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The Relationship Between ϵ-Kronecker Sets and Sidon Sets

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Abstract. A subset E of a discrete abelian group is called ϵ-Kronecker if all E-functions of modulus one can be approximated to within ϵ by characters. E is called a Sidon set if all bounded E-functions can be interpolated by the Fourier transform of measures on the dual group. As ϵ-Kronecker sets with $\epsilon < 2$ possess the same arithmetic properties as Sidon sets, it is natural to ask if they are Sidon. We use the Pisier net characterization of Sidonicity to prove this is true.

1 Introduction

A subset E of the dual of a compact, abelian group G is called an ϵ-*Kronecker set* if for every function ϕ mapping E into the set of complex numbers of modulus one, there exists $x \in G$ such that

$$\left|\phi(\gamma) - \gamma(x)\right| < \epsilon \text{ for all } \gamma \in E.$$

The infimum of such ϵ is called the *Kronecker constant* of E and is denoted $\kappa(E)$. Trivially, $\kappa(E) \leq 2$ for all sets E, and this is sharp if the identity of the dual group belongs to E. ϵ-Kronecker sets have been studied for over 50 years since the concept was introduced by Kahane in [9], and the terminology was coined by Varapoulos in [14]. Examples of recent work include [1,2] (where they are called ϵ-free) and [3–7,10].

If $\kappa(E) < \sqrt{2}$, then E is known to be an example of a Sidon set, meaning every bounded E-function is the restriction to E of the Fourier transform of a measure on G. In fact, the interpolating measure can be chosen to be discrete, and $\sqrt{2}$ is sharp with this additional property. Like ϵ-Kronecker sets, Sidon sets have also been extensively studied for many years; we refer the reader to [8] or [12] for an overview of what was known prior to the early 1970's and to [5] for more recent results. But many fundamental problems remain open, including a full understanding of the connections between these two classes of interpolation sets.

As sets with Kronecker constant less than 2 possess many of the known arithmetic properties satisfied by Sidon sets, it was asked in [5] whether all such sets are Sidon. Here we answer this question affirmatively by using Pisier's remarkable net characterization of Sidon sets. We also construct non-trivial examples of Sidon sets with Kronecker constant 2.

As well, we define a weaker interpolation property than ϵ-Kronecker by only requiring the approximation of target functions whose range lies in the set of n-th roots

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of unity. Sets that satisfy a suitable quantitative condition for this less demanding interpolation property are also shown to be Sidon.

2 Kronecker-like Sets that are Sidon

Let G be a compact abelian group and Γ its discrete abelian dual group. An example of such a group G is the circle group $\mathbb{T}$, the complex numbers of modulus one, whose discrete dual is the group of integers, $\mathbb{Z}$.

Definition 2.1 (i) A subset $E \subseteq \Gamma$ is said to be ϵ-*Kronecker* if for every $\phi: E \to \mathbb{T}$ there exists $x \in G$ such that

$$\left|\phi(\gamma) - \gamma(x)\right| < \epsilon \text{ for all } \gamma \in E. \tag{2.1}$$

By the *Kronecker constant* of E, $\kappa(E)$, we mean the infimum of the constants ϵ for which (2.1) is satisfied.

(ii) A subset $E \subseteq \Gamma$ is said to be *Sidon* if for every bounded function $\phi: E \to \mathbb{C}$ there is a measure μ on G with $\widehat{\mu}(\gamma) = \phi(\gamma)$ for all $\gamma \in E$. If the interpolating measure μ can always be chosen to be discrete, then the set E is said to be I_0.

Hadamard sets $E = \{n_j\} \subseteq \mathbb{N}$ with $\inf n_{j+1}/n_j = q > 2$ are known to satisfy $\kappa(E) \leq |1 - e^{i\pi(q-1)}|$, and this tends to 0 as q tends to infinity. More generally, every infinite subset of a torsion-free dual group Γ contains subsets of the same cardinality that are ϵ-Kronecker for any given $\epsilon > 0$. If Γ is not torsion-free, but the subset E does not contain "too many" elements of order 2, then E will contain a subset F of the same cardinality, having $\kappa(F) = 1$ (see [3, 4]).

Obviously, every I_0 set is Sidon, but the converse is not true. It is unknown whether every Sidon set is a finite union of I_0 sets.

