COMPOSITION OPERATORS

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1. Preliminaries. The object of this note is to report on some of the properties of a class of operators induced by inner functions. If m is normalized Lebesgue measure on the unit circle X in the complex plane and ϕ is an inner function (a complex function on X of unit modulus almost everywhere whose Poisson integral is a non-constant holomorphic function in the open unit disk), then an operator C_ϕ on $L^2(m)$ is defined by

$$C_\phi f = f \circ \phi.$$

The first task is to prove that C_ϕ is a bounded operator and to determine its norm. In §2 we shall relate properties of C_ϕ to the existence of fixed points of the Poisson integral of ϕ, and in §3 we shall describe the spectrum of C_ϕ in the case when ϕ is a linear fractional transformation of the unit disk onto itself. Throughout this paper ϕ will be used to denote an inner function.

An inner function ϕ induces a measurable transformation of the unit circle X onto itself and hence a measure $m\phi^{-1}$ on X. The Radon–Nikodym derivative of $m\phi^{-1}$ is identified as a Poisson kernel in the following lemma.

LEMMA 1. *If* $a = \int \phi\, dm$, *then* $(dm\, \phi^{-1}/dm)(u) = P_a(u) = \operatorname{Re}(u+a)/(u-a)$.

Proof. The conclusion will follow from the fact that $m\phi^{-1}$ and P_a have the same Fourier coefficients. Let $e_n(u) = u^n$ for $n = 0, \pm 1, \pm 2, \ldots$. Then for $n \geqslant 0$

$$\int e_n\, dm\, \phi^{-1} = \int e_n \circ \phi\, dm = \int \phi^n\, dm = a^n$$

and

$$\int e_n P_a\, dm = a^n.$$

Thus $m\phi^{-1}$ and P_a have the same Fourier coefficients of negative index, and the remaining ones are disposed of by taking complex conjugates in the above equations. Hence the lemma is proved.

COROLLARY. *If* $\int \phi\, dm = 0$, *then* ϕ *is a measure-preserving transformation.*

Proof. $P_0(u) = 1$.

Let H^2 be the subspace of $L^2(m)$ spanned by the functions $e_0, e_1, e_2, \ldots$. We remark that by use of the transformation of integral formula, the well-known

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multiplicative property of the measure $P_a\, dm$ on H^2 may be derived from Lemma 1 by reduction to the corresponding property of m.

THEOREM 1. *Let* $a = \int \phi\, dm$ *and* $f \in L^2(m)$. *Then*:
(a) C_ϕ *is a bounded operator on* $L^2(m)$.
(b) $||C_\phi|| = [(1 + |a|)/(1 - |a|)]^{1/2}$.
(c) $[(1 - |a|)/(1 + |a|)]^{1/2}||f|| \leqslant ||C_\phi f|| \leqslant [(1 + |a|)/(1 - |a|)]^{1/2}||f||$.

Proof. Part (c), and hence part (a), follows from the calculation

$$(1) \qquad ||C_\phi f||^2 = \int |f \circ \phi|^2\, dm = \int |f|^2\, dm\, \phi^{-1} = \int |f|^2 P_a\, dm$$

and the inequality

$$(1 - |a|)/(1 + |a|) \leqslant P_a \leqslant (1 + |a|)/(1 - |a|).$$

This also shows that $||C_\phi|| \leqslant [(1 + |a|)/(1 - |a|)]^{1/2}$. Since P_a is continuous with $(1 + |a|)/(1 - |a|)$ as its maximum, it is easily seen from (1) that equality must hold in the last inequality, and the proof is complete.

COROLLARY. *A necessary and sufficient condition for* C_ϕ *to be an isometry is that* $\int \phi\, dm = 0$.

2. Fixed points. If f is in $L^2(m)$ and

$$\tilde{f}(z) = \int f P_z\, dm \qquad (|z| < 1),$$

then the function $\tilde{f}$ is harmonic on the open unit disk, and if f is in H^2, then $\tilde{f}$ is holomorphic in the disk. In particular if ϕ is inner, then $\tilde{\phi}$ is a holomorphic mapping of the disk into itself. The corollary above may be rephrased using this language as follows: C_ϕ is an isometry if and only if 0 is a fixed point of $\tilde{\phi}$, i.e., $\tilde{\phi}(0) = 0$. We have in addition:

THEOREM 2. *If* $\tilde{\phi}$ *has a fixed point in the open unit disk, then* C_ϕ *is similar to an isometry.*

The proof will depend on three lemmas.

