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Prime and Primary Ideals in a Prüfer Order in a Simple Artinian Ring with Finite Dimension over its Center

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Abstract. Let Q be a simple Artinian ring with finite dimension over its center. An order R in Q is said to be *Prüfer* if any one-sided R-ideal is a progenerator. We study prime and primary ideals of a Prüfer order under the condition that the center is Prüfer. Also we characterize branched and unbranched prime ideals of a Prüfer order.

0 Introduction

Let D be a domain with quotient field F and let Q be a simple Artinian ring with finite dimension over its center F. A subring R with $D = Z(R)$, the center of R, is called an *order* in Q if $FR = Q$. Then, of course, R is a prime Goldie ring with quotient ring Q. Following [AD], R is a *Prüfer* order in Q if any one-sided R-ideal is a progenerator.

In this paper, we shall study prime and primary ideals of a Prüfer order R in Q under the condition that $D = Z(R)$ is Prüfer. Particularly we give in Theorem 2.7 a generalization of well-known results about branched and unbranched prime ideals of commutative Prüfer domains (*cf.* [Gi, Theorem 23.3]). If $D = Z(R)$ is a Prüfer domain, then R is a Prüfer order in Q if and only if R_m is a semi-local Bezout order in Q for any maximal ideal m of D (*cf.* [D3, Theorem 3] and [M2, Theorem 2.5]). In [G2], Gräter has characterized a semi-local Bezout order R as follows; $R = R_1 \cap \cdots \cap R_n$, where $R_1, \ldots, R_n$ are incomparable Dubrovin valuation rings of Q having the intersection property. By using this property, it is shown in Theorem 1.5 that there exists a bijective correspondence between the set of all primary ideals of R and the set of all primary ideals of R_i, $1 \leq i \leq n$. This theorem will be applied in Section 2 to characterize branched and unbranched prime ideals of a Prüfer order.

We use $\subset$ for proper inclusion and $\subseteq$ for inclusion.

1 The Case of Semi-Local Bezout Orders

In this section, we shall study prime and primary ideals in a semi-local Bezout order in a simple Artinian ring with finite dimension over its center.

First, we shall investigate primary ideals and prime radicals of a prime Goldie ring and its central localization. An element a of a ring R is called *strongly nilpotent* if every sequence $a_0, a_1, a_2, \ldots$, such that $a_0 = a, a_{n+1} \in a_nRa_n$ is ultimately zero. Clearly, every strongly nilpotent element is nilpotent. Let A be an ideal of R. Then we denote by $\sqrt{A}$ the prime radical of A, that is, $\sqrt{A} = \bigcap\{P : \text{prime ideals of } R \mid P \supseteq A\}$. It is well known that the

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prime radical $\sqrt{A}$ of A is the set of all elements of R which are strongly nilpotent modulo A (*cf.* [L, p. 56, Proposition 1]). So, we have

Lemma 1.1 *Let $R \subseteq S$ be rings and let A' be an ideal of S with $A = A' \cap R$. Then $\sqrt{A'} \cap R \subseteq \sqrt{A}$.*

An ideal A of a ring R is called a *right* $(\sqrt{A})$*-primary* ideal if $xRy \subseteq A$ and $y \notin \sqrt{A}$, then $x \in A$. It is easily shown that an ideal A is right primary if and only if $BC \subseteq A$ implies $B \subseteq A$ or $C \subseteq \sqrt{A}$ for ideals B and C of R. Similarly, a *left primary ideal* is defined. An ideal A of a ring R is said to be $(\sqrt{A})$*-primary* if it is right and left primary.

Lemma 1.2 *Let R be a prime Goldie ring, let $\mathcal{S}(\not\ni 0)$ be a multiplicatively closed subset of $Z(R)$ and let A be an ideal of R such that $A \cap \mathcal{S} = \varnothing$. Then*

(1) $\sqrt{A} \cap \mathcal{S} = \varnothing$,
(2) if A is one-sided primary, then
  (i) $A_{\mathcal{S}} \cap R = A$,
  (ii) $\sqrt{A_{\mathcal{S}}} = (\sqrt{A})_{\mathcal{S}}$ and
  (iii) $\sqrt{A_{\mathcal{S}}} \cap R = \sqrt{A}$.

Proof (1) is clear since any element of $\sqrt{A}$ is nilpotent modulo A.

