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MULTIPLE POSITIVE SOLUTIONS FOR A CRITICAL GROWTH PROBLEM WITH HARDY POTENTIAL

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Abstract In this paper we study the existence and nonexistence of multiple positive solutions for the Dirichlet problem:

$$-\Delta u - \mu\frac{u}{|x|^2} = \lambda(1+u)^p, \quad u > 0, \quad u \in H_0^1(\Omega), \tag{$*$}$$

where $0 \leqslant \mu < (\frac{1}{2}(N-2))^2$, $\lambda > 0$, $1 < p \leqslant (N+2)/(N-2)$, $N \geqslant 3$. Using the sub–supersolution method and the variational approach, we prove that there exists a positive number λ^* such that problem $(*)$ possesses at least two positive solutions if $\lambda \in (0, \lambda^*)$, a unique positive solution if $\lambda = \lambda^*$, and no positive solution if $\lambda \in (\lambda^*, \infty)$.

Keywords: positive solution; subsolution and supersolution; Palais–Smale condition; critical Sobolev exponent

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1. Introduction and main results

Let $\Omega \subset R^N$ $(N \geqslant 3)$ be a bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$. We are concerned with the following semilinear elliptic problem:

$$\left.\begin{aligned} -\Delta u - \mu\frac{u}{|x|^2} &= \lambda(1+u)^p \quad \text{in } \Omega, \\ u &> 0 \qquad\qquad\quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1_λ}$$

where $0 < \mu < (\frac{1}{2}(N-2))^2$, $\lambda > 0$, $1 < p \leqslant (N+2)/(N-2)$.

A positive function $u \in H_0^1(\Omega)$ is said to be a solution of problem $(1.1)_\lambda$ if u satisfies

$$\int_\Omega \left(\nabla u \cdot \nabla v - \mu\frac{uv}{|x|^2} - \lambda(1+u)^p v\right) \mathrm{d}x = 0 \quad \forall v \in H_0^1(\Omega).$$

It easily follows from standard regularity theory that, for any solution u of problem $(1.1)_\lambda$, $u \in C^2(\Omega\backslash\{0\}) \cap C^1(\bar{\Omega}\backslash\{0\})$.

Set

$$D^{1,2}(R^N) = \{u \in L^{2^*}(R^N) \mid |\nabla u| \in L^2(R^N)\}.$$

Let $\bar{\mu} = (\frac{1}{2}(N-2))^2$. We then define the constant S_μ, for all $\mu \in [0, \bar{\mu})$, as follows:

$$S_\mu := \inf_{u \in D^{1,2}(R^N)\setminus\{0\}} \frac{\int_{R^N}(|\nabla u|^2 - \mu(u^2/|x|^2))\,\mathrm{d}x}{(\int_{R^N}|u|^{2^*}\,\mathrm{d}x)^{2/2^*}},$$

where $2^* = 2N/(N-2)$ is the so-called critical Sobolev exponent for the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$.

From [**11**, **12**], S_μ is independent of any $\Omega \subset R^N$ in the sense that, if

$$S_\mu(\Omega) := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega(|\nabla u|^2 - \mu(u^2/|x|^2))\,\mathrm{d}x}{(\int_\Omega |u|^{2^*}\,\mathrm{d}x)^{2/2^*}},$$

then $S_\mu(\Omega) = S_\mu(R^N) = S_\mu$.

Set

$$\gamma = \sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}, \qquad \gamma' = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}.$$

Catrina and Wang [**7**] and Terracini [**14**] proved that, for $\varepsilon > 0$,

$$U_\varepsilon(x) = \frac{(4\varepsilon^2 N(\bar{\mu} - \mu)/(N-2))^{(N-2)/4}}{(\varepsilon^2|x|^{\gamma'/\sqrt{\bar{\mu}}} + |x|^{\gamma/\sqrt{\bar{\mu}}})^{\sqrt{\bar{\mu}}}}$$

satisfies

$$\left.\begin{aligned} -\Delta u = |u|^{2^*-2}u + \mu\frac{u}{|x|^2} \quad &\text{in } R^N\setminus\{0\}, \\ u \to 0 \quad \text{as } |x| \to \infty.& \end{aligned}\right\} \tag{1.1}$$

From Theorem B in [**8**], all the positive solutions of problem (1.1) must have the form of U_ε. Moreover, U_ε achieves S_μ.

In the case $\mu = 0$, problem $(1.1)_\lambda$ has been studied extensively. In the celebrated paper by Brezis and Nirenberg [**5**], it was proved that there exists a $\Lambda^* > 0$ such that problem $(1.1)_\lambda$ (with $\mu = 0$) admits at least two positive solutions if $0 < \lambda < \Lambda^*$, a unique positive solution if $\lambda = \Lambda^*$, and no positive solutions if $\lambda > \Lambda^*$. To prove this, two important facts are needed. One is that any solution of problem $(1.1)_\lambda$ (with $\mu = 0$) belongs to $L^\infty(\Omega)$; the other is that, if $u \in H_0^1(\Omega)$ is a solution of $(1.1)_\lambda$ (with $\mu = 0$), then a standard regularity argument shows that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. The maximum principle implies that u has a positive lower bound in any neighbourhood of zero, which is the key point in the energy estimate (see the related papers [**2**, **10**, **13**], and the references therein).

By the Hardy inequality (see [**3**])

$$\int_\Omega \frac{u^2}{|x|^2}\,\mathrm{d}x \leqslant \frac{1}{\bar{\mu}}\int_\Omega |\nabla u|^2\,\mathrm{d}x \quad \forall u \in H_0^1(\Omega),$$

we infer that, for $0 < \mu < \bar{\mu}$, the operator $-\Delta - (\mu/|x|^2)$ is positive. In this paper we use the sub–supersolution method and the variational approach to deal with problem $(1.1)_\lambda$

for $\lambda > 0$. Due to the presence of the term $\mu(u/|x|^2)$, problem $(1.1)_\lambda$ has a strong singularity at zero, and we do not expect any solution of problem $(1.1)_\lambda$ to be bounded. So there is some new difficulty to overcome in studying problem $(1.1)_\lambda$.

Our main results are as follows.

Theorem 1.1. *Let $0 < \mu < \bar{\mu}$. Then there exists a $\lambda^* > 0$ such that*

(1) *problem $(1.1)_\lambda$ possesses a minimal solution u_λ if $\lambda \in (0, \lambda^*)$, and there are no solutions for problem $(1.1)_\lambda$ if $\lambda > \lambda^*$;*

(2) *there is a unique solution of problem $(1.1)_\lambda$ for $\lambda = \lambda^*$;*

(3) *$u_\lambda(x)$ is increasing with respect to $\lambda \in (0, \lambda^*)$ for all $x \in \Omega$;*

(4) *$\|u_\lambda\|_{H_0^1(\Omega)} \to 0$ as $\lambda \to 0$; and*

(5) *there exists a $c_0 > 0$ such that $u_\lambda(x) \geqslant c_0|x|^{-(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})}$ for any $x \in B_{r_0}(0)\backslash\{0\}$, where $0 < r_0 < \inf_{x\in\partial\Omega}|x|$.*

Remark. As we see below (Lemma 2.6), any solution of problem $(1.1)_\lambda$ with $\lambda \in (0, \lambda^*]$ has the property (5) in Theorem 1.1.