LEMMA 2. *The subspace* H^2 *is invariant under each* C_ϕ.

Proof. Clearly $C_\phi\, e_n = \phi^n \in H^2$ for $n = 0, 1, 2, \ldots$. Since the functions e_n with $n = 0, 1, 2, \ldots$ span H^2, the corollary follows.

Remark. The invariance of H^2 under C_ϕ allows us to consider the operator obtained by restricting C_ϕ to H^2. Since every positive $L^2(m)$ function with integrable logarithm is the modulus of some H^2 function (**4**, p. 53), the calculation in the proof of Theorem 1 may be used to show that the norm of this restriction is the same as the norm of C_ϕ.

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LEMMA 3. *If f is in H^2 and $|z| < 1$, then* $(f \circ \phi)^\sim(z) = \tilde{f}(\tilde{\phi}(z))$.

Proof. Choose a sequence of polynomials f_n in H^2 that converges to f. Thus $f_n \circ \phi$ converges to $f \circ \phi$ in H^2. By the multiplicativeness of the measure $P_z\, dm$ and the fact that the f_n are polynomials,

$$(f_n \circ \phi)^\sim(z) = \int f_n \circ \phi P_z\, dm = \tilde{f}_n\left(\int \phi P_z\, dm\right) = \tilde{f}_n(\tilde{\phi}(z)).$$

Since P_z and $P_{\tilde{\phi}(z)}$ are in $L^2(m)$, the $L^2(m)$ convergence of $f_n \circ \phi$ to $f \circ \phi$ and f_n to f implies the convergence of $(f_n \circ \phi)^\sim(z)$ to $(f \circ \phi)^\sim(z)$ and $\tilde{f}_n(\tilde{\phi}(z))$ to $\tilde{f}(\tilde{\phi}(z))$. Hence $(f \circ \phi)^\sim(z) = \tilde{f}(\tilde{\phi}(z))$.

LEMMA 4. *If ϕ and ψ are inner functions, then $\phi \circ \psi$ is inner and* $C_{\phi\circ\psi} = C_\psi C_\phi$.

Proof. Since $m\psi^{-1} \ll m$, it follows that $\phi \circ \psi$ has unit modulus almost everywhere, and $\phi \circ \psi = C_\psi \phi$ shows that $\phi \circ \psi$ is in H^2. By Lemma 3, $(\phi \circ \psi)^\sim = \tilde{\phi} \circ \tilde{\psi}$, so the elementary fact that the holomorphic function $\tilde{\phi} \circ \tilde{\psi}$ is non-constant implies that $\phi \circ \psi$ is non-constant. Hence $\phi \circ \psi$ is inner. The asserted equality follows from an obvious calculation.

Proof of Theorem 2. If a is the fixed point of $\tilde{\phi}$ in the open disk, then let β be the inner function defined by $\beta(u) = (a - u)/(1 - a^*u)$ and let $\psi = \beta^{-1} \circ \phi \circ \beta$, where a^* is the complex conjugate of a and β^{-1} is the inverse mapping of β. By Lemma 4, ψ is inner,

$$C_\psi = C_\beta\, C_\phi\, C_{\beta^{-1}} = C_\beta\, C_\phi\, C_\beta^{-1},$$

and it only remains to prove that C_ψ is an isometry. This follows from the calculation

$$\int \psi\, dm = \int \beta^{-1} \circ \phi \circ \beta\, dm = \int \beta^{-1} \circ \phi P_a\, dm$$

$$= \tilde{\beta}^{-1}\left(\int \phi P_a\, dm\right) = \tilde{\beta}^{-1}(a) = 0,$$

where the third equality follows from Lemma 3. The corollary to Theorem 1 implies that C_ψ is isometric, which proves the theorem.