For a set E to be Sidon (or I_0), it is enough that there be a constant $\delta < 1$ such that for every E-function ϕ with $|\phi(\gamma)| \leq 1$ for all γ, there is a (discrete) measure μ such that

$$|\phi(\gamma) - \widehat{\mu}(\gamma)| < \delta \text{ for all } \gamma \in E.$$

Since $\gamma(x) = \widehat{\delta_x}(\gamma)$ for δ_x the point mass measure at x, it is easy to see that if E is ϵ-Kronecker for some $\epsilon < 1$, then E is I_0. With more work this can be improved: if $\kappa(E) < \sqrt{2}$, then E is I_0. This result is sharp, as there are non-I_0 sets that are $\sqrt{2}$-Kronecker; see [3].

It is well known that Sidon sets satisfy a number of arithmetic properties such as not containing large squares or long arithmetic progressions. In [3] (or see the discussion in [5, p. 35]), it was shown that sets E with $\kappa(E) < 2$ also satisfy these conditions, thus it is natural to ask if such sets are always Sidon. Here we answer this question affirmatively.

Theorem 2.2 *If the Kronecker constant of $E \subseteq \Gamma$ is less than two, then E is Sidon.*

Proof We use Pisier's ε-net condition, which states that a subset E is Sidon if and only if there is some $\varepsilon > 0$ such that for each finite subset $F \subset E$ there is a set $Y \subset G$

with $|Y| \geq 2^{\varepsilon|F|}$, and whenever $x \neq y \in Y$,

$$\varepsilon \leq \sup_{\gamma\in F} \big| \gamma(x) - \gamma(y) \big|.$$

This was proven by Pisier in [13]. Proofs can also be found in [5, Thm. 9.2.1] and [11, Thm. V.5].

Since we are assuming that $\kappa(E) < 2$, we can choose $\varepsilon > 0$ such that $\kappa(E) + \varepsilon < 2$. Let F be any finite subset of E.

For all $g \in G$ and $\lambda > 0$, the sets

$$U(g,\lambda) = \Big\{ h \in G : \lambda > \sup_{\gamma\in F} \big| \gamma(h) - \gamma(g) \big| \Big\}$$

are among the basic open sets for the topology on G (the topology of pointwise convergence as functions on Γ). We claim there is a finite maximal set $S \subset G$ such that

$$x \neq y \in S \Longrightarrow \varepsilon \leq \sup_{\gamma\in F} |\gamma(x) - \gamma(y)|.$$

This is a consequence of the compactness of G. If it was not true, one could choose an infinite set S having this separation property. As G is compact, S would have a cluster point $z \in G$. The open set $U(z, \varepsilon/2)$ would then contain infinitely many members of S, violating the required separation assumption.

By the maximality of S, for each $g \in G$ there is some $h \in S$ such that $g \in U(h, \varepsilon)$.

Consider any function $\phi: F \to \mathbb{T}$. By the Kronecker property, there is some $g \in G$ such that $\sup_{\gamma\in F} |\gamma(g) - \phi(\gamma)| \leq \kappa(E)$. Since there is some $h \in S$ such that $g \in U(h, \epsilon)$, we have that $\phi \in W(h)$, where

$$W(h) := \Big\{ \psi: F \to \mathbb{T} : \sup_{\gamma\in F} \big| \gamma(h) - \psi(\gamma) \big| \leq \kappa(E) + \varepsilon < 2 \Big\}.$$

Consequently,

$$\mathbb{T}^F = \bigcup_{h\in S} W(h).$$

We identify $\mathbb{T}^F$ with $[0, 2\pi)^F$, with the group operation being addition $\bmod 2\pi$, and in this way put $|F|$-dimensional Euclidean volume on $\mathbb{T}^F$. With this identification,

$$W(h) \subseteq \prod_{\gamma\in F} \big[\gamma(h) - \eta, \gamma(h) + \eta \big],$$

where $\eta < \pi$ depends only on the number $\kappa(E) + \varepsilon$ (and not on h or F). Thus, the $|F|$-dimensional volume of each set $W(h)$ is bounded by $(2\eta)^{|F|}$, while the volume of $\mathbb{T}^F$ is $(2\pi)^{|F|}$. It follows that

$$\operatorname{card}(S) \geq \Big(\frac{2\pi}{2\eta} \Big)^{|F|} = 2^{\varepsilon'|F|}$$

for a suitable choice of $\varepsilon' > 0$.