Theorem 1.2. *Let $0 < \mu \leqslant \bar{\mu} - 1$. If $\lambda \in (0, \lambda^*)$, then problem $(1.1)_\lambda$ admits the second solution U_λ satisfying $U_\lambda > u_\lambda$ in Ω.*

Throughout this paper we denote the norm of $H_0^1(\Omega)$ by

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x\right)^{1/2},$$

the norm of $L^l(\Omega)$ $(1 \leqslant l < \infty)$ by

$$\|u\|_{L^l(\Omega)} = \left(\int_\Omega |u|^l \, \mathrm{d}x\right)^{1/l},$$

and positive constants (possibly different) by $C, C_1, C_2, \dots$.

2. Proof of Theorem 1.1

Before giving the proof of Theorem 1.1, we introduce some notation and preliminary lemmas.

Lemma 2.1. *There exists a $\lambda_0 > 0$ such that problem $(1.1)_\lambda$ has a solution for all $\lambda \in (0, \lambda_0)$.*

Proof. Obviously, 0 is a subsolution of problem $(1.1)_\lambda$ for any $\lambda > 0$. By the sub–supersolution principle, we know that to obtain a solution of problem $(1.1)_\lambda$, we only need to find an $H_0^1(\Omega)$ supersolution of $(1.1)_\lambda$. Set $Q(t) = t^2 + (N-2)t + \mu$, $0 < \mu < \bar{\mu}$.

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A direct computation then shows that $-\gamma$, $-\gamma'$ are two roots of $Q(t) = 0$, and that $\gamma' p < p + 2$, where

$$\gamma = \sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}, \qquad \gamma' = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}.$$

Furthermore, we can choose $t_0 \in (\gamma', \gamma)$ and t_0 close enough to γ', such that $t_0 p < p + 2$, $|x|^{-t_0} \in H^1(\Omega)$ and $Q(-t_0) < 0$.

Set $w = A|x|^{-t_0}$ ($A > 0$ will be fixed below). After a simple calculation, we have

$$-\Delta w - \mu \frac{w}{|x|^2} = -AQ(-t_0)|x|^{-t_0-2}.$$

We want to show that

$$-AQ(-t_0)|x|^{-t_0-2} > \lambda(1 + w)^p = \lambda(1 + A|x|^{-t_0})^p \quad \text{in } \Omega. \tag{2.1}$$

This will be true when

$$\left.\begin{aligned} -\tfrac{1}{2}AQ(-t_0)|x|^{-t_0-2} &> \lambda 2^p, \\ -\tfrac{1}{2}AQ(-t_0)|x|^{-t_0-2} &> \lambda 2^p A^p |x|^{-t_0 p}. \end{aligned}\right\} \tag{2.2}$$

Choose

$$\lambda_0 = -\frac{1}{2^{p+1}} Q(-t_0)(\operatorname{diam}\Omega)^{-t_0-2} \quad \text{and} \quad A < \left(-\frac{Q(-t_0)}{\lambda_0 2^{p+1} (\operatorname{diam}\Omega)^{t_0+2-t_0 p}} \right)^{1/p}.$$

Then (2.2) can be satisfied for any $\lambda \in (0, \lambda_0)$ with such a choice of λ_0 and A, and we have constructed $w \in H^1(\Omega)$ such that, for any $\lambda \in (0, \lambda_0)$,

$$\begin{aligned} -\Delta w - \mu \frac{w}{|x|^2} &> \lambda(1 + w)^p \quad \text{in } \Omega, \\ w &> 0 \qquad\qquad\quad \text{on } \bar{\Omega}. \end{aligned}$$

Finally, we need to construct an $H_0^1(\Omega)$ supersolution of $(1.1)_\lambda$ for $\lambda \in (0, \lambda_0)$. Let w_1 be the unique positive solution of

$$\begin{aligned} -\Delta w_1 - \mu \frac{w_1}{|x|^2} &= 0 \quad \text{in } \Omega, \\ w_1 &= w \quad \text{on } \partial\Omega. \end{aligned}$$

Set $\tilde{w} = w - w_1$; $\tilde{w}$ then satisfies

$$\begin{aligned} -\Delta \tilde{w} - \mu \frac{\tilde{w}}{|x|^2} > \lambda(1 + w)^p &> 0 \quad \text{in } \Omega, \\ \tilde{w} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

By the maximum principle, $\tilde{w} > 0$ in Ω, and then

$$-\Delta \tilde{w} - \mu \frac{\tilde{w}}{|x|^2} > \lambda(1 + w)^p = \lambda(1 + \tilde{w} + w_1)^p > \lambda(1 + \tilde{w})^p,$$

which implies that $\tilde{w}$ is an $H_0^1(\Omega)$ supersolution of $(1.1)_\lambda$ for $\lambda \in (0, \lambda_0)$. Then the sub–supersolution method shows that $(1.1)_\lambda$ has a solution $\tilde{u}_\lambda$ with the property $0 \leqslant \tilde{u}_\lambda \leqslant \tilde{w}$ for $\lambda \in (0, \lambda_0)$. By the maximum principle, we conclude that $\tilde{u}_\lambda > 0$ in Ω. □

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Set

$$\lambda^* = \sup\{\tilde{\lambda} \in \mathbb{R}^+ \mid \text{problem } (1.1)_\lambda \text{ has at least one solution for } \lambda \in (0, \tilde{\lambda})\}.$$

We then have the following lemma.

Lemma 2.2. $0 < \lambda^* < \infty$.

Proof. It follows from Lemma 2.1 that $\lambda^* > 0$. Now we show $\lambda^* < \infty$.

Let $\lambda_1(\mu)$ be the first eigenvalue of the operator: $-\Delta - (\mu/|x|^2)$ with $0 < \mu < \bar{\mu}$, and $\varphi_1 > 0$ (the corresponding eigenfunction). Assume that $u \in H_0^1(\Omega)$ is a solution of $(1.1)_\lambda$. We then have

$$\lambda_1(\mu) \int_\Omega u\varphi_1 \,\mathrm{d}x = \lambda \int_\Omega (1+u)^p \varphi_1 \,\mathrm{d}x \geqslant \frac{\lambda p^p}{(p-1)^{p-1}} \int_\Omega u\varphi_1 \,\mathrm{d}x, \tag{2.3}$$

where we use the inequality

$$(1+s)^p \geqslant \frac{p^p}{(p-1)^{p-1}} s$$

for all $s \geqslant 0$.

Therefore, from (2.3), we infer that

$$\lambda \leqslant \frac{(p-1)^{p-1}}{p^p} \lambda_1(\mu),$$

which implies that $\lambda^* < \infty$. □

Lemma 2.3. *For any $\lambda \in (0, \lambda^*)$, problem $(1.1)_\lambda$ has a minimal solution u_λ. Moreover, u_λ is increasing with respect to λ.*

Proof. We already know that there exists a solution u of $(1.1)_\lambda$ for every $\lambda \in (0, \lambda^*)$. Obviously, 0 is a subsolution of $(1.1)_\lambda$. Using the method of monotone iteration and the maximum principle, it follows that there exists a solution u_λ of $(1.1)_\lambda$ such that $0 < u_\lambda \leqslant u$ for all $x \in \Omega$, and u_λ is a minimal solution of problem $(1.1)_\lambda$ for $\lambda \in (0, \lambda^*)$. Similarly, we can also prove that u_λ is increasing with respect to λ. □

Let u_λ be the minimal solution of problem $(1.1)_\lambda$ obtained in Lemma 2.3. We consider the following eigenvalue problem with respect to m:

$$\left.\begin{aligned} -\Delta\psi - \mu\frac{\psi}{|x|^2} &= m\lambda p(1+u_\lambda)^{p-1}\psi \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.4}$$

We then have the following lemma.

Lemma 2.4. *The first eigenvalue of problem (2.4),*

$$m(\lambda) = \inf\left\{ \int_\Omega \left(|\nabla\psi|^2 - \mu\frac{\psi^2}{|x|^2} \right) \mathrm{d}x \;\middle|\; \psi \in H_0^1(\Omega),\ \lambda p \int_\Omega (1+u_\lambda)^{p-1}\psi^2 \,\mathrm{d}x = 1 \right\}, \tag{2.5}$$

can be achieved by a function $\psi_1 > 0$ in $H_0^1(\Omega)$ if $\lambda \in (0, \lambda^*)$. Furthermore, $m(\lambda) > 1$.

Proof. Observe that, for any $\psi \in H_0^1(\Omega)\backslash\{0\}$,

$$\begin{aligned}
&\int_\Omega \left(|\nabla\psi|^2 - \mu\frac{\psi^2}{|x|^2}\right)\mathrm{d}x - \varepsilon\lambda p\int_\Omega (1+u_\lambda)^{p-1}\psi^2\,\mathrm{d}x \\
&\quad\geqslant \int_\Omega \left(|\nabla\psi|^2 - \mu\frac{\psi^2}{|x|^2}\right)\mathrm{d}x - \varepsilon\lambda p\biggl(\int_\Omega (1+u_\lambda)^{p+1}\,\mathrm{d}x\biggr)^{(p-1)/(p+1)}\biggl(\int_\Omega \psi^{p+1}\,\mathrm{d}x\biggr)^{2/(p+1)} \\
&\quad\geqslant (1-\varepsilon C(\lambda,p))\left(1-\frac{\mu}{\bar\mu}\right)\|\psi\|^2_{H_0^1(\Omega)} \\
&\quad> 0 \quad \text{with } 0<\varepsilon<1/C(\lambda,p),
\end{aligned}$$

where $C(\lambda,p)>0$ is a constant depending only on λ and p.

Thus we conclude that $m(\lambda)\geqslant\varepsilon>0$ with $0<\varepsilon<1/C(\lambda,p)$. Since $(1+u_\lambda)^{p-1}\in L^{N/2}(\Omega)$, by choosing a minimizing sequence in $H_0^1(\Omega)$, we can easily prove that the infimum in (2.5) can be achieved by a function $\psi_1>0$ in Ω. So it remains to prove that $m(\lambda)>1$. For any $\lambda\in(0,\lambda^*)$, there exists a $\bar\lambda\in(\lambda,\lambda^*)$. Suppose that u_λ, $u_{\bar\lambda}$ are the minimal solutions of $(1.1)_\lambda$, $(1.1)_{\bar\lambda}$, respectively. Then, by Lemma 2.3, we have $0<u_\lambda<u_{\bar\lambda}$ in Ω, and

$$\begin{aligned}
-\Delta(u_{\bar\lambda}-u_\lambda) - \mu\frac{u_{\bar\lambda}-u_\lambda}{|x|^2} &= \bar\lambda(1+u_{\bar\lambda})^p - \lambda(1+u_\lambda)^p \\
&> \lambda((1+u_\lambda+(u_{\bar\lambda}-u_\lambda))^p-(1+u_\lambda)^p) \\
&\geqslant \lambda p(1+u_\lambda)^{p-1}(u_{\bar\lambda}-u_\lambda). \qquad (2.6)
\end{aligned}$$

Thus, from (2.5) and (2.6), we deduce that

$$\begin{aligned}
\lambda m(\lambda)p\int_\Omega (1+u_\lambda)^{p-1}(u_{\bar\lambda}-u_\lambda)\psi_1\,\mathrm{d}x &= \int_\Omega\left(-\Delta\psi_1-\mu\frac{\psi_1}{|x|^2}\right)(u_{\bar\lambda}-u_\lambda)\,\mathrm{d}x \\
&= \int_\Omega\left(-\Delta(u_{\bar\lambda}-u_\lambda)-\mu\frac{u_{\bar\lambda}-u_\lambda}{|x|^2}\right)\psi_1\,\mathrm{d}x \\
&> \lambda p\int_\Omega (1+u_\lambda)^{p-1}(u_{\bar\lambda}-u_\lambda)\psi_1\,\mathrm{d}x,
\end{aligned}$$

which implies that $m(\lambda)>1$. □

Lemma 2.5. *There exists a unique solution of problem* $(1.1)_\lambda$ *for* $\lambda=\lambda^*$.

Proof. First we prove the existence of a minimal solution of problem $(1.1)_\lambda$ for $\lambda=\lambda^*$. Multiplying both sides of $(1.1)_\lambda$ by u_λ for $\lambda\in(0,\lambda^*)$, we get

$$\int_\Omega\left(|\nabla u_\lambda|^2-\mu\frac{u_\lambda^2}{|x|^2}\right)\mathrm{d}x = \lambda\int_\Omega(1+u_\lambda)^p u_\lambda\,\mathrm{d}x.$$

On the other hand, using Lemma 2.4, we derive

$$\int_\Omega\left(|\nabla u_\lambda|^2-\mu\frac{u_\lambda^2}{|x|^2}\right)\mathrm{d}x > \lambda p\int_\Omega(1+u_\lambda)^{p-1}u_\lambda^2\,\mathrm{d}x.$$

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Therefore, we obtain

$$p\int_\Omega (1+u_\lambda)^{p-1}u_\lambda^2\,\mathrm{d}x < \int_\Omega (1+u_\lambda)^p u_\lambda\,\mathrm{d}x = \int_\Omega (1+u_\lambda)^{p-1}(u_\lambda+u_\lambda^2)\,\mathrm{d}x.$$

We then have

$$(p-1)\int_\Omega u_\lambda^{p+1}\,\mathrm{d}x < (p-1)\int_\Omega (1+u_\lambda)^{p-1}u_\lambda^2 < \int_\Omega (1+u_\lambda)^{p-1}u_\lambda\,\mathrm{d}x \leqslant \varepsilon\int_\Omega u_\lambda^{p+1}\,\mathrm{d}x + C(\varepsilon). \tag{2.7}$$