(2) (i) is obvious. To prove (ii), let P' be any prime ideal of $R_{\mathcal{S}}$ such that $A_{\mathcal{S}} \subseteq P'$. Then we have $A \subseteq A_{\mathcal{S}} \cap R \subseteq P' \cap R$ and $P' \cap R$ is a prime ideal of R, because $R_{\mathcal{S}}$ is a central localization. Hence $\sqrt{A} \subseteq P' \cap R$ and so $(\sqrt{A})_{\mathcal{S}} \subseteq (P' \cap R)_{\mathcal{S}} = P'$, which implies $(\sqrt{A})_{\mathcal{S}} \subseteq \sqrt{A_{\mathcal{S}}}$. To prove the converse inclusion, let $x = as^{-1} \in \sqrt{A_{\mathcal{S}}}$, where $a \in R$ and $s \in \mathcal{S}$. Then, since $a = xs \in \sqrt{A_{\mathcal{S}}}$, a is strongly nilpotent modulo $A_{\mathcal{S}}$, and so is modulo A, because $A_{\mathcal{S}} \cap R = A$. Thus $a \in \sqrt{A}$ and hence $x \in (\sqrt{A})_{\mathcal{S}}$. (iii) follows from Lemma 1.1 and (ii).

Let $\mathrm{Spec}(R)$ be the set of all prime ideals of a ring R and let $\mathrm{Pr}(R)$ be the set of all right primary ideals of R. Then, from Lemma 1.2, we easily obtain the following.

Lemma 1.3 *Let R be a prime Goldie ring and let $\mathcal{S}(\not\ni 0)$ be a multiplicatively closed subset of $Z(R)$. Then*

(1) The mappings $P \to P' = P_{\mathcal{S}}$ and $P' \to P = P' \cap R$ give a bijective correspondence between $\{P \in \mathrm{Spec}(R) \mid P \cap \mathcal{S} = \varnothing\}$ and $\mathrm{Spec}(R_{\mathcal{S}})$, where $P \in \mathrm{Spec}(R)$ with $P \cap \mathcal{S} = \varnothing$ and $P' \in \mathrm{Spec}(R_{\mathcal{S}})$.
(2) The mappings $A \to A' = A_{\mathcal{S}}$ and $A' \to A = A' \cap R$ give a bijective correspondence between $\{A \in \mathrm{Pr}(R) \mid A \cap \mathcal{S} = \varnothing\}$ and $\mathrm{Pr}(R_{\mathcal{S}})$, where $A \in \mathrm{Pr}(R)$ with $A \cap \mathcal{S} = \varnothing$ and $A' \in \mathrm{Pr}(R_{\mathcal{S}})$.

In the remainder of this section, *let Q be a simple Artinian ring with finite dimension over its center F and let R be an order in Q*. An order R in Q is said to be *Bezout* if any one-sided finitely generated R-ideal is principal. We say that R is *semi-local* if $R/J(R)$ is a semi-simple Artinian ring, where $J(R)$ is the Jacobson radical of R.

If R is a semi-local Bezout order in Q, then, by [G2, Corollary 3.5], we have $R = R_1 \cap \cdots \cap R_n$, where each R_i is a Dubrovin valuation ring of Q and $R_1, \ldots, R_n$ have the intersection

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property, that is, the mapping $S \to J(S) \cap R$ is a well-defined anti-ordered isomorphism between $\mathcal{B}(R_1) \cup \cdots \cup \mathcal{B}(R_n)$ and $\mathrm{Spec}(R)$, where $S \in \mathcal{B}(R_i) = \{S : \text{ overring of } R_i\}$ $(1 \le i \le n)$. Further, by [G2, Theorem 2.6], for any prime ideal P of R, $C(P) = \{c \in R \mid [c+P] \text{ is regular in } R/P\}$ is a regular Ore set of R and R_P is a Dubrovin valuation ring of Q such that $J(R_P) \cap R = P$. Then we have the following.

Lemma 1.4 *Let $R = R_1 \cap \cdots \cap R_n$ be a semi-local Bezout order in Q. Assume that A and A' are right primary ideals of R and R_i (for some i) respectively satisfying $A = A' \cap R$ and $P = \sqrt{A} = \sqrt{A'} \cap R$, and P is a prime ideal of R. Then $A_P = A'$ and it is a $J(R_P)$-primary ideal of R_P.*

Proof Set $P' = \sqrt{A'}$, a prime ideal of R_i by [MMU, Lemma 1], and set $S = R_{iP'}$. Then it follows from [D2, Section 2, Theorem 1], [MMU, Lemmas 6 and 8] and [G2, Theorem 2.5] that $P' = J(S)$, $S = R_P$ and A' is a $J(R_P)$-primary ideal of S. Hence $A_P = (A' \cap R)_P = A'_P \cap R_P = A' \cap R_P = A'$.

If $R = R_1 \cap \cdots \cap R_n$ is a semi-local Bezout order in Q, where $R_1, \ldots, R_n$ are Dubrovin valuation rings of Q having the intersection property, then, by [D2, Section 2, Theorem 1] and the definition of the intersection property, the mapping $P' \to P = P' \cap R$ is a well-defined inclusion preserving bijective correspondence between $\mathrm{Spec}(R_1) \cup \cdots \cup \mathrm{Spec}(R_n)$ and $\mathrm{Spec}(R)$. Concerning primary ideals, we have the following.