The minimum of ε and ε' meet the Pisier net condition and are independent of F. Thus, E is Sidon. ■

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Remark 2.3 In number theory, a set $E \subseteq \Gamma$ is sometimes called a Sidon set if whenever $\gamma_j \in E$, $\gamma_1\gamma_2 = \gamma_3\gamma_4$ if and only if $\{\gamma_3, \gamma_4\}$ is a permutation of $\{\gamma_1, \gamma_2\}$. This is a different class of sets from the Sidon sets defined above. ε-Kronecker sets need not be Sidon in this sense; indeed, any finite subset $E \subseteq \mathbb{Z}$ that does not contain 0 has $\kappa(E) < 2$. However, if E is ε-Kronecker for some $\varepsilon < \sqrt{2}$, then there are a bounded number of pairs with common product, with the bound depending only on ε (see [3]).

Next, we alter the definition of the Kronecker constant by only considering target functions whose range is restricted to a finite subgroup of $\mathbb{T}$. This is a natural variation to consider, for if Γ is a torsion group, the characters of G take on only the values in a suitable finite subgroup of $\mathbb{T}$. Moreover, there are even subsets E of $\mathbb{Z}$ (including all subsets of size 2 and many of size 3) whose Kronecker constant is realized with target functions ϕ mapping E into $\{-1, +1\}$ (*cf.* [7]).

Definition 2.4 Let $\mathbf{T}_n$ denote the set of n-th roots of unity in $\mathbb{T}$ for $n \geq 2$. Let $\kappa_n(E)$ be the infimum of $\epsilon \geq 0$ such that E is (ϵ, n)-Kronecker, where $E \subseteq \Gamma$ is (ϵ, n)-Kronecker if for every $\phi: E \to \mathbf{T}_n$ there exists $x \in G$ such that

$$\gamma \in E \Longrightarrow \left|\phi(\gamma) - \gamma(x)\right| < \epsilon.$$

Theorem 2.5 *Let $E \subset \Gamma$. If $\kappa_n(E) < |1 - e^{i\pi(1-1/n)}|$, then E is Sidon.*

Proof Choose $\varepsilon > 0$ such that $\kappa_n(E) + \varepsilon < |1 - e^{i\pi(1-1/n)}|$. Let $F \subset E$ be finite. Choose $S \subset G$ as in the proof of Theorem 2.2. Arguing in a similar fashion to that proof, we again deduce that for every $\phi: E \to \mathbf{T}_n$, there is some $h \in S$ such that $\phi \in V(h)$, where

$$V(h) := \left\{ \psi: F \to \mathbf{T}_n : \sup_{\gamma\in F}\left|\gamma(h) - \psi(\gamma)\right| \leq \kappa_n(E) + \varepsilon \right\}.$$

Consequently,

$$(\mathbf{T}_n)^F = \bigcup_{h\in S} V(h).$$

For each $h \in S$ and every $\gamma \in F$, there is an n-th root of unity, $\omega \in \mathbf{T}_n$, such that $|\gamma(h) - \omega| \geq |1 - e^{i\pi(1-1/n)}|$. Whenever $\phi_h(\gamma) = \omega$, it follows that $\phi_h \notin V(h)$. Thus, each $V(h)$ has at most $(n-1)^{|F|}$ elements. Consequently, there is some $\varepsilon' > 0$, independent of F, such that

$$\operatorname{card}(S) \geq \frac{n^{|F|}}{(n-1)^{|F|}} = 2^{\epsilon'|F|}.$$

Again, the minimum of ε and ε' meets the Pisier net condition to be Sidon. ∎

It is sometimes more convenient to measure angular distances when comparing elements of $\mathbb{T}$ and to express Kronecker constants in those terms. Towards this, put $\mathbf{Z}_n = \{2\pi j/n : j = 0, 1, \ldots, n-1\}$, and for $z \in \mathbb{T}$, let $\arg(z)$ be the angle $\theta \in [0, 2\pi)$ such that $\exp(i\theta) = z$. Let $\alpha_n(E)$ be the infimum of $\epsilon \geq 0$ such that for every $\phi: E \to \mathbf{Z}_n$ there exists $x \in G$ such that

$$\gamma \in E \Longrightarrow \left|\phi(\gamma) - \arg\gamma(x)\right| \leq \epsilon.$$

A set E satisfying this condition is called weak (ϵ, n)-angular Kronecker. Here $|\phi(\gamma) - \arg\gamma(x)|$ should be understood mod 2π, so $\alpha_n(E) \in [0, \pi]$.

