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ON THE UMBILICITY OF HYPERSURFACES IN THE HYPERBOLIC SPACE

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Abstract

In this paper, we establish new characterization results concerning totally umbilical hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$, under suitable constraints on the behavior of the Lorentzian Gauss map of complete hypersurfaces having some constant higher order mean curvature. Furthermore, working with different warped product models for $\mathbb{H}^{n+1}$ and supposing that certain natural inequalities involving two consecutive higher order mean curvature functions are satisfied, we study the rigidity and the nonexistence of complete hypersurfaces immersed in $\mathbb{H}^{n+1}$.

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1. Introduction

The study of the geometry of complete hypersurfaces with constant mean curvature in a Riemannian space form constitutes a classical and fruitful theme in the theory of geometric analysis. In this branch, do Carmo and Lawson [12] used the well-known Alexandrov reflexion method to show that a complete hypersurface properly embedded with constant mean curvature in the $(n + 1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ with a single point at the asymptotic boundary must be a horosphere. Moreover, they also observed that the statement is no longer true if we replace embedded by immersed. Later on, Alías and Dajczer [2] proved that the horospheres are the only surfaces properly immersed in $\mathbb{H}^3$ with constant mean curvature $-1 \leq H \leq 1$ and which are contained in a slab (that is, the region between two horospheres that share the same point in the asymptotic boundary).

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In [3], Alías *et al.* proved that a bounded, complete hypersurface in $\mathbb{H}^{n+1}$ with normal curvatures greater than -1 must be diffeomorphic to a Euclidean sphere $\mathbb{S}^n$. Afterwards, Wang and Xia [21] showed that a closed hypersurface in $\mathbb{H}^{n+1}$ and whose second fundamental form has constant norm must be totally umbilical. In [20], Shu proved that a complete hypersurface in $\mathbb{H}^{n+1}$ with constant normalized scalar curvature and nonnegative sectional curvature must be either totally umbilical or isometric to a hyperbolic cylinder of $\mathbb{H}^{n+1}$. Next, the third author and Caminha [11] studied complete vertical graphs of constant mean curvature in $\mathbb{H}^{n+1}$. Under appropriate restrictions on the values of the mean curvature and the growth of the height function, they established necessary conditions for the existence of such a graph Σ^n and, when $n = 2$, they proved that Σ^2 must be a horosphere. By extending a technique due to Yau [22], these same authors jointly with Camargo [9] obtained another rigidity result concerning the horospheres of $\mathbb{H}^{n+1}$, without the assumption of the constancy of the mean curvature. Moreover, they also obtained a characterization theorem for the horospheres of $\mathbb{H}^{n+1}$ under suitable constraints on two consecutive higher order mean curvatures. More recently, the first and third authors [7] used some generalized maximum principles in order to obtain several new characterization results for horospheres of $\mathbb{H}^{n+1}$ via suitable restrictions on the mean curvature function.

Meanwhile, in [5], these same authors jointly with Barros improved previous results of [6], showing that the only complete constant mