and upper bound on $E_i[\sum_{k=0}^{\tau_0 - 1} V^{(p)}(\Phi_k)]$.

On the one hand, using (3.2), Lemma 3.1, and the fact that, for any $p, k \in \mathbb{N}$,

$$\Delta((k+1)^p) = (k+1)^p - k^p = p \int_k^{k+1} x^{p-1}\, dx \leq p(k+1)^{p-1},$$

we have, for any $n \in \mathbb{N}$,

$$\begin{aligned} E_n\Bigg[\sum_{k=0}^{\tau_0 - 1} (k+1)^p f(\Phi_k)\Bigg] &= \sum_{i=0}^{n-1} \frac{1}{b_i \mu_i} \sum_{j=i+1}^{\infty} V^{(p)}(j)\mu_j \\ &\leq p \sum_{i=0}^{n-1} \frac{1}{b_i \mu_i} \sum_{j=i+1}^{\infty} E_j\Bigg[\sum_{k=0}^{\tau_0 - 1} (k+1)^{p-1} f(\Phi_k)\Bigg]\mu_j \\ &\leq p^p g_n^{(p)}(f). \end{aligned} \tag{3.3}$$

On the other hand, we have

$$(k+1)^{p-1} k \geq k^{p-1} k$$

for any $k \in \mathbb{Z}_+$ and $1 \leq p \leq \ell$, which is equivalent to

$$\Delta((k+1)^p) = (k+1)^p - k^p \geq (k+1)^{p-1}. \tag{3.4}$$

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Using (3.2), (3.4), and Lemma 3.1, we have, for any $n \in \mathbb{N}_+$,

$$\mathrm{E}_n\Bigg[\sum_{k=0}^{\tau_0-1}(k+1)^p f(\Phi_k)\Bigg] \geq \sum_{i=0}^{n-1}\frac{1}{b_i\mu_i}\sum_{j=i+1}^{\infty}\mathrm{E}_j\Bigg[\sum_{k=0}^{\tau_0-1}(k+1)^{p-1}f(\Phi_k)\Bigg]\mu_j \geq g_n^{(p)}(f). \quad (3.5)$$

If follows from (3.3) and (3.5) that, for any $n \in \mathbb{N}_+$, we have $\mathrm{E}_n[\sum_{k=0}^{\tau_0-1}(k+1)^p f(\Phi_k)] < \infty$ if and only if $g_n^{(p)}(f) < \infty$. Using Theorem 2.1(i), we obtain

$$\mathrm{E}_0\Bigg[\sum_{k=0}^{\tau_0-1}(k+1)^{\ell} f(\Phi_k)\Bigg] = b_0\,\mathrm{E}_1\Bigg[\sum_{k=0}^{\tau_0-1}(k+1)^{\ell} f(\Phi_k)\Bigg] + V^{(\ell)}(0). \quad (3.6)$$

We are now ready to complete the proof of this theorem. For the sufficiency, if

$$\sum_{j=1}^{\infty} g_j^{(\ell-1)}(f)\mu_j < \infty$$

then $g_1^{(\ell)}(f) < \infty$, and, hence, $\mathrm{E}_1[\sum_{k=0}^{\tau_0-1}(k+1)^{\ell} f(\Phi_k)] < \infty$. Owing to the assumption that the chain Φ_n is f-ergodic (i.e. $\mathrm{E}_0[\sum_{k=0}^{\tau_0-1} f(\Phi_k)] < \infty$), we obtain, by induction,

$$\mathrm{E}_0\Bigg[\sum_{k=0}^{\tau_0-1}(k+1)^{\ell-1} f(\Phi_k)\Bigg] < \infty,$$

or, equivalently, $V^{(\ell)}(0) < \infty$. By (3.6) we have $\mathrm{E}_0[\sum_{k=0}^{\tau_0-1}(k+1)^{\ell} f(\Phi_k)] < \infty$, i.e. the chain Φ_n is (f, ℓ)-ergodic.

For the necessity, we assume that Φ_n is (f, ℓ)-ergodic, i.e. $\mathrm{E}_0[\sum_{k=0}^{\tau_0-1}(k+1)^{\ell} f(\Phi_k)] < \infty$. From (3.5) and (3.6), we have

$$\mathrm{E}_0\Bigg[\sum_{k=0}^{\tau_0-1}(k+1)^{\ell} f(\Phi_k)\Bigg] \geq b_0\,\mathrm{E}_1\Bigg[\sum_{k=0}^{\tau_0-1}(k+1)^{\ell} f(\Phi_k)\Bigg] \geq b_0 g_1^{(\ell)}(f).$$

Hence, $g_1^{(\ell)}(f) < \infty$, or, equivalently, $\sum_{j=1}^{\infty} g_j^{(\ell-1)}(f)\mu_j < \infty$.

Remark 3.2. We expect that Lemma 3.1 and Theorem 3.1 can be extended to a more complex Markov chain with the lower-Hessenberg stochastic matrix (or single-birth chains), i.e. $p_{ij} = 0$ if $j \geq i+1$, using the notation given in [11, p. 220].

Example 3.1. Let Φ_n be a DTBD process with

$$a_i = \frac{1}{2}, \qquad b_i = \frac{1}{2} - \frac{\alpha}{2i}, \qquad r_i = \frac{\alpha}{2i}, \qquad i \gg 1.$$

It is easy to calculate $\mu_n = (1-\alpha)(1-\alpha/2)\cdots(1-\alpha/n) \sim n^{-\alpha}$. For two sequence $\{c_n\}$ and $\{d_n\}$, the notation $c_n \sim d_n$ means that the limit of c_n/d_n as $n \to \infty$ is a positive number.

(i) Let $f(n) = n^{\beta_1}(\log n)^{\beta_2}$, $\beta_1, \beta_2 \geq 0$. Then we have $g_n^{(p)}(f) \sim n^{2(p+1)+\beta_1}(\log n)^{\beta_2}$. Hence, the chain is (f, ℓ)-ergodic if $\alpha > 2\ell + \beta_1 + 1$.

(ii) Let $f(n) = \mathrm{e}^{\beta n}$, $\beta > 0$. Then $g_1^{(0)}(f) = \infty$ and the chain is not (f, ℓ)-ergodic for any $\ell \geq 1$.

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Example 3.2. Let Φ_n be a DTBD process with

$$b_i = \frac{1}{1+c}, \quad i \geq 0, \qquad a_i = \frac{1}{1+c} c^{\sqrt{i}-\sqrt{i-1}}, \quad i \geq 1.$$

Obviously, $\mu_i = c^{-\sqrt{i}}$, $i \geq 0$. By Theorem 2 of [13], we can easily show that the chain is not geometrically ergodic. Let β be any number in $(1, c)$ and $f(i) = \beta^{\sqrt{i}}$, $i \geq 0$. It is not hard to derive

$$(n+1)^{\alpha} \gamma^{\sqrt{n+1}} - n^{\alpha} \gamma^{\sqrt{n}} \sim n^{\alpha-1/2} \gamma^{\sqrt{n}}$$

for any $\alpha \geq 0$ and positive number $\gamma \neq 1$, which shows that if $\gamma < 1$ then

$$\sum_{j=n+1}^{\infty} j^{\alpha} \gamma^{\sqrt{j}} \sim n^{\alpha+1/2} \gamma^{\sqrt{j}}$$

and if $\gamma > 1$ then

$$\sum_{j=0}^{n-1} j^{\alpha} \gamma^{\sqrt{j}} \sim n^{\alpha+1/2} \gamma^{\sqrt{j}}.$$

Using the above facts and the Stolz–Cesáro theorem, we obtain

$$g_n^{(\ell)}(f) \sim n^{\ell+1} \beta^{\sqrt{n}}$$

for any $\ell \geq 0$. Hence, the chain is (f, ℓ)-ergodic for any $\ell \geq 0$.

3.2. Central limit theorems

The following proposition, which is taken from Chapter 17 of [15], gives a sufficient condition for the existence for a CLT and an expression of the asymptotic variance. In this section we set f to be a real-valued function which may be negative.

Proposition 3.1. *Suppose that $\pi(|f|) < \infty$. If*

$$\mathrm{E}_i\left[\left(\sum_{k=0}^{\tau_i} |\bar{f}(\Phi_k)|\right)^2\right] < \infty \tag{3.7}$$

for some (then for any) $i \in \mathbb{E}$, then (1.1) holds and the asymptotic variance is given by

$$\sigma_f^2 = \pi_i \,\mathrm{E}_i\left[\left(\sum_{k=1}^{\tau_i} \bar{f}(\Phi_k)\right)^2\right] = \sum_{n\in\mathbb{E}} \pi_n \left(2\,\mathrm{E}_n\left[\sum_{k=0}^{\sigma_i} \bar{f}(\Phi_k)\right] \bar{f}(n) - \bar{f}^2(n)\right).$$

Lemma 3.2. *Suppose that $\pi(|f|) < \infty$. Then (3.7) holds for the DTBD process if and only if*

$$\sum_{j=1}^{\infty} |\bar{f}(j)| \mu_j \left[\sum_{n=0}^{j-1} \frac{1}{b_n \mu_n} \sum_{k=n+1}^{\infty} |\bar{f}(k)| \mu_k - |\bar{f}(j)|\right] < \infty. \tag{3.8}$$

Proof. Since $\pi(|f|) < \infty$, it follows from Lemma 3.1 that

$$T_{i0} = \mathrm{E}_i\left[\sum_{k=0}^{\tau_0-1} |\bar{f}(\Phi_k)|\right] = \sum_{n=0}^{i-1} \frac{1}{b_n \mu_n} \sum_{k=n+1}^{\infty} |\bar{f}(k)| \mu_k < \infty \quad \text{for any } i \geq 1$$

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and $T_{00}=\mathrm{E}_0[\sum_{k=0}^{\tau_0-1}|\bar f(\Phi_k)|]<\infty$. In Theorem 2.1, let $p=2$ and $V(i)=|\bar f(i)|$. Then $T_{i0}^{(2)}=\mathrm{E}_i[(\sum_{k=0}^{\tau_0}|\bar f(\Phi_k)|)^2]$. It follows from Theorem 2.1(ii) that the sequence $\{x_i^*,\ i\ge 0\}$ given by $x_i^*=T_{i0}^{(2)}$, $i\ge 1$, and $x_0^*=0$ is the minimal nonnegative solution to the equations

$$x_i=\sum_{k\ge 1}p_{ik}x_k+2|\bar f(i)|(T_{i0}-|\bar f(i)|),\quad i\ge 1,\qquad x_0=0.$$

From Lemma 3.1 and Remark 3.1, we find that, for any $n\ge 1$, $T_{n0}^{(2)}<\infty$ if and only if (3.8) holds. From Theorem 2.1(i) we obtain

$$T_{00}^{(2)}=b_0T_{10}^{(2)}+2|\bar f(0)|T_{00}-|\bar f(0)|^2.$$

Hence, $T_{00}^{(2)}<\infty$ if and only if $T_{10}^{(2)}<\infty$, completing the proof.

Example 3.3. In Example 3.2, let $f(i)=\beta^{\sqrt{i}/2}$, where β is any number in $(1,c)$. Then $\pi(f^2)<\infty$. It is well known (see, e.g. [10]) that if a Markov chain is reversible and geometrically ergodic, then the CLT holds for any f such that $\pi(f^2)<\infty$. Note that a positive recurrent DTBD process is reversible. This DTBD process is not geometrically ergodic, but condition (3.8) can be easily checked. Hence, the CLT holds.

Theorem 3.2. *Suppose that $\pi(|f|)<\infty$ and that (3.8) holds for the DTBD process. Then the asymptotic variance is given by*

$$\sigma_f^2=-\pi_0\bar f^2(0)+\sum_{n=1}^{\infty}\pi_n[2g_n^{(0)}(\bar f)\bar f(n)-\bar f(n)^2],$$

where $\{\pi_n,\ n\in\mathbb{E}\}$ is given by (3.1) and

$$g_n^{(0)}(\bar f)=\sum_{i=0}^{n-1}\frac{1}{b_i\mu_i}\sum_{j=i+1}^{\infty}\mu_j\bar f(j),\qquad n\ge 1.$$

Proof. It is easy to derive

$$\sigma_f^2=-\pi_0\bar f^2(0)+\sum_{n=1}^{\infty}\pi_n\biggl(2\,\mathrm{E}_n\biggl[\sum_{k=0}^{\tau_0-1}\bar f(\Phi_k)\biggr]\bar f(n)-\bar f(n)^2\biggr).\tag{3.9}$$

We now give the expression for $\mathrm{E}_n[\sum_{k=0}^{\tau_0-1}\bar f(\Phi_k)]$, where $\bar f(\Phi_k)$ may be a negative number. For a real-valued function $h(i)$, define $h_+(i)=\max\{h(i),0\}$ and $h_-(i)=\max\{-h(i),0\}$. Since $\pi(|f|)<\infty$, we have $\pi(|\bar f|)<\infty$, which implies that $\pi(|\bar f_+|)<\infty$ and $\pi(|\bar f_-|)<\infty$. From Lemma 3.1 we obtain

$$\mathrm{E}_n\biggl[\sum_{k=0}^{\tau_0-1}\bar f_+(\Phi_k)\biggr]=\sum_{i=0}^{n-1}\frac{1}{b_i\mu_i}\sum_{j=i+1}^{\infty}\mu_j\bar f_+(j)$$

and

$$\mathrm{E}_n\biggl[\sum_{k=0}^{\tau_0-1}\bar f_-(\Phi_k)\biggr]=\sum_{i=0}^{n-1}\frac{1}{b_i\mu_i}\sum_{j=i+1}^{\infty}\mu_j\bar f_-(j).$$

Hence,

$$\mathrm{E}_n\Bigg[\sum_{k=0}^{\tau_0-1}\bar f(\Phi_k)\Bigg]=\mathrm{E}_n\Bigg[\sum_{k=0}^{\tau_0-1}\bar f_+(\Phi_k)\Bigg]-\mathrm{E}_n\Bigg[\sum_{k=0}^{\tau_0-1}\bar f_-(\Phi_k)\Bigg]=\sum_{i=0}^{n-1}\frac{1}{b_i\mu_i}\sum_{j=i+1}^{\infty}\mu_j\bar f(j).$$

The assertion follows by substituting the value of $\mathrm{E}_n[\sum_{k=0}^{\tau_0}\bar f(\Phi_k)]$ into (3.9).