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It is easy to see that $\kappa_n(E) = |1 - e^{i\alpha_n(E)}|$, thus the previous theorem can be restated as: E is Sidon if $\alpha_n(E) < \pi(1 - 1/n)$.

We can similarly define weak angular ϵ-Kronecker sets and the angular Kronecker constant, $\alpha(E)$, by considering the approximation problem for functions $\phi: E \to [0, 2\pi)$. One can easily check that $\kappa(E) = |1 - e^{i\alpha(E)}|$, hence Theorem 2.2 can be restated as: E is Sidon if $\alpha(E) < \pi$.

Example 2.6 Let $n > 1$ be any integer. The set $E = 1 + n\mathbb{Z}$ is not a Sidon subset of $\mathbb{Z}$ being a coset of an infinite subgroup, but $\alpha_n(E) = \pi - \pi/n$. That shows Theorem 2.5 is sharp. In fact, for odd n, $\alpha_n(E) \le \pi - \pi/n$ for all subsets E of any discrete abelian group Γ. This is because the n-th root of unity farthest from 1 is $e^{i\pi(1-1/n)}$, so that if we let 1 denote the identity element of G, then for all $\mathbf{T}_n$-valued functions ϕ, and any $\gamma \in \Gamma$ we have $|\phi(\gamma) - \gamma(1)| \le |1 - e^{i\pi(1-1/n)}|$.

To see that $\alpha_n(1 + n\mathbb{Z}) \le \pi - \pi/n$ for n even, take $g = \exp(\pi i/n)$. For any character $\gamma = 1 + nk \in E$, we have $\arg \gamma(g) = \pi(nk + 1)/n$ with $nk + 1$ an odd integer. Thus, $|z - \arg \gamma(x)| \le \pi - \pi/n$ for any $z \in \mathbf{Z}_n$.

3 Some Examples of Sidon Sets with Kronecker Constant Equal to 2

Since any subset of Γ that contains the identity character 1 has Kronecker constant equal to 2, we are interested in constructing examples of Sidon subsets E of $\Gamma \setminus \{1\}$ with $\kappa(E) = 2$ and $\kappa_n(E) \ge |1 - e^{i\pi(1-1/n)}|$. We give one example with a set of elements of finite order and a second example where all the elements of E have infinite order.

Example 3.1 Let $\Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{0, 1\}$. Then $E = \Gamma \setminus \{(0, 0, 0)\}$ is Sidon, but $\kappa(E) = 2$ and $\kappa_n(E) \ge |1 - e^{i\pi(1-1/n)}|$ for $n \ge 2$.

Proof Being a finite set, $\Gamma \setminus \{(0, 0, 0)\}$ is Sidon. Let e_j be the standard basis vectors of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and let $E' = \{e_2, e_3, e_1 + e_2, e_1 + e_3\}$.

We will first show that $\kappa(E') = 2$, whence $\kappa(E) = 2$. Define ϕ by $\phi(e_2) = \phi(e_3) = \phi(e_1 + e_2) = 1$ and $\phi(e_1 + e_3) = -1$. Suppose that $g \in G$ and $\epsilon > 0$ satisfies

$$|\gamma(g) - \phi(\gamma)| < 2 - \epsilon \quad \text{for all } \gamma \in E.$$

Because $\gamma(g) \in \{-1, +1\}$ for every $\gamma \in \Gamma$, we must have

$$e_2(g) = e_3(g) = 1 = (e_1 + e_2)(g) \quad \text{and} \quad (e_1 + e_3)(g) = -1.$$

This forces $e_1(g)$ to be equal to both -1 and 1, a contradiction. Hence $\kappa(E) = 2$.

Since ϕ takes on only n-th roots of unity for even n, this argument also proves $\kappa_n(E) = 2$ when n is even.