COROLLARY. *If $\tilde{\phi}$ has a fixed point in the open unit disk, then the spectrum of C_ϕ is included in the closed unit disk.*

Proof. If $\tilde{\phi}$ has a fixed point in the open disk, then C_ϕ is similar to an isometry. Since the spectra of isometries are included in the closed unit disk, and since similar operators have the same spectra, the corollary follows.

Hilbert space isometries have a natural decomposition into pure and unitary parts (see **1**). An isometry V on a Hilbert space $\mathfrak{H}$ is said to be pure if

$$\bigcup_{n=0}^{\infty} V^n(V\mathfrak{H})^\perp$$

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spans $\mathfrak{H}$. For an arbitrary isometry U there is a subspace $\mathfrak{M}$ such that the restriction of U to $\mathfrak{M}$ is unitary and the restriction of U to the orthogonal complement of $\mathfrak{M}$ is a pure isometry. It may be shown **(3)** that

$$\mathfrak{M} = \bigcap_{n=0}^{\infty} U^n\mathfrak{H}.$$

We shall identify the unitary part of an isometric composition operator as its restriction to the set of constant functions.

LEMMA 5. *If ϕ is an inner function other than a linear fractional transformation such that $\tilde{\phi}$ has a fixed point, then*

$$\mathfrak{M}_\phi = \bigcap_{n=0}^{\infty} C_\phi^n H^2$$

contains only the constant functions.

Proof. It suffices to consider the case where 0 is the fixed point. For if $\tilde{\phi}$ has a fixed point $a \neq 0$, then define the isometry C_ψ as in the proof of Theorem 2. Since $C_\beta^{-1}H^2 = H^2$,

$$\mathfrak{M}_\phi = C_\beta^{-1} \bigcap_{n=0}^{\infty} C_\beta\, C_\phi^n C_\beta^{-1}H^2 = C_\beta^{-1} \bigcap_{n=0}^{\infty} C_\psi^n H^2 = C_\beta^{-1}\mathfrak{M}_\psi.$$

Hence if $\mathfrak{M}_\psi$ contains only the constant functions, then this is also true of $\mathfrak{M}_\phi$ because of the invariance of these functions under C_β^{-1}. Moreover, since ϕ is not a linear fractional transformation, ψ is not merely multiplication by a constant of unit modulus.

Suppose then that $\tilde{\phi}(0) = 0$. Since it is also true that $|\tilde{\phi}(z)| < 1$ whenever $|z| < 1$, and $\tilde{\phi}$ is not multiplication by a constant of modulus 1, the Schwarz lemma may be applied to conclude that $|\tilde{\phi}(z)| < |z|$ whenever $0 < |z| < 1$. Thus it follows, if we define $\tilde{\phi}^{(1)} = \tilde{\phi}$ and $\tilde{\phi}^{(n+1)} = \tilde{\phi}^{(n)} \circ \tilde{\phi}$ for $n = 1, 2, \ldots,$ that $\lim_{n\to\infty} \tilde{\phi}^{(n)}(z) = 0$ for each z in the open disk.

It may be shown that for each point z in the open disk the functional $f \to \tilde{f}'(z)$ is bounded on H^2. In addition, if z is restricted to a compact subset K of the open disk, then the corresponding set of such functionals is bounded, i.e., there is a constant M such that

$$|\tilde{f}'(z)| \leqslant M||f|| \qquad (z \in K, f \in H^2).$$

Hence if K is a disk centred at 0 and of radius less than one, then by the mean value theorem

$$|\tilde{f}(z) - \tilde{f}(0)| \leqslant |z|\, M\, ||f|| \qquad (z \in K, f \in H^2).$$

Suppose now that h is in $\mathfrak{M}_\phi$, and a is a point in the open disk such that $\tilde{h}(a) \neq \tilde{h}(0)$. Determine M as above with K the disk centred at 0 and of radius $|a|$, and choose n large enough that

$$|\tilde{\phi}^{(n)}(a)| < |\tilde{h}(a) - \tilde{h}(0)|/M||h||.$$

Since h is in $\mathfrak{M}_\phi$, $h = C_\phi^n g$ for some g in H^2, and since C_ϕ is an isometry, $||g|| = ||h||$. Using Lemma 4, we now obtain the contradiction:

$$|\tilde{h}(a) - \tilde{h}(0)| = |\tilde{g}(\tilde{\phi}^{(n)}(a)) - \tilde{g}(0)| \leqslant |\tilde{\phi}^{(n)}(a)|\ M\ ||g|| < |\tilde{h}(a) - \tilde{h}(0)|.$$

Hence the only functions in $\mathfrak{M}_\phi$ are constants, and the lemma is proved.

THEOREM 3. *If C_ϕ is an isometry and ϕ is not a linear fractional transformation, then the unitary and purely isometric parts of C_ϕ are obtained by restricting it to the constant functions and the orthogonal complement of the constant functions respectively.*

Proof. It must be shown that if f is in

$$\bigcap_{n=0}^{\infty} C_\phi^n L^2(m),$$

then f is constant. Suppose that $f = f_1 + f_2$, where f_1 is in $H_0^2 = \{g \in H^2: \int g\, dm = 0\}$ and f_2 is in $H_0^{2\perp} = H^{2*}$. By Lemma 2 and the corollary to Lemma 1, H_0^2 is invariant under C_ϕ; by Lemma 2 and the relation $C_\phi f^* = (C_\phi f)^*$, $H_0^{2\perp}$ is invariant under C_ϕ. For each n there is an h in $L^2(m)$ such that $f = C_\phi{}^n h$, and h, like f, has a decomposition $h = h_1 + h_2$. The invariance of H_0^2 and $H_0^{2\perp}$ under C_ϕ implies that $f_1 = C_\phi{}^n h_1$ and $f_2^* = (C_\phi{}^n h_2)^* = C_\phi{}^n h_2^*$, which in turn imply that f_1 and f_2^* are in $\mathfrak{M}_\phi$. Thus, by Lemma 5, f is constant, and the proof is complete.

3. Linear fractional transformations and spectra. Every linear fractional transformation ϕ of the unit disk onto itself may be written in the form

$$\phi(z) = (az + c^*)/(cz + a^*)$$

where $|a|^2 - |c|^2 = 1$ (see **2**, Chapter 1). We assume ϕ has this form henceforth. The fixed points of ϕ may be determined from the formula

$$z = (i \operatorname{Im} a \pm (|c|^2 - (\operatorname{Im} a)^2)^{1/2})/c \tag{2}$$

if $c \neq 0$. The nature of the fixed points is then determined by the relative magnitudes of c and $\operatorname{Im} a$. There are three cases: (1) $|c| < |\operatorname{Im} a|$, in which case there are two fixed points, one interior and one exterior to the unit circle, and ϕ is said to be elliptic (note that the case $c = 0$ is also included here); (2) $|c| = |\operatorname{Im} a|$, in which case there is one fixed point on the unit circle, and ϕ is said to be parabolic; (3) $|c| > |\operatorname{Im} a|$, in which case there are two fixed points on the unit circle, and ϕ is said to be hyperbolic. The spectral behaviour of C_ϕ varies considerably from case to case, as will be shown in this section.

THEOREM 4. *If ϕ is elliptic, then the spectrum of C_ϕ is the closure of*

$$\{K^n: n = 0, \pm 1, \pm 2, \ldots\},$$

where

$$K = (\operatorname{Re} a + i((\operatorname{Im} a)^2 - |c|^2)^{1/2})/(\operatorname{Re} a - i((\operatorname{Im} a)^2 - |c|^2)^{1/2}).$$

Proof. We shall assume without loss of generality that $\operatorname{Im} a > 0$. In case $c = 0$, then $\phi(z) = az/a^* = Kz$, and the result is well known. Suppose then that $c \neq 0$. Let z_i and z_e be the fixed points of ϕ with $|z_i| < 1$ and $|z_e| > 1$. The invariance of the cross ratio under linear fractional transformations may be used to show that

$$(\phi(z) - z_i)/(\phi(z) - z_e) = K(z - z_i)/(z - z_e), \tag{3}$$

where

$$\begin{aligned} K &= (a - cz_i)/(a - cz_e) \\ &= (\operatorname{Re} a + i((\operatorname{Im} a)^2 - |c|^2)^{1/2})/(\operatorname{Re} a - i((\operatorname{Im} a)^2 - |c|^2)^{1/2}). \end{aligned}$$

Define $\beta(z) = k(z - z_i)/(z - z_e)$, where k is a constant yet to be specified. Whatever k is, we may rewrite (3) as

$$\beta(\phi(z)) = K\beta(z).$$

Since ϕ leaves X fixed, $\beta(X) = K\beta(X)$. Since $\beta(X)$ is a circle, and since multiplication by K, which has modulus one, leaves $\beta(X)$ fixed, it follows that $\beta(X)$ must be centred at the origin. Thus k can be chosen so that $\beta(X) = X$. Thus β is a linear fractional transformation of the unit disk onto itself such that

$$\beta \circ \phi \circ \beta^{-1}(z) = Kz,$$

which implies that C_ϕ is similar to a composition operator of the type mentioned in the first line of this proof. Thus the result follows.

THEOREM 5. *If ϕ is parabolic, then the spectrum of C_ϕ is the unit circle.*

Proof. We shall show that the spectral radii of C_ϕ and C_ϕ^{-1} are both one and that the unit circle is in the point spectrum of C_ϕ.

First we may assume that the fixed point of ϕ is 1. For if it is λ, then $\psi(z) = \lambda^*\phi(\lambda z)$ is a parabolic linear fractional transformation of the unit disk onto itself with fixed point 1, and C_ϕ is unitarily equivalent to C_ψ.

Define a linear fractional transformation γ of the disk onto the upper half-plane by $\gamma(z) = i(1 + z)/(1 - z)$. Then $\delta = \gamma \circ \phi \circ \gamma^{-1}$ is a linear fractional transformation of the upper half-plane onto itself, and ∞ is the only fixed point of δ since $\gamma(1) = \infty$. Thus $\delta(w) = w + b$ for some real number $b \neq 0$. Since

$$\delta^{(n)}(w) = w + nb = \gamma \circ \phi^{(n)} \circ \gamma^{-1},$$

it follows that

$$\phi^{(n)}(0) = \gamma^{-1} \circ \delta^{(n)}(\gamma(0)) = \gamma^{-1}(i + nb) = nb/(2i + nb),$$

and hence

$$\begin{aligned} ||C_\phi^n||^2 &= (1 + |nb/(2i + nb)|)/(1 - |nb/(2i + nb)|) \\ &= (|2i + nb| + n|b|)^2/(|2i + nb|^2 - n^2b^2) \\ &\leqslant (n|b| + 1)^2. \end{aligned}$$

Thus, if $r(C_\phi)$ is the spectral radius of C_ϕ, then

$$r(C_\phi) \leqslant \limsup_{n\to\infty} (n|b| + 1)^{1/n} = 1,$$

and since 1 is in the point spectrum of every composition operator, $r(C_\phi) = 1$. The inverse of ϕ is also parabolic, so $r(C_\phi^{-1}) = 1$, and hence the spectrum of C_ϕ is included in the unit circle.

To obtain the opposite inclusion, let $\lambda \neq 1$ be a point of the unit circle. Choose a real number d of the same sign as b such that $e^{id} = \lambda$, and let

$$f(z) = \exp(idb^{-1}\gamma(z)).$$

For $|z| \leqslant 1$, since γ carries the disk onto the upper half-plane,

$$|f(z)| = \exp(-db^{-1}\operatorname{Im}\gamma(z)) \leqslant 1,$$

and consequently f is in H^2. Since $\gamma \circ \phi = \delta \circ \gamma = \gamma + b$,

$$C_\phi f = \exp(idb^{-1}\gamma \circ \phi) = \exp(id + idb^{-1}\gamma) = \lambda f.$$

Hence λ is in the point spectrum of C_ϕ, and the proof is complete.

THEOREM 6. *Let ϕ be hyperbolic, and assume without loss of generality that* $\operatorname{Re} a > 0$. *Then the spectrum of C_ϕ is the annulus centred at* 0 *of outer radius $K^{1/2}$ and inner radius $K^{-1/2}$, where*

$$K = (\operatorname{Re} a + (|c|^2 - (\operatorname{Im} a)^2)^{1/2})/(\operatorname{Re} a - (|c|^2 - (\operatorname{Im} a)^2)^{1/2}).$$

Proof. Let z_- be the fixed point of ϕ with the negative square root taken in formula (2) and let z_+ be that with the positive square root. By the same procedure used in the proof of Theorem 4, we obtain

$$\beta(\phi(z)) = K\beta(z), \tag{4}$$

where K is the constant of the assertion and β is the linear fractional transformation of the disk onto the upper half-plane defined by

$$\beta(z) = e^{i\theta}(z - z_-)/(z - z_+)$$

with θ chosen so that $\beta(X)$ is the real axis. If γ is defined as in the proof of Theorem 5, then $\gamma^{-1} \circ \beta$ is a linear fractional transformation of the unit disk onto itself such that

$$\psi(z) = (\gamma^{-1} \circ \beta) \circ \phi \circ (\gamma^{-1} \circ \beta)^{-1}(z) = (z + r)/(1 + rz), \tag{5}$$

where $r = (K - 1)/(K + 1)$. Thus C_ϕ is similar to C_ψ, and it is sufficient to show that the spectrum of C_ψ is the annulus centred at 0 with outer radius $K^{1/2} = ((1 + r)/(1 - r))^{1/2}$ and inner radius $K^{-1/2}$.

First note that $||C_\psi|| = ((1 + r)/(1 - r))^{1/2} = K^{1/2}$, so the spectrum is contained in the disk of radius $K^{1/2}$. It is easy to see that $\psi^{-1}(z) = (z - r)/(1 - rz)$, so $||C_\psi^{-1}|| = K^{1/2}$. It follows from the spectral mapping theorem again that the spectrum of C_ψ is contained in the annulus in question.

We shall conclude by showing that the interior of this annulus lies in the point spectrum. Let λ be a point of this interior, i.e., $\lambda = K^{\alpha+i\omega}$ with $-\frac{1}{2} < \alpha < \frac{1}{2}$ and $0 \leqslant \omega < 2\pi/(\log K)$. It is convenient at this point to identify the space H^2 with the space of holomorphic functions f on the open unit disk such that

$$\sup_{0 \leqslant r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty .$$

We obtain an eigenvector of C_ψ corresponding to λ by choosing a branch of the logarithm that is real valued on the real axis and holomorphic in the upper half-plane and setting

$$f = \exp((\alpha + i\omega) \log \gamma).$$

By a result in (**4**, p. 106), f is in H^2 if $f \circ \gamma^{-1}(w)/(1 - iw)$ is in H^2 of the upper half-plane, and this follows from an elementary calculation. Hence f is in H^2. Finally note by (4) and (5) that $\gamma \circ \psi = K\gamma$, and hence

$$\begin{aligned} C_\psi f &= \exp((\alpha + i\omega)\log \gamma \circ \psi) = \exp((\alpha + i\omega)(\log K + \log \gamma)) \\ &= K^{\alpha+i\omega} \exp((\alpha + i\omega) \log \gamma) = \lambda f. \end{aligned}$$

This concludes the proof.

Remarks. (1) With slight modifications and with the exception of Theorem 3, the results of this paper are valid with arbitrary $L^p(m)$ $(1 \leqslant p < \infty)$ in place of $L^2(m)$. For example, part (b) of Theorem 1 becomes

$$||C_\phi||_p = ((1 + |a|)/(1 - |a|))^{1/p},$$

and in Theorem 6 the inner and outer radii of the annulus become $K^{-1/p}$ and $K^{1/p}$ respectively.

(2) The domain of C_ϕ might equally well be taken to be H^2 rather than $L^2(m)$. With this modification all the statements of results can remain the same, and most proofs require little or no modification.

After submitting this paper I learned of an earlier paper of J. V. Ryff, *Subordinate H^p functions*, Duke Math. J., *33* (1966), 347–354, in which more general composition operators are studied and some of our results are obtained by other methods. The question raised in Ryff's first theorem is settled by our Theorem 1 and the remark following Lemma 2.

References

1. A. Brown, *On a class of operators*, Proc. Amer. Math. Soc., *4* (1953), 723–728.
2. L. R. Ford, *Automorphic functions* (New York, 1929).
3. P. R. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math., *208* (1961), 102–112.
4. K. Hoffman, *Banach spaces of analytic functions* (Englewood Cliffs, N.J., 1962).

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