Theorem 1.5 *Let $R = R_1 \cap \cdots \cap R_n$ be a semi-local Bezout order in a simple Artinian ring Q with finite dimension over its center F, where $R_1, \ldots, R_n$ are incomparable Dubrovin valuation rings of Q having the intersection property. Then the prime radical of any right primary ideal of R is a prime ideal, and the mappings $A' \to A = A' \cap R$ and $A \to A' = A_P$ give a bijective correspondence between $\mathrm{Pr}(R_1) \cup \cdots \cup \mathrm{Pr}(R_n)$ and $\mathrm{Pr}(R)$ satisfying $\sqrt{A} = \sqrt{A'} \cap R$, where $A' \in \mathrm{Pr}(R_i)$ for some i and $P = \sqrt{A}$.*

Proof We define $\varphi\colon \mathrm{Pr}(R_1) \cup \cdots \cup \mathrm{Pr}(R_n) \to \mathrm{Pr}(R)$ by $\varphi(A') = A' \cap R$, where $A' \in \mathrm{Pr}(R_i)$ for some i. First of all, we have to show that φ is well-defined. To do this, let $A' \in \mathrm{Pr}(R_i)$ and let $P' = \sqrt{A'}$, a prime ideal of R_i by [MMU, Lemma 1]. If we set $S = R_{iP'}$, then S is a Dubrovin valuation ring of Q with $J(S) = P'$ and A' is a P'-primary ideal of S by [D2, Section 2, Theorem 1] and [MMU, Lemmas 6 and 8]. Thus $P = P' \cap R$ is a prime ideal of R and $R_P = S$ by [G2, Theorem 2.5]. Set $A = A' \cap R$. Then it is clear that $\sqrt{A} \subseteq P$. On the other hand, $P = \sqrt{A'} \cap R \subseteq \sqrt{A}$ by Lemma 1.1. Hence $P = \sqrt{A}$, a prime ideal. To prove that A is a right primary ideal of R, suppose that $aRb \subseteq A$ and $b \notin P$. Since R is a PI ring, $RbR \cap C(P) \neq \varnothing$. This implies that $a \in aR_P = a(RbR)_P \subseteq (aRbR)_P \subseteq A_P = A'$. Thus $a \in A' \cap R = A$, proving that A is right primary. Hence φ is well-defined.

To prove that φ is one-to-one, suppose that $A' \cap R = A = A'_1 \cap R$, where $A' \in \mathrm{Pr}(R_i)$ and $A'_1 \in \mathrm{Pr}(R_j)$ for some i and j. Set $P = \sqrt{A}$. Then, by Lemma 1.4, $A' = A_P = A'_1$, proving that φ is one-to-one.

Next, we shall prove that φ is onto by induction on $[Q : F]$. Since the case of $[Q : F] = 1$ is clear, we may assume that $[Q : F] > 1$ and let A be a right primary ideal of R. Let $D = Z(R)$ and let $m_1, \ldots, m_k$ be the full set of maximal ideals of D. Since $A = A_{m_1} \cap \cdots \cap A_{m_k}$, we may assume that $A_{m_1} \subset R_{m_1}$, that is, $A \cap (D \setminus m_1) = \varnothing$. Then, by Lemma 1.3, A_{m_1} is right primary with $A_{m_1} \cap R = A$, and $\sqrt{A_{m_1}} \cap R = \sqrt{A}$ by Lemma 1.2. There are two cases.

Case 1 In the case R_{m_1} is a Dubrovin valuation ring of Q. Then, since $R_{1m_1}, \dots, R_{nm_1}$ are linearly ordered by inclusion, $R_{m_1} = R_{im_1}$ for some i. By [MMU, Lemma 1], $P' = \sqrt{A_{m_1}}$ is a prime ideal of R_{im_1}. Set $S = (R_{i\,m_1})_{P'}$, a Dubrovin valuation ring with $J(S) = P'$ by [D2, Section 2, Theorem 1], and A_{m_1} is a $J(S)$-primary ideal of $R_{i\,m_1}$ which is an ideal of S by [MMU, Lemma 6]. Hence $A_{m_1} \in \operatorname{Pr}(R_i)$ by [MMU, Lemma 6] and $A = A_{m_1} \cap R$ by Lemma 1.3. $P = \sqrt{A}$ is a prime ideal of R, because $\sqrt{A} = \sqrt{A_{m_1}} \cap R = J(S) \cap R$ by Lemma 1.2 and the intersection property.

Case 2 In the case R_{m_1} is not a Dubrovin valuation ring. Then $R_{m_1} = R_{1m_1} \cap \cdots \cap R_{nm_1} = R_{1m_1} \cap \cdots \cap R_{lm_1}$, where $R_{1m_1}, \dots, R_{lm_1}$ are incomparable and $Z(R_{m_1}) = D_{m_1}$ is a valuation ring. By [G1, p. 835, Case 2], there exists a Dubrovin valuation ring S of Q integral over $W = Z(S)$ such that

(a) $S \supseteq R_{1m_1}, \dots, R_{lm_1}$ and
(b) $[Q : F] > [\bar{S} : Z(\bar{S})]$, where $\bar{S} = S/J(S)$.

By [G1, Lemma 6.4], we have the following two cases:

(i) In the case $A_{m_1} \supseteq J(S)$. If $A_{m_1} = J(S)$, then $A_{m_1} \in \operatorname{Pr}(R_i)$, $1 \leq i \leq l$. Thus we may assume that $A_{m_1} \supset J(S)$. Since $\widetilde{R_{1\,m_1}}, \dots, \widetilde{R_{n\,m_1}}$ have the intersection property by [G1, Proposition 6.3] where $\widetilde{R_{i\,m_1}} = R_{i\,m_1}/J(S)$, there exists an $\tilde{A}' \in \operatorname{Pr}(\widetilde{R_{i\,m_1}})$ for some i with $\widetilde{A_{m_1}} = \tilde{A}' \cap \widetilde{R_{m_1}}$ and $\sqrt{\widetilde{A_{m_1}}} = \sqrt{\tilde{A}'} \cap \widetilde{R_{m_1}}$ by induction hypothesis. It follows from [MMU, Lemmas 6 and 8] that there exists an overring $\tilde{T}$ of $\widetilde{R_{i\,m_1}}$ such that $\tilde{A}'$ is a $J(\tilde{T})$-primary ideal of $\tilde{T}$. By [D2, Section 1, Proposition 2], there exists a Dubrovin valuation ring T of Q such that $S \supseteq T \supseteq R_{i\,m_1}$ with $\tilde{T} = T/J(S)$. Let A' be the inverse image of $\tilde{A}'$ in T. Then A' is a $J(T)$-primary ideal of T since $J(\tilde{T}) = J(T)/J(S)$. Hence $A' \in \operatorname{Pr}(R_i)$ by [MMU, Lemma 6]. It is easy to see that $A_{m_1} = A' \cap R_{m_1}$ and $\sqrt{A_{m_1}} = J(T) \cap R_{m_1}$, because $\widetilde{\sqrt{A_{m_1}}} = \sqrt{\widetilde{A_{m_1}}} = J(\tilde{T}) \cap \widetilde{R_{m_1}} = \widetilde{J(T)} \cap \widetilde{R_{m_1}}$. Therefore, by Lemmas 1.2 and 1.3, we have $A = A_{m_1} \cap R = A' \cap R_{m_1} \cap R = A' \cap R$ and $\sqrt{A} = \sqrt{A_{m_1}} \cap R = J(T) \cap R$, a prime ideal of R by [G2, Theorem 2.5].

(ii) In the case $J(S) \supset A_{m_1}$. Since $\sqrt{A_{m_1}}$ is a semi-prime ideal of R_{m_1} and $J(S) \supseteq \sqrt{A_{m_1}}$, $\sqrt{A_{m_1}}$ is an ideal of S by [G1, Lemma 6.4]. We claim that A_{m_1} is also an ideal of S. Before proving this claim, we note that $S = (R_{m_1})_p$, where $p = J(W)$. It is clear that $S \supseteq (R_{m_1})_p$ and $Z\big((R_{m_1})_p\big) = (D_{m_1})_p = W$. Thus $(R_{m_1})_p$ is a Bezout W-order and hence $S = (R_{m_1})_p$ by [M1, Theorem 3.4]. Thus, to prove the claim, it is enough to show that $c^{-1}A_{m_1} \subseteq A_{m_1}$ for any $c \in D_{m_1} \setminus p$. Since $c^{-1}A_{m_1} \subseteq S \cdot J(S) = J(S) \subseteq R_{m_1}$, $c^{-1}A_{m_1}$ is an ideal of R_{m_1}. Now $A_{m_1} = c \cdot c^{-1}A_{m_1}$ and $c \notin \sqrt{A_{m_1}}$ imply that $c^{-1}A_{m_1} \subseteq A_{m_1}$. In particular, $A_{m_1} = (A_{m_1})_p$. Hence, by Lemma 1.3, A_{m_1} is a primary ideal of $S = (R_{m_1})_p$. Set $P' = \sqrt{A_{m_1}}$, a prime ideal of S by [MMU, Lemma 1], and set $T = S_{P'}$. Then $J(T) = \sqrt{A_{m_1}}$ by [D2, Section 2, Theorem 1] and thus, by Lemma 1.2, $\sqrt{A} = \sqrt{A_{m_1}} \cap R = J(T) \cap R$, a prime ideal of R by [G2, Theorem 2.5]. Since $T \supseteq S \supseteq R_{i\,m_1} \supseteq R_i (i = 1, \dots, l)$, it is clear from [MMU, Lemma 6] that $A_{m_1} \in \operatorname{Pr}(R_i)$ and $A = A_{m_1} \cap R$. Thus φ is onto. We have also proved that $P = \sqrt{A}$ is a prime ideal of R for any $A \in \operatorname{Pr}(R)$.

To completes the proof, it only remains to show that $A' = A_P$ for any $A \in \operatorname{Pr}(R)$ and for any $A' \in \operatorname{Pr}(R_i)$ with $A' \cap R = A$, where $P = \sqrt{A}$. However, this always holds by Lemma 1.4, completing the proof.

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We conclude this section with the following results deriving from Lemma 1.4 and Theorem 1.5.

Corollary 1.6 *Let A be a right primary ideal of a semi-local Bezout order R in Q with $P = \sqrt{A}$. Then A_P is a $J(R_P)$-primary ideal of R_P with $A = A_P \cap R$.*

Corollary 1.7 *Let A be an ideal of a semi-local Bezout order R in Q. Then A is right primary if and only if A is left primary. In this case, $A_P = {}_PA$ holds, where $P = \sqrt{A}$.*

Proof Let A be right primary and assume that $aRb \subseteq A$ for $a \in R \setminus P$ and $b \in R$. Then ${}_P(RaR) = {}_PR = R_P$. Hence $b \in R_Pb = {}_P(RaR)b = R_PRaRb \subseteq R_PA \subseteq R_PA_P = A_P$ since A_P is an ideal of R_P, and so $b \in A_P \cap R = A$ by Corollary 1.6. Therefore, A is left primary. The converse is proved similarly. Because A_P and ${}_PA$ are both ideals of R_P, we have $A_P = {}_PA$.

2 Prime and Primary Ideals of a Prüfer Order

Throughout this section, *let R be a Prüfer order in a simple Artinian ring Q with finite dimension over its center F and suppose that $Z(R)$ is Prüfer*. Then we note that R_M exists and is a Dubrovin valuation ring for any maximal ideal M of R by [D3, Theorem 3]. In this section, we shall study primary ideals of R and characterize branched and unbranched prime ideals of R.

Lemma 2.1 *Let A be an ideal of R. Then A is right primary if and only if it is left primary. In this case, $\sqrt{A}$ is prime.*

Proof Assume that A is right primary. Then for any maximal ideal m of D with $A \cap (D \setminus m) = \varnothing$, A_m is right primary by Lemma 1.3, so A_m is left primary by Corollary 1.7. Hence, by the left version of Lemma 1.3, $A = A_m \cap R$ is left primary. The converse is proved similarly. Further, $\sqrt{A_m}$ is prime by Theorem 1.5, and so $\sqrt{A} = \sqrt{A_m} \cap R$ is also prime by Lemmas 1.2 and 1.3.

Lemma 2.2 *Let A be an ideal of R with $P = \sqrt{A}$ prime. Suppose that A_M is an ideal of R_M and is P_M-primary for every maximal ideal M of R. Then A is P-primary.*

Proof Assume that $xRy \subseteq A$, where $x \in R$ and $y \in R \setminus P$. Let M be any maximal ideal of R. If $P \nsubseteq M$, then $P \cap C(M) \neq \varnothing$ and so $A \cap C(M) \neq \varnothing$. Hence $A_M = R_M \ni x$. If $P \subseteq M$, then $C(M) \subseteq C(P)$ and $R_M \subseteq R_P$ by [MM2, Lemmas 1 and 2]. So P_M is a prime ideal of R_M. Since R_M is a Dubrovin valuation ring and $R_P = (R_M)_{P_M}$, A_M is an ideal of R_P by the assumption and [MMU, Lemma 6]. Hence we obtain that $A_M = A_P$. Now $RyR \cap C(P) \neq \varnothing$ because R is a PI ring, and so we have $(RyR)_P = R_P$. It follows that $x \in xR_P = xR(RyR)_P \subseteq (xRyR)_P \subseteq A_P = A_M$. Thus $x \in \bigcap A_M = A$ by [M2, Lemma 2.4], and so A is P-primary.

Lemma 2.3 *Let A be a P-primary ideal of R. Then $A_M = {}_MA$ for any maximal ideal M of R. Further, if M is a maximal ideal of R with $P \subseteq M$, then A_M is a P_M-primary ideal of R_M.*

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Proof Let A be a P-primary ideal of R and let M be a maximal ideal of R. If $P \nsubseteq M$, then $A \nsubseteq M$. So $A \cap C(M) \neq \varnothing$ and we have $A_M = R_M = {}_MA$. Next assume that $P \subseteq M$ and set $m = M \cap D$, a maximal ideal of D. By [M2, Theorem 2.5] and [M1, Lemma 2.4], R_m is a semi-local Bezout order. Thus, by Lemma 1.3, we may assume that R is a semi-local Bezout order with D a valuation ring. By Lemma 1.4, Theorem 1.5 and Corollary 1.7, A_P is P'-primary and ${}_PA = A_P$, where $P' = J(R_P)$. Set $p = D \cap P$. Then $R_P = (R_M)_{P_M} = (R_M)_p$ by [D2, Section 2, Theorem 1] (note that $Z(R_M) = D = Z(R)$). Thus we have $A_P = (A_M)_{P_M} = (A_M)_p$. Now we show that $A_P \cap R_M = (A_M)_p \cap R_M = A_M$. To show this, let $x \in (A_M)_p \cap R_M$. Then $xd \in A_M$ for some $d \in D \setminus p$. By the Ore condition, there exists $c \in C(M)$ such that $xc \in R$ and $xdc \in A$. Then $A \supseteq xdcR = xcRd$ and $d \notin P$ imply $xc \in A$, and $x \in A_M$ follows. In a similar way, we have ${}_PA \cap R_P = {}_MA$ and hence ${}_MA = {}_PA \cap R_M = A_P \cap R_M = A_M$. Further, by Lemma 1.3, A_M is a P_M-primary ideal of R_M, because A_P is P'-primary.

Lemma 2.4 *Let A_1 and A_2 be P-primary ideals of R. Then A_1A_2 is also a P-primary ideal.*

Proof It is clear that $\sqrt{A_1A_2} = P$. Let M be any maximal ideal of R. If $P \nsubseteq M$, then $A_1A_2 \nsubseteq M$ and so $(A_1A_2)_M = R_M = {}_M(A_1A_2)$. If $P \subseteq M$, then, by Lemma 2.3, we have $(A_1A_2)_M = A_1A_2 \cdot R_M = A_1R_M \cdot A_2R_M = R_MA_1 \cdot R_MA_1 = R_M(A_1A_2) = {}_M(A_1A_2)$, an ideal of R_M. By Lemma 2.3, $A_{1\,M}$ and $A_{2\,M}$ are P_M-primary ideals of a Dubrovin valuation ring R_M. This implies that $(A_1A_2)_M = A_{1\,M} \cdot A_{2\,M}$ is a P_M-primary ideal of R_M by [MMU, Corollary 7]. Therefore A_1A_2 is P-primary by Lemma 2.2.

Lemma 2.5 *Let P be a prime ideal of R and let $P' = P_P$. Then the mappings*

$$P_1 \to P_1' = P_{1\,P} \quad \textit{and} \quad P_1' \to P_1 = P_1' \cap R$$

give a bijective correspondence between the set $\{P_1 \in \operatorname{Spec}(R) \mid P_1 \subseteq P\}$ and the set $\{P_1' \in \operatorname{Spec}(R_P) \mid P_1' \subseteq P'\}$. In particular, the set $\{P_1 \in \operatorname{Spec}(R) \mid P_1 \subseteq P\}$ is linearly ordered by inclusion.

Proof If P is a prime ideal of R such that $P_1 \subseteq P$, then $C(P) \subseteq C(P_1)$ by [MMU, Lemma 1]. Hence $P_{1\,P} \cap R = P_1$ and $P_{1\,P}$ is a prime ideal of R_P. Conversely, let $P_1'(\subseteq P')$ be a prime ideal of R_P. Then there exists a Dubrovin valuation ring $S(\supseteq R_P)$ with $J(S) = P_1'$ by [D2, Theorem 1, Section 2], and so $P_1 := P_1' \cap R(\subseteq P)$ is a prime ideal of R such that $P_1' = P_{1\,P}$ by [M2, Proposition 2.7]. The last statement follows from [D1, Section 2, Theorem 4].

A prime ideal P of R is said to be *branched* if there exists a P-primary ideal A of R such that $A \neq P$. Otherwise, P is called an *unbranched* prime ideal.

Lemma 2.6 *Let P be a prime ideal of R. Then*

(1) P is branched if and only if P_P is branched.
(2) P is idempotent if and only if P_P is idempotent.

Proof As in the proof of Lemma 2.3, we may assume that R is semi-local Bezout. Then (1) follows from Theorem 1.5 and Corollary 1.6.

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(2) If P is idempotent, then it is clear that P_P is idempotent. The converse follows from Corollary 1.6 and Lemma 2.4.

Now we are going to prove the main theorem of this paper concerning branched and unbranched prime ideals of a Prüfer order which extend our earlier results in the case of Dubrovin valuation rings.

Theorem 2.7 *Let R be a Prüfer order in a simple Artinian ring Q with finite dimension over its center F. Suppose that the center of R is a Prüfer domain. Let P be a prime ideal of R.*

(1) If P is branched and $P \neq P^2$, then

- *(i) $\{P^k \mid k > 0\}$ is the full set of P-primary ideals of R, and*
- *(ii) $P_0 = \bigcap_{n=1}^{\infty} P^n$ is a prime ideal and there are no prime ideals P_1 such that $P_0 \subset P_1 \subset P$.*

(2) If P is branched and $P = P^2$, then

- *(i) for any P-primary ideal $A(\neq P)$,*

$$P_0 = \bigcap_{n=1}^{\infty} A^n = \bigcap \{A_\lambda \mid A_\lambda : P\text{-primary ideal}\},$$

- *(ii) P_0 is a prime ideal of R, and*
- *(iii) there are no prime ideals P_1 with $P_0 \subset P_1 \subset P$.*

(3) The following are equivalent:

- *(i) P is branched.*
- *(ii) There exists an ideal C of R with $\sqrt{C} = P$ and $C \neq P$.*
- *(iii) There exists $x \in R$ such that P is a minimal prime ideal over RxR.*
- *(iv) $P \neq \bigcup \{P_\lambda \mid P_\lambda \in \operatorname{Spec}(R) \text{ with } P_\lambda \subset P\}$.*
- *(v) There is a prime ideal P_0 of R such that $P_0 \subset P$ and there are no prime ideals P_1 with $P_0 \subset P_1 \subset P$.*

(4) P is unbranched if and only if $P = \bigcup \{P_\lambda \mid P_\lambda \in \operatorname{Spec}(R) \text{ with } P_\lambda \subset P\}$.

Proof As noted in the proof of Lemma 2.3, we may assume that R is semi-local Bezout.

(1) (i) By Lemma 2.6, $P \neq P^2$ if and only if $P_P \neq P_P^2$. Thus $\{{P_P}^k \mid k > 0\}$ is the full set of P_P-primary ideals of R_P by [MMU, Theorem 12]. It follows from Corollary 1.6 and Lemma 2.4 that $\{P^k \mid k > 0\}$ is the full set of P-primary ideals of R. (ii) is clear from Lemma 2.5 and [MMU, Theorem 12].

(2) (i) Let A be a P-primary ideal of R with $A \neq P$. By Corollary 1.6, $A_P \neq P_P$. Also, by Corollary 1.6 and Theorem 1.5, $\{A_{\lambda P} \mid A_\lambda : P\text{-primary ideal of } R\}$ is the full set of P_P-primary ideals of R_P. Hence $\bigcap_\lambda A_{\lambda P} = \bigcap_{n=1}^{\infty} {A_P}^n$ by [MMU, Theorem 12]. So we have $\bigcap_\lambda A_\lambda = (\bigcap_\lambda A_{\lambda P}) \cap R = (\bigcap_{n=1}^{\infty} {A_P}^n) \cap R = \bigcap_{n=1}^{\infty} A^n = A_0$ by Lemma 2.4 and Theorem 1.5. (ii) By [MMU, Theorem 12], $\bigcap_{n=1}^{\infty} {A_P}^n$ is a prime ideal of R_P and so A_0 is a prime ideal of R by Lemma 2.5. (iii) follows immediately from [MMU, Theorem 12] and Lemma 2.5.

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(3) (i) $\Rightarrow$ (ii) is clear from definition.

(ii) $\Rightarrow$ (iii): Let C be an ideal of R such that $C \subset P$ and $\sqrt{C} = P$. Then there exists a maximal ideal M of R such that $C_M \subset P_M$ by [G2, Proposition 3.1]. It follows that $P \subseteq M$, because $C_N = R_N = P_N$ for any maximal ideal N of R with $P \nsubseteq N$. Take any element $a \in P \setminus C$ with $a \notin C_M$ and $c_N \in C \cap C(N)$ for any maximal ideal N of R with $P \nsubseteq N$. Set $I = RaR + \sum Rc_N R (\subseteq P)$. Then $I = RbR$ for any $b \in I$ such that $bR = aR + \sum c_N R$ (note that we assume R is semi-local Bezout). To prove that $\sqrt{I} = P$, let P_1 be any prime ideal of R with $P_1 \supseteq I$ and let M_1 be a maximal ideal of R containing P_1. If $M_1 \nsupseteq P$, then $R_{M_1} = I_{M_1} \subseteq P_{1\,M_1} \subset R_{M_1}$, a contradiction. If $M_1 \supseteq P$, then either $P_1 \supseteq P$ or $P \supseteq P_1$ by Lemma 2.5. If $P \supseteq P_1$, then $P_M \supseteq P_{1\,M} \supseteq I_M \supset C_M$ by [D1, Section 2. Corollary to Lemma 2] and so $P \supseteq P_1 \supseteq C$. Hence $P = P_1$, proving that $P = \sqrt{I}$, that is, P is the minimal prime ideal over I.

(iii) $\Rightarrow$ (iv): Let a be an element of R such that P is the minimal prime ideal over RaR. Then $a \notin \bigcup\{P_\lambda \mid P_\lambda \in \operatorname{Spec}(R)$ with $P_\lambda \subset P\}$. Hence $P \neq \bigcup\{P_\lambda \mid P_\lambda \in \operatorname{Spec}(R)$ with $P_\lambda \subset P\}$.

(iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i) follow from [MMU, Theorem 12], Lemmas 2.5 and 2.6.

(4) follows from (3).

Let R be a Prüfer order in a simple Artinian ring with finite dimension over its center. If R is integral over its center $Z(R)$, then $Z(R)$ is a Prüfer domain by [MM1, Theorem 1.3] and so Theorem 2.7 is valid for such Prüfer orders. But there exists a Prüfer order with $Z(R)$ not Prüfer (*e.g.* [G2, Section 3 Example 1]). In the case when $Z(R)$ is not Prüfer, we do not know whether Theorem 2.7 still holds or not.

Next we give some examples of Prüfer orders.

Example 1 Let $\tilde{Q}$ be the field of all algebraic numbers, let $\tilde{Z}$ be the ring of all algebraic integers, and let $D = \tilde{Z}_S$, where $S = \{2^n \mid n = 0, 1, 2, 3, \dots\}$. Let σ be the automorphism of $\tilde{Q}$ defined by $\sigma(a+bi) = a - bi$ and let $G = \langle\sigma\rangle$ be the cyclic group generated by σ. Now let $R = D * G$ be the skew group ring of G over D. If p is a prime ideal of D, then we set $p^\sigma = \{\sigma(a) \mid a \in p\}$ and $p_0 = p \cap p^\sigma$. Then we have $D*G = \bigcap_{p_0}(D*G)_{p_0} = \bigcap_{p_0}(D_{p_0}*G)$, where $p_0 = p \cap p^\sigma$ and p runs over all prime ideals of D. It is checked that D_{p_0} satisfies the conditions of [MY, Theorem 3.5], and so $D_{p_0} * G$ is a Dubrovin valuation ring. Hence, for any finitely generated right R-ideal I, we have $I^{-1}I = \bigcap_{p_0}(I^{-1}I)_{p_0} = \bigcap_{p_0}(I_{p_0})^{-1}I_{p_0} = \bigcap_{p_0}(D_{p_0} * G) = R$. Also, we have $II^{-1} = O_l(I)$. Similarly, we have $J^{-1}J = O_r(J)$ and $JJ^{-1} = R$ for any finitely generated left R-ideal J. Thus $R = D * G$ is a Prüfer order.

***Example 2* (*cf.* [G2, Section 3])** Let F be a commutative field and let K be a finite cyclic Galois extension of F with Galois group $\langle\sigma\rangle$ and $n = |\langle\sigma\rangle|$. Let V be a valuation ring of F whose maximal ideal p is branched and idempotent (*e.g.* [H, Example 31]) and let W be the integral closure of V in K, which is a semilocal Bezout domain. Consider the division ring $D = K\big((x, \sigma)\big)$ of all twisted Laurent series where multiplication is defined by $xk = \sigma(k)x$ for all $k \in K$. Then $F\big((x^n, \sigma^n)\big)$ $(= F\big((x^n)\big))$ is the center of D and D is finite dimensional over its center. Now let $B = \{k_0 + k_1 x + \cdots \mid k_i \in K\}$. Then B is an invariant valuation ring of D, that is, for any non-zero element $d \in D$, either $d \in B$ or $d^{-1} \in B$ holds and we have $dBd^{-1} = B$. xB is the unique maximal ideal of B. Further $C = \{f_0 + f_n x^n + f_{2n}x^{2n} + \cdots \mid f_i \in F\}$ is the center of B and $C = B \cap F\big((x^n)\big)$. Then $R = W + xB$ is a Prüfer (actually

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Bezout) order in D and $S = V + x^n C$ is the center of R which is a valuation ring. If m is a maximal ideal of W, then $m + xB$ is a branched and idempotent prime ideal of R by [MMU, Corollary 13] and Lemma 2.6 because $(m+xB)_{(m+xB)} \cap S = (m_m + xB) \cap S = p + x^n C$, which is a branched and idempotent prime ideal of S. If we take V to be a valuation ring whose maximal ideal is unbranched (*e.g.* [H, Example 36]), then we can construct an unbranched prime ideal in a similar way.

We close this paper with the following.

Proposition 2.8 *Let A_1 and A_2 be primary ideals of a Prüfer order R. Then*

$$A_1 + A_2 = R \quad \text{or} \quad A_1 \supseteq A_2 \quad \text{or} \quad A_1 \subseteq A_2.$$

Proof Assume that $R \supset A_1 + A_2$. Then these exists a maximal ideal M of R such that $M \supseteq A_1 + A_2$. Let $P_i = \sqrt{A_i}$ $(i = 1, 2)$. Then $A_{i\,M}$ is a $P_{i\,M}$-primary ideal of R_M by Lemma 2.3. So, by [D1, Theorem 4, Section 2], we have $A_{1\,M} \supseteq A_{2\,M}$ or $A_{1\,M} \subseteq A_{2\,M}$. Now, by [MM2, Lemma 1] and Theorem 1.5, $A_i = A_{i\,P_i} \cap R \supseteq A_{i\,M} \cap R \supseteq A_i$, so $A_i = A_{i\,M} \cap R$. Hence we have either $A_1 \supseteq A_2$ or $A_1 \subseteq A_2$.

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