Remark 3.3. The existence of a CLT is related to the (f,ℓ)-ergodicity. Another applicable sufficient condition for (3.7) is that, for some $i\in\mathbb{E}$,

$$\mathrm{E}_i\Bigg[\Bigg(\sum_{k=0}^{\tau_i}|f(\Phi_k)|\Bigg)^2\Bigg]<\infty\quad\text{and}\quad \mathrm{E}_i[\tau_i^2]<\infty.$$

The second part of this condition corresponds to the (1,2)-ergodicity investigated in Section 3.1.

Example 3.4. Let Φ_n be a DTBD process with

$$b_i=b,\quad i\ge 0,\qquad a_i=a,\quad i\ge 1,\qquad b<a,\qquad a+b=1.$$

Let $\rho=b/a$. Then $\mu_i=\rho^i$, $i\ge 0$. This chain is geometrically ergodic with invariant distribution $\pi_i=\rho^i(1-\rho)$, $i\ge 0$. Let $f(i)=i$. Then, obviously, (3.8) holds. We now compute the interesting quantity σ_f^2. By routine calculations we have $\pi(f)=\rho/(1-\rho)$,

$$\mathrm{E}_n\Bigg[\sum_{k=0}^{\tau_0-1}\bar f(\Phi_k)\Bigg]=\sum_{i=0}^{n-1}\frac{1}{b\rho^i}\sum_{j=i+1}^{\infty}\Big(j-\frac{\rho}{1-\rho}\Big)\rho^j=\frac{\rho n(n+1)}{b(1-\rho)},$$

and, finally,

$$\sigma_f^2=-\frac{\rho^2}{1-\rho}-\frac{\rho-\rho^2+\rho^3}{(1-\rho)^2}+\frac{2\rho^2(2\rho+2)}{b(1-\rho)^4}.$$

Appendix A

To assist the reader, in this appendix we state some known definitions and results about the theory of the minimal nonnegative solution, which are taken from Chapter 3 of [8].

We consider the following system of nonnegative linear equations:

$$x_i=\sum_{k\in\mathbb{E}}c_{ik}x_k+b_i,\qquad i\in\mathbb{E},\tag{A.1}$$

where $0\le c_{ik}<\infty$ and $0\le b_i\le\infty$ for any $i,k\in\mathbb{E}$.

Definition A.1. The solution $\{x_i^*,\ i\in\mathbb{E}\}$ of (A.1) such that $0\le x_i^*\le\infty$ is called the minimal nonnegative solution if, for any solution $\{x_i,\ i\in\mathbb{E}\}$ of (A.1) such that $0\le x_i\le\infty$, we have $x_i^*\le x_i$, $i\in\mathbb{E}$.

It is known that the minimal nonnegative solution of (A.1) exists and is unique. Moreover, $x_i^*=\lim_{n\to\infty}x_i^{(n)}$, where $\{x_i^{(n)},\ i\in\mathbb{E}\}$ is given recursively by

$$x_i^{(0)}\equiv 0,\qquad x_i^{(n+1)}=\sum_{k\in\mathbb{E}}c_{ik}x_k^{(n)}+b_i,\quad n\ge 0.$$

In the following theorems, the sequence $\{x_i^*,\ i \in \mathbb{E}\}$ denotes the minimal nonnegative solution of (A.1). The first theorem, called the localization theorem, is Theorem 3.4.1 of [8]. The second theorem, called the comparison theorem, is Theorem 3.3.1 of [8]. The generalized localization theorem and comparison theorem can be found in Chapter 2 of [2].

Theorem A.1. (Localization theorem.) *Let G be a nonempty set of $\mathbb{E}$. The minimal non-negative solution of the equations*

$$x_i = \sum_{k \in G} c_{ik} x_k + \Bigg(\sum_{k \in E \setminus G} c_{ik} x_k^* + b_i \Bigg), \qquad i \in G,$$

is $\tilde{x}_i^* = x_i^*,\ i \in G$.

Definition A.2. The inequality system

$$X_i \geq \sum_{k \in \mathbb{E}} C_{ik} X_k + B_i, \qquad i \in \mathbb{E}, \tag{A.2}$$

is called a major system of (A.1) if $c_{ik} \leq C_{ik}$ and $b_i \leq B_i$ for any $i, k \in \mathbb{E}$.

Theorem A.2. (Comparison theorem.) *Let $\{X_i,\ i \in \mathbb{E}\}$ be any solution of the major system (A.2). Then $x_i^* \leq X_i$ for any $i \in \mathbb{E}$.*

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