Take $\varepsilon = \frac{1}{2}(p-1)$ in (2.7), we conclude that $\int_\Omega u_\lambda^{p+1}\,\mathrm{d}x \leqslant C$. Furthermore, by the Hardy inequality, we infer that

$$\begin{aligned}\int_\Omega |\nabla u_\lambda|^2\,\mathrm{d}x &\leqslant \left(1-\frac{\mu}{\bar\mu}\right)^{-1}\int_\Omega \left(|\nabla u_\lambda|^2 - \mu\frac{u_\lambda^2}{|x|^2}\right)\mathrm{d}x \\ &= \left(1-\frac{\mu}{\bar\mu}\right)^{-1}\lambda\int_\Omega (1+u_\lambda)^p u_\lambda\,\mathrm{d}x \\ &\leqslant \left(1-\frac{\mu}{\bar\mu}\right)^{-1}\lambda^* 2^p\int_\Omega (u_\lambda+u_\lambda^{p+1})\,\mathrm{d}x \\ &\leqslant C.\end{aligned} \tag{2.8}$$

Suppose that $\{\lambda_j\}_{j\geqslant 1}$ is an increasing sequence in $(0,\lambda^*)$ that satisfies $\lim_{j\to\infty}\lambda_j = \lambda^*$. The corresponding sequence of minimal solutions is $\{u_{\lambda_j}\}_{j\geqslant 1} \subset H_0^1(\Omega)$. From (2.8), and up to a subsequence, we may assume that, as $j\to\infty$,

$$\begin{aligned} u_{\lambda_j} &\rightharpoonup \bar u \quad \text{weakly in } H_0^1(\Omega), \\ u_{\lambda_j} &\rightharpoonup \bar u \quad \text{weakly in } L^2(\Omega, |x|^{-2}\,\mathrm{d}x), \\ u_{\lambda_j} &\rightharpoonup \bar u \quad \text{weakly in } L^{p+1}(\Omega) \text{ if } p = \frac{N+2}{N-2}, \\ u_{\lambda_j} &\to \bar u \quad \text{strongly in } L^{p+1}(\Omega) \text{ if } 1 < p < \frac{N+2}{N-2}, \\ u_{\lambda_j} &\to \bar u \quad \text{a.e. on } \Omega. \end{aligned}$$

Thus, for any $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned} 0 &= \int_\Omega \left(\nabla u_{\lambda_j}\cdot\nabla\varphi - \mu\frac{u_{\lambda_j}\varphi}{|x|^2} - \lambda_j(1+u_{\lambda_j})^p\varphi\right)\mathrm{d}x \\ &\to \int_\Omega \left(\nabla\bar u\cdot\nabla\varphi - \mu\frac{\bar u\varphi}{|x|^2} - \lambda^*(1+\bar u)^p\varphi\right)\mathrm{d}x \quad \text{as } j\to\infty, \end{aligned}$$

and hence

$$\int_\Omega \left(\nabla\bar u\cdot\nabla\varphi - \mu\frac{\bar u\varphi}{|x|^2} - \lambda^*(1+\bar u)^p\varphi\right)\mathrm{d}x = 0,$$

i.e. $\bar u$ is a solution of $(1.1)_{\lambda^*}$. Obviously, 0 is a subsolution of $(1.1)_{\lambda^*}$. For any solution u of $(1.1)_{\lambda^*}$, using the method of monotone iteration and the maximum principle, it follows that there exists a minimal solution u_{λ^*} of $(1.1)_{\lambda^*}$.

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Now we prove the uniqueness of problem $(1.1)_{\lambda^*}$. Suppose to the contrary that there is a different solution u^* of $(1.1)_{\lambda^*}$. That is,

$$\begin{aligned} -\Delta u^* - \mu\frac{u^*}{|x|^2} &= \lambda^*(1+u^*)^p && \text{in } \Omega,\\ u^* &> 0 && \text{in } \Omega,\\ u^* &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Moreover, $u_{\lambda^*} \leqslant u^*$ and $u_{\lambda^*} \not\equiv u^*$. Set

$$L(u^*, u_{\lambda^*}) = \int_0^1 (1 + u_{\lambda^*} + s(u^* - u_{\lambda^*}))^{p-1}\, \mathrm{d}s.$$

We then obtain

$$L(u^*, u_{\lambda^*}) > (1 + u_{\lambda^*})^{p-1} > 0, \tag{2.9}$$

and

$$\left.\begin{aligned} -\Delta(u^* - u_{\lambda^*}) - \mu\frac{u^* - u_{\lambda^*}}{|x|^2} &= p\lambda^* L(u^*, u_{\lambda^*})(u^* - u_{\lambda^*}) && \text{in } \Omega,\\ u^* - u_{\lambda^*} &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.10}$$

We claim that λ^* is the first eigenvalue of

$$\left.\begin{aligned} -\Delta u - \mu\frac{u}{|x|^2} &= ptL(u^*, u_{\lambda^*})u && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.11}$$

In fact, assume that $t_1 > 0$ is the first eigenvalue of (2.11) and that $\varphi_1^* > 0$ is the corresponding eigenfunction. Then, from (2.10), (2.11), we get

$$\begin{aligned} pt_1 \int_\Omega L(u^*, u_{\lambda^*})(u^* - u_{\lambda^*})\varphi_1^*\, \mathrm{d}x &= \int_\Omega \left(\nabla(u^* - u_{\lambda^*})\nabla\varphi_1^* - \mu\frac{(u^* - u_{\lambda^*})\varphi_1^*}{|x|^2}\right) \mathrm{d}x\\ &= p\lambda^* \int_\Omega L(u^*, u_{\lambda^*})(u^* - u_{\lambda^*})\varphi_1^*\, \mathrm{d}x. \end{aligned} \tag{2.12}$$

Since $u^* - u_{\lambda^*} \geqslant 0$ and $u^* - u_{\lambda^*} \not\equiv 0$ in Ω, from (2.9), (2.12), we infer that

$$\int_\Omega L(u^*, u_{\lambda^*})(u^* - u_{\lambda^*})\varphi_1^*\, \mathrm{d}x > 0, \quad \text{and then } \lambda^* = t_1.$$

Suppose that $s(\lambda^*) > 0$ is the first eigenvalue of

$$-\Delta e - \mu\frac{e}{|x|^2} = ps(1 + u_{\lambda^*})^{p-1}e, \quad e \in H_0^1(\Omega),$$

and $e_1 > 0$ is the corresponding eigenfunction. Then, from (2.9), (2.11), we deduce that

$$\begin{aligned}\lambda^* &= \inf_{u\in H_0^1(\Omega)\backslash\{0\}} \frac{\int_\Omega(|\nabla u|^2-\mu(u^2/|x|^2))\,\mathrm{d}x}{p\int_\Omega L(u^*,u_{\lambda^*})u^2\,\mathrm{d}x}\\ &\leqslant \frac{\int_\Omega(|\nabla e_1|^2-\mu(e_1^2/|x|^2))\,\mathrm{d}x}{p\int_\Omega L(u^*,u_{\lambda^*})e_1^2\,\mathrm{d}x}\\ &< \frac{\int_\Omega(|\nabla e_1|^2-\mu(e_1^2/|x|^2))\,\mathrm{d}x}{p\int_\Omega (1+u_{\lambda^*})^{p-1}e_1^2\,\mathrm{d}x}\\ &= s(\lambda^*). \end{aligned} \tag{2.13}$$

On the other hand, we claim that there is a non-trivial solution of the linear problem

$$-\Delta w-\mu\frac{w}{|x|^2} = p\lambda^*(1+u_{\lambda^*})^{p-1}w,\quad w\in H_0^1(\Omega).$$

In fact, it follows from Lemma 2.4 that

$$\int_\Omega\left(|\nabla\psi|^2-\mu\frac{\psi^2}{|x|^2}\right)\mathrm{d}x-\lambda p\int_\Omega(1+u_\lambda)^{p-1}\psi^2\,\mathrm{d}x\geqslant 0\quad\forall\psi\in H_0^1(\Omega),$$

and letting $\lambda\to\lambda^*$ we have

$$\int_\Omega\left(|\nabla\psi|^2-\mu\frac{\psi^2}{|x|^2}\right)\mathrm{d}x-\lambda^* p\int_\Omega(1+u_{\lambda^*})^{p-1}\psi^2\,\mathrm{d}x\geqslant 0\quad\forall\psi\in H_0^1(\Omega).$$

This implies that $\mu_1\geqslant 0$, where μ_1 denotes the first eigenvalue of the linear problem

$$-\Delta w-\mu\frac{w}{|x|^2}-p\lambda^*(1+u_{\lambda^*})^{p-1}w=\mu w,\quad w\in H_0^1(\Omega).$$

Now we prove that $\mu_1 = 0$. Suppose, by way of contradiction, that $\mu_1 > 0$ and introduce the function $F: H_0^1(\Omega)\times R\to H^{-1}(\Omega)$ defined by

$$F(v,\lambda) = -\Delta v-\mu\frac{v}{|x|^2}-\lambda((1+v)^+)^p.$$

Then, with $\mu_1 > 0$ we infer that the linear operator

$$w\mapsto\langle F_v'(u_{\lambda^*},\lambda^*),w\rangle = -\Delta w-\mu\frac{w}{|x|^2}-p\lambda^*(1+u_{\lambda^*})^{p-1}w$$

is an isomorphism and the 'implicit function theorem' applies, contradicting the maximality of λ^* (see, for example, [**9**], [**10**] and §§ 2 and 7 in [**6**]).

Hence, by the definition of $s(\lambda^*)$, we infer that $s(\lambda^*)\leqslant\lambda^*$, which contradicts (2.13). So the uniqueness of solutions of $(1.1)_{\lambda^*}$ is proved. □

Lemma 2.6. *For any solution u of problem $(1.1)_\lambda$ with $\lambda\in(0,\lambda^*]$, there exists a $c_0 > 0$ such that*

$$u(x)\geqslant c_0|x|^{-(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}$$

for any $x\in B_{r_0}(0)\backslash\{0\}$, where $0 < r_0 < \inf_{x\in\partial\Omega}|x|$.

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Proof. Set

$$v(x) = |x|^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}u(x).$$

As mentioned in § 1, $u \in C^2(\Omega\backslash\{0\})$. Then, after a direct computation, we have $v \in C^2(\Omega\backslash\{0\})$ and

$$\left.\begin{aligned} -\operatorname{div}(|x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\nabla v) &= \lambda|x|^{-(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}(1+|x|^{-(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}v)^p \quad &&\text{in } \Omega,\\ v &> 0 &&\text{in } \Omega,\\ v &= 0 &&\text{on } \partial\Omega. \end{aligned}\right\} \tag{2.14}$$

Let $0 < t_0 < r_0 < \inf_{x\in\partial\Omega}|x|$ and let $n(t) = \min_{|x|=t} v(x)$, $t_0 \leqslant t \leqslant r_0$, such that

$$n(t_0) = At_0^{-2\sqrt{\bar\mu-\mu}} + B, \qquad n(r_0) = Ar_0^{-2\sqrt{\bar\mu-\mu}} + B,$$

where

$$A = \frac{n(r_0)-n(t_0)}{r_0^{-2\sqrt{\bar\mu-\mu}} - t_0^{-2\sqrt{\bar\mu-\mu}}}, \qquad B = \frac{n(r_0)t_0^{-2\sqrt{\bar\mu-\mu}} - n(t_0)r_0^{-2\sqrt{\bar\mu-\mu}}}{t_0^{-2\sqrt{\bar\mu-\mu}} - r_0^{-2\sqrt{\bar\mu-\mu}}}.$$

It is easy to verify that

$$-\operatorname{div}(|x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\nabla(|x|^{-2\sqrt{\bar\mu-\mu}})) = 0 \quad \forall x \in \Omega\backslash\{0\}. \tag{2.15}$$

Combining (2.14) with (2.15) we get

$$\begin{aligned} -\operatorname{div}(|x|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}&\nabla(v - (A|x|^{-2\sqrt{\bar\mu-\mu}} + B)))\\ &= \lambda|x|^{-(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}(1+|x|^{-(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}v)^p\\ &> 0 \quad \text{in } \Omega\backslash\{0\}, \end{aligned}$$

and $v(x) - \min_{x\in\partial(B_{r_0}(0)\backslash B_{t_0}(0))} v(x) \geqslant 0$, $\forall x \in \partial(B_{r_0}(0)\backslash B_{t_0}(0))$.

Therefore, by the maximum principle, we obtain

$$\begin{aligned} v(x) &\geqslant A|x|^{-2\sqrt{\bar\mu-\mu}} + B\\ &= \frac{|x|^{-2\sqrt{\bar\mu-\mu}} - r_0^{-2\sqrt{\bar\mu-\mu}}}{t_0^{-2\sqrt{\bar\mu-\mu}} - r_0^{-2\sqrt{\bar\mu-\mu}}}n(t_0) + \frac{t_0^{-2\sqrt{\bar\mu-\mu}} - |x|^{-2\sqrt{\bar\mu-\mu}}}{t_0^{-2\sqrt{\bar\mu-\mu}} - r_0^{-2\sqrt{\bar\mu-\mu}}}n(r_0)\\ &\geqslant \frac{|x|^{2\sqrt{\bar\mu-\mu}} - t_0^{2\sqrt{\bar\mu-\mu}}}{|x|^{2\sqrt{\bar\mu-\mu}} - t_0^{2\sqrt{\bar\mu-\mu}}r_0^{-2\sqrt{\bar\mu-\mu}}|x|^{2\sqrt{\bar\mu-\mu}}}n(r_0) \quad \text{for all } x \in B_{r_0}(0)\backslash B_{t_0}(0). \end{aligned}$$