If n is odd, then, instead, define $\phi(e_1 + e_3) = \omega_n$, where $\omega_n = e^{i\pi(1-1/n)}$, an n-th root of unity nearest to -1. If $\kappa_n(E) < |1 - e^{i\pi(1-1/n)}|$, then we obtain the same contradiction as before by noting that the identity $|1 - \phi(e_1 + e_3)| = |1 - e^{i\pi(1-1/n)}|$ forces $(e_1 + e_3)(g) = -1$. ∎

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Example 3.2 Let $\Gamma = \mathbb{Z} \oplus \Gamma_2$ where Γ_2 is the countable direct sum of copies of $\mathbb{Z}_2$. Let e_n be the character $e^{2\pi in(\cdot)}$ on $\mathbb{T}$ and let γ_n be the projection onto the n-th-$\mathbb{Z}_2$ factor, both viewed as elements of Γ in the canonical way. Set

$$E = \{(e_n, \gamma_n)\}_{n=1}^{\infty} \cup \{(e_n^{-1}, \gamma_n)\}_{n=1}^{\infty}.$$

Then E is Sidon, but $\kappa(E) = 2$ and $\kappa_n(E) \geq |1 - e^{i\pi(1-1/n)}|$ for $n \geq 2$.

Proof We argue first that $E_1 = \{(e_n, \gamma_n)\}_{n=1}^{\infty}$ and $E_2 = \{(e_n^{-1}, \gamma_n)\}_{n=1}^{\infty}$ both satisfy algebraic conditions to be Sidon. Let $f: \mathbb{N} \to \{-1, 0, 1\}$ be finitely non-zero and satisfy

$$\prod_n (e_n, \gamma_n)^{f(n)} = 1.$$

By the algebraic independence of the factors of Γ this implies $\gamma_n^{f(n)} = 1$ for all n and hence $f(n) = 0$. Therefore, E_1 is quasi-independent and such sets are well known to be Sidon. Likewise, E_2 is Sidon, and hence the union, $E = E_1 \cup E_2$, is Sidon.

Let $\epsilon > 0$ and suppose E is $(2 - \epsilon)$-Kronecker. Define ϕ to be -1 on E_1 and 1 on E_2. The compact group $G = \mathbb{T} \otimes G_2$, where G_2 is the direct product of countably many copies of (the multiplicative group) $\mathbb{Z}_2$, is the dual of Γ. Choose $g \in G$ such that for all $\gamma \in E$,

$$|\phi(\gamma) - \gamma(g)| < 2 - \epsilon.$$

Write $g = (u, (g_n))$ where $u \in \mathbb{T}$ and g_n is the projection of g onto the n-th-$\mathbb{Z}_2$ factor. With this notation, $(e_n^{\pm 1}, \gamma_n)(g) = e^{\pm 2\pi inu} g_n$, hence for all n,

$$|-e^{-2\pi inu} - g_n| = |-1 - e^{2\pi inu} g_n| < 2 - \epsilon \quad \text{and}$$
$$|e^{2\pi inu} - g_n| = |1 - e^{-2\pi inu} g_n| < 2 - \epsilon.$$

If u is rational, then $e^{2\pi inu} = e^{-2\pi inu} = 1$ periodically as a function of n. For these infinitely many n, we have $|-1 - g_n| < 2 - \epsilon$ and $|1 - g_n| < 2 - \epsilon$. But $g_n = \pm 1$, so this is impossible.

Otherwise, $\{e^{2\pi inu}\}_{n=1}^{\infty}$ is dense in $\mathbb{T}$. Choose n such that

$$|1 - e^{2\pi inu}| = |1 - e^{-2\pi inu}| < \epsilon/2.$$

But then $|-1 - g_n| < 2 - \epsilon/2$ and $|1 - g_n| < 2 - \epsilon/2$, and again these cannot be simultaneously satisfied for $g_n = \pm 1$. This impossibility proves $\kappa(E) = 2$ and also establishes $\kappa_n(E) = 2$ for n even.

If, instead, we define $\phi = \omega_n$ on E_1, where ω_n is an n-th root of unity nearest -1, then similar arguments show that $\kappa_n(E) = |1 - e^{i\pi(1-1/n)}|$ for n odd and $\kappa_n(E) = 2$ for n even. ∎

Remark 3.3 It would be interesting to know whether non-trivial examples of 2-Kronecker Sidon sets could be found in a torsion-free group and also whether every Sidon set is a finite union of sets that are ϵ-Kronecker for some $\epsilon < 2$.

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