Let $t_0 \to 0$. We conclude that $v(x) \geqslant n(r_0) = \min_{|x|=r_0} v(x) > 0$, $\forall x \in B_{r_0}(0)\backslash\{0\}$. □

Proof of Theorem 1.1. Parts (1)–(3) and (5) of Theorem 1.1 are direct consequences of Lemmas 2.2, 2.3, 2.5 and 2.6. Now we prove (4). By (2.8) and the Sobolev inequality, we derive that $\int_\Omega |u_\lambda|^{p+1}\,\mathrm{d}x \leqslant C$ for all $\lambda \in (0, \lambda^*)$. Therefore,

$$\begin{aligned} \|u_\lambda\|^2_{H_0^1(\Omega)} &\leqslant \left(1 - \frac{\mu}{\bar\mu}\right)^{-1}\int_\Omega\left(|\nabla u_\lambda|^2 - \mu\frac{u_\lambda^2}{|x|^2}\right)\mathrm{d}x\\ &= \left(1 - \frac{\mu}{\bar\mu}\right)^{-1}\lambda\int_\Omega(1+u_\lambda)^p u_\lambda\,\mathrm{d}x \leqslant C\lambda \to 0, \end{aligned}$$

as $\lambda \to 0$. □

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3. Proof of Theorem 1.2

In this section, we deal only with the case in which $p = (N+2)/(N-2)$, because the subcritical case (i.e. $1 < p < (N+2)/(N-2)$) is trivial. In order to find the second solution of problem $(1.1)_\lambda$, we consider the following problem:

$$\left.\begin{aligned} -\Delta u - \mu\frac{u}{|x|^2} &= \lambda(1+u_\lambda+u)^p - \lambda(1+u_\lambda)^p && \text{in } \Omega,\\ u &> 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.1$_\lambda$}$$

Clearly, if problem $(3.1)_\lambda$ possesses a solution u, then $u + u_\lambda$ is another solution of problem $(1.1)_\lambda$. So we only need to prove that problem $(3.1)_\lambda$ admits one solution for $\lambda \in (0, \lambda^*)$.

Set $g(x,u) = (1+u_\lambda+u)^p - (1+u_\lambda)^p - u^p$ for $u \geqslant 0$, and $a(x) = p(1+u_\lambda)^{p-1}$. Since the values of $g(x,u)$ for $u < 0$ are irrelevant and we may assume that $g(x,u) = 0$ for $u < 0$, it will suffice to prove that

$$\left.\begin{aligned} -\Delta u - \mu\frac{u}{|x|^2} &= \lambda u^p + \lambda g(x,u) && \text{in } \Omega,\\ u &> 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.2$_\lambda$}$$

It is well known that solutions of problem $(3.2)_\lambda$ are equivalent to the non-zero critical points of the variational functional of problem $(3.2)_\lambda$:

$$I_{\lambda,\mu}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right)\mathrm{d}x - \frac{\lambda}{p+1}\int_\Omega (u^+)^{p+1}\,\mathrm{d}x - \lambda\int_\Omega G(x,u)\,\mathrm{d}x, \quad u \in H_0^1(\Omega),$$

where $u^+ = \max\{u, 0\}$, $G(x,u) = \int_0^u g(x,s)\,\mathrm{d}s$.

The functional $I_{\lambda,\mu} \in C^1(H_0^1(\Omega), R)$ is said to satisfy the Palais–Smale condition at the level c ($(\mathrm{PS})_c$ for short) if any sequence $\{u_n\} \subset H_0^1(\Omega)$ such that, as $n \to \infty$,

$$I_{\lambda,\mu}(u_n) \to c, \quad \mathrm{d}I_{\lambda,\mu}(u_n) \to 0 \text{ strongly in } H^{-1}(\Omega),$$

contains a subsequence converging in $H_0^1(\Omega)$ to a critical point of $I_{\lambda,\mu}$.

Lemma 3.1. *The functional $I_{\lambda,\mu}$ satisfies condition $(\mathrm{PS})_c$ with*

$$c < (1/N\lambda^{(N-2)/2})S_\mu^{N/2}.$$

Proof. Assume that $\{u_n\} \subset H_0^1(\Omega)$ is a Palais–Smale sequence of $I_{\lambda,\mu}$ at the level c, i.e.

$$I_{\lambda,\mu}(u_n) \to c, \quad \mathrm{d}I_{\lambda,\mu}(u_n) \to 0.$$

We then have

$$\begin{aligned}\lambda\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega (u_n^+)^{p+1}\,\mathrm{d}x &= I_{\lambda,\mu}(u_n)-\tfrac{1}{2}\langle \mathrm{d}I_{\lambda,\mu}(u_n),u_n\rangle+\lambda\int_\Omega (G(u_n^+)-\tfrac{1}{2}g(x,u_n^+)u_n^+)\,\mathrm{d}x\\ &= c+o(1)(1+\|u_n\|_{H_0^1(\Omega)})+\lambda\int_\Omega (G(u_n^+)-\tfrac{1}{2}g(x,u_n^+)u_n^+)\,\mathrm{d}x. \qquad (3.1)\end{aligned}$$

Observe that

$$\lim_{u\to+\infty}\frac{g(x,u)}{u^p}=0 \quad \text{uniformly in } x\in\Omega, \qquad (3.2)$$

$$\lim_{u\to 0^+}\frac{g(x,u)}{u}=a(x) \quad \text{uniformly in } x\in\Omega. \qquad (3.3)$$

Then, from (3.2) and (3.3), we deduce that, for any $\varepsilon>0$,

$$\int_\Omega (G(u_n^+)-\tfrac{1}{2}g(x,u_n^+)u_n^+)\,\mathrm{d}x \leqslant \frac{\varepsilon}{p+1}\int_\Omega (u_n^+)^{p+1}\,\mathrm{d}x+C(\varepsilon)\left(1+\int_\Omega (1+u_\lambda)^{p+1}\,\mathrm{d}x\right). \qquad (3.4)$$

Inserting (3.4) into (3.1), and taking $\varepsilon=\frac{1}{4}(p-1)$, we derive

$$\int_\Omega (u_n^+)^{p+1}\,\mathrm{d}x \leqslant C+o(\|u_n\|_{H_0^1(\Omega)}). \qquad (3.5)$$

Furthermore,

$$\begin{aligned}&\left(\frac{1}{2}-\frac{1}{p+1}\right)\|u_n\|^2_{H_0^1(\Omega)}\\ &\leqslant \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(1-\frac{\mu}{\bar\mu}\right)^{-1}\int_\Omega \left(|\nabla u_n|^2-\mu\frac{u_n^2}{|x|^2}\right)\mathrm{d}x\\ &= \left(1-\frac{\mu}{\bar\mu}\right)^{-1}\left(I_{\lambda,\mu}(u_n)-\frac{1}{p+1}\langle \mathrm{d}I_{\lambda,\mu}(u_n),u_n\rangle+\lambda\int_\Omega \left(G(u_n^+)-\frac{1}{p+1}g(x,u_n^+)u_n^+\right)\mathrm{d}x\right)\\ &= \left(1-\frac{\mu}{\bar\mu}\right)^{-1}\left(c+o(1)(1+\|u_n\|_{H_0^1(\Omega)})+\lambda\int_\Omega \left(G(u_n^+)-\frac{1}{p+1}g(x,u_n^+)u_n^+\right)\mathrm{d}x\right)\\ &\leqslant C+o(\|u_n\|_{H_0^1(\Omega)}).\end{aligned}$$

which implies that $\|u_n\|_{H_0^1(\Omega)}\leqslant C$.

Thus, up to a subsequence, we may assume that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } H_0^1(\Omega),\\ u_n &\rightharpoonup u \quad \text{weakly in } L^2(\Omega,|x|^{-2}\,\mathrm{d}x),\\ u_n &\rightharpoonup u \quad \text{weakly in } L^{p+1}(\Omega),\\ u_n &\to u \quad \text{a.e. on } \Omega.\end{aligned}$$

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It is then easy to verify that $u \in H_0^1(\Omega)$ is a non-negative critical point of $I_{\lambda,\mu}$. Let $v_n = u_n - u$. By the Brezis–Lieb lemma [**4**], we conclude that, as $n \to \infty$,

$$\begin{aligned} c - I_{\lambda,\mu}(u) &= I_{\lambda,\mu}(u_n) - I_{\lambda,\mu}(u) + o(1) \\ &= \frac{1}{2}\int_\Omega \left(|\nabla v_n|^2 - \mu\frac{v_n^2}{|x|^2}\right) \mathrm{d}x - \frac{\lambda}{p+1}\int_\Omega (v_n^+)^{p+1}\,\mathrm{d}x + o(1) \end{aligned} \tag{3.6}$$

and

$$o(\|u_n\|_{H_0^1(\Omega)}) = \langle \mathrm{d}I_{\lambda,\mu}(u_n), u_n\rangle = \int_\Omega \left(|\nabla v_n|^2 - \mu\frac{v_n^2}{|x|^2}\right) \mathrm{d}x - \lambda\int_\Omega (v_n^+)^{p+1}\,\mathrm{d}x + o(1). \tag{3.7}$$

Thus, from (3.7), we may assume that

$$\lim_{n\to\infty}\int_\Omega \left(|\nabla v_n|^2 - \mu\frac{v_n^2}{|x|^2}\right) \mathrm{d}x = a, \qquad \lim_{n\to\infty}\lambda\int_\Omega (v_n^+)^{p+1}\,\mathrm{d}x = a,$$

where a is a non-negative number.

By the Sobolev inequality, we obtain

$$\left(\int_\Omega (v_n^+)^{p+1}\,\mathrm{d}x\right)^{2/(p+1)} \leqslant \left(\int_\Omega |v_n|^{p+1}\,\mathrm{d}x\right)^{2/(p+1)} \leqslant S_\mu^{-1}\int_\Omega \left(|\nabla v_n|^2 - \mu\frac{v_n^2}{|x|^2}\right) \mathrm{d}x. \tag{3.8}$$

Passing to a limit in (3.8), we derive

$$\left(\frac{a}{\lambda}\right)^{2/(p+1)} \leqslant S_\mu^{-1} a.$$

If $a = 0$, the proof is complete. Assume that $a > 0$. We then get

$$a \geqslant \frac{S_\mu^{N/2}}{\lambda^{(N-2)/2}}.$$

Note that $I_{\lambda,\mu}(u) \geqslant 0$, and, from (3.6), we deduce that

$$c = \tfrac{1}{2}a - \frac{1}{p+1}a + I_{\lambda,\mu}(u) \geqslant \left(\frac{1}{2} - \frac{1}{p+1}\right)a \geqslant \frac{1}{N\lambda^{(N-2)/2}} S_\mu^{N/2},$$

which contradicts the assumption concerning c at the beginning of this proof. □

Lemma 3.2. *Let $0 < \mu \leqslant \bar{\mu} - 1$. There then exists a non-negative function $v \in H_0^1(\Omega)\backslash\{0\}$ such that*

$$\sup_{t\geqslant 0} I_{\lambda,\mu}(tv) < \frac{1}{N\lambda^{(N-2)/2}} S_\mu^{N/2}.$$

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Proof. Let $u_\varepsilon(x)=\eta(x)U_\varepsilon(x)$, where $\eta\in C_0^\infty(\Omega)$, satisfying $0\leqslant\eta\leqslant 1$. Following [**5**], and after a simple calculation, we have

$$\int_\Omega\left(|\nabla u_\varepsilon|^2-\mu\frac{u_\varepsilon^2}{|x|^2}\right)\mathrm{d}x=S_\mu^{N/2}+O(\varepsilon^{N-2}),$$

$$\int_\Omega|u_\varepsilon|^{p+1}\,\mathrm{d}x=S_\mu^{N/2}+O(\varepsilon^N),$$

$$\int_\Omega|u_\varepsilon|^2\,\mathrm{d}x\geqslant\begin{cases}C_1\varepsilon^{2\sqrt{\bar\mu}/\sqrt{\bar\mu-\mu}} & \text{if } 0<\mu<\bar\mu-1,\\ C_2\varepsilon^{N-2}|\ln\varepsilon| & \text{if } \mu=\bar\mu-1.\end{cases}$$

Since, for $s>0$,

$$g(x,s)=(1+u_\lambda+s)^p-(1+u_\lambda)^p-s^p\geqslant c(p)(1+u_\lambda)^{p-1}s,$$

we have, for $s>0$,

$$G(x,s)\geqslant\tfrac12c(p)(1+u_\lambda)^{p-1}s^2,\tag{3.9}$$

where we use the following inequality: for all $p>1$, there exists a $c(p)>0$ such that

$$(a+b)^p\geqslant a^p+b^p+c(p)a^{p-1}b\quad\forall a,b\geqslant0.$$

Therefore, using (3.9), we infer that

$$\begin{aligned}I_{\lambda,\mu}(tu_\varepsilon)&=\tfrac12t^2\int_\Omega\left(|\nabla u_\varepsilon|^2-\mu\frac{u_\varepsilon^2}{|x|^2}\right)\mathrm{d}x-\frac{\lambda t^{p+1}}{p+1}\int_\Omega|u_\varepsilon|^{p+1}\,\mathrm{d}x-\lambda\int_\Omega G(x,tu_\varepsilon)\,\mathrm{d}x\\
&\leqslant\tfrac12t^2\int_\Omega\left(|\nabla u_\varepsilon|^2-\mu\frac{u_\varepsilon^2}{|x|^2}-\tfrac12\lambda c(p)(1+u_\lambda)^{p-1}u_\varepsilon^2\right)\mathrm{d}x-\frac{\lambda t^{p+1}}{p+1}\int_\Omega|u_\varepsilon|^{p+1}\,\mathrm{d}x\\
&\leqslant\frac1N\left(\frac{\int_\Omega(|\nabla u_\varepsilon|^2-\mu(u_\varepsilon^2/|x|^2)-\frac12\lambda c(p)(1+u_\lambda)^{p-1}u_\varepsilon^2)\,\mathrm{d}x}{(\lambda\int_\Omega|u_\varepsilon|^{p+1}\,\mathrm{d}x)^{2/(p+1)}}\right)^{N/2}\\
&\leqslant\frac{1}{N\lambda^{(N-2)/2}}\left(\frac{S_\mu^{N/2}-\frac12\lambda c(p)\alpha(\varepsilon)+O(\varepsilon^{N-2})}{(S_\mu^{N/2}+O(\varepsilon^N))^{2/(p+1)}}\right)^{N/2}\\
&<\frac{1}{N\lambda^{(N-2)/2}}S_\mu^{N/2},\end{aligned}$$

where

$$\alpha(\varepsilon)=\begin{cases}C_1\varepsilon^{2\sqrt{\bar\mu}/\sqrt{\bar\mu-\mu}} & \text{if } 0<\mu<\bar\mu-1,\\ C_2\varepsilon^{N-2}|\ln\varepsilon| & \text{if } \mu=\bar\mu-1,\end{cases}$$

and the following fact has been used:

$$\max_{t\geqslant0}\left(\tfrac12t^2A-\frac{t^{p+1}}{p+1}B\right)=\frac1NA\left(\frac AB\right)^{(N-2)/2},\quad A,B>0.$$

□

Proof of Theorem 1.2. Set $c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t))$, where

$$\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) \mid \gamma(0) = 0 \text{ and } I_{\lambda,\mu}(\gamma(1)) < 0\}.$$

Then we claim that $c > 0$. In fact, by (3.2), (3.3), we derive

$$\int_\Omega (G(x,u) - \tfrac{1}{2}a(x)u^2)\,\mathrm{d}x \leqslant \tfrac{1}{2}\varepsilon\int_\Omega a(x)u^2\,\mathrm{d}x + \frac{C(\varepsilon)}{p+1}\int_\Omega |u|^{p+1}\,\mathrm{d}x, \quad \varepsilon > 0,$$

and then, for any $u \in H_0^1(\Omega)\backslash\{0\}$, by Lemma 2.4, we have

$$\begin{aligned}
I_{\lambda,\mu}(u) &= \frac{1}{2}\int_\Omega \biggl(|\nabla u|^2 - \mu\frac{u^2}{|x|^2} - \lambda a(x)u^2\biggr)\mathrm{d}x - \frac{\lambda}{p+1}\int_\Omega (u^+)^{p+1}\,\mathrm{d}x \\
&\qquad - \lambda\int_\Omega (G(x,u) - \tfrac{1}{2}a(x)u^2)\,\mathrm{d}x \\
&\geqslant \frac{1}{2}\biggl(1 - \frac{1}{m(\lambda)}\biggr)\int_\Omega \biggl(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\biggr)\mathrm{d}x - \frac{\lambda}{p+1}\int_\Omega (u^+)^{p+1}\,\mathrm{d}x \\
&\qquad - \tfrac{1}{2}\lambda\varepsilon\int_\Omega a(x)u^2\,\mathrm{d}x - \frac{\lambda C(\varepsilon)}{p+1}\int_\Omega |u|^{p+1}\,\mathrm{d}x \\
&\geqslant \frac{1}{2}\biggl(1 - \frac{1}{m(\lambda)}\biggr)\biggl(1 - \frac{\mu}{\bar\mu}\biggr)\|u\|^2_{H_0^1(\Omega)} \\
&\qquad - \frac{\varepsilon\lambda p}{2S_\mu}\biggl(1 - \frac{\mu}{\bar\mu}\biggr)\biggl(\int_\Omega (1+u_\lambda)^{p+1}\,\mathrm{d}x\biggr)^{(p-1)/(p+1)}\|u\|^2_{H_0^1(\Omega)} \\
&\qquad - \frac{\lambda(1+C(\varepsilon))}{(p+1)S_\mu^{(p+1)/2}}\biggl(1 - \frac{\mu}{\bar\mu}\biggr)^{(p+1)/2}\|u\|^{(p+1)/2}_{H_0^1(\Omega)}.
\end{aligned}$$

Setting

$$\begin{aligned}
C_3 &= \frac{1}{2}\biggl(1 - \frac{1}{m(\lambda)}\biggr)\biggl(1 - \frac{\mu}{\bar\mu}\biggr), \\
C_4 &= \frac{\lambda p}{2S_\mu}\biggl(1 - \frac{\mu}{\bar\mu}\biggr)\biggl(\int_\Omega (1+u_\lambda)^{p+1}\,\mathrm{d}x\biggr)^{(p-1)/(p+1)}, \\
C_5 &= \frac{\lambda(1+C(\varepsilon))}{(p+1)S_\mu^{(p+1)/2}}\biggl(1 - \frac{\mu}{\bar\mu}\biggr)^{(p+1)/2},
\end{aligned}$$

and taking $\varepsilon = C_3/2C_4$, we obtain

$$I_{\lambda,\mu}(u) \geqslant \tfrac{1}{2}C_3\|u\|^2_{H_0^1(\Omega)} - C_5\|u\|^{(p+1)/2}_{H_0^1(\Omega)}.$$

If

$$\|u\|_{H_0^1(\Omega)} = \rho = \biggl(\frac{C_3}{4C_5}\biggr)^{2/(p-1)} > 0,$$

we obtain

$$I_{\lambda,\mu}(u) \geqslant \tfrac{1}{4}C_3\biggl(\frac{C_3}{4C_5}\biggr)^{4/(p-1)} > 0 = I_{\lambda,\mu}(0).$$

So $c > 0$. In addition, $I_{\lambda,\mu}(tv) \to -\infty$ as $t \to \infty$, where v is from Lemma 3.2. Hence there exists a $t_0 > 0$ such that $\|t_0 v\|_{H_0^1(\Omega)} > \rho$ and $I_{\lambda,\mu}(t_0 v) < 0$. By the mountain pass lemma (see [**1**]), there is a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$I_{\lambda,\mu}(u_n) \to c, \quad \mathrm{d}I_{\lambda,\mu}(u_n) \to 0.$$

Observe that

$$c \leqslant \sup_{t\in[0,1]} I_{\lambda,\mu}(tt_0 v) \leqslant \sup_{t\geqslant 0} I_{\lambda,\mu}(tv) < \frac{1}{N\lambda^{(N-2)/2}} S_\mu^{N/2}.$$

By Lemma 3.1, we infer that there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a function $u \in H_0^1(\Omega)$, which satisfy

$$u_n \to u \quad \text{strongly in } H_0^1(\Omega),$$

and then c is a positive critical value of $I_{\lambda,\mu}$, and, by the maximum principle, u is a solution of problem $(3.1)_\lambda$ for $\lambda \in (0, \lambda^*)$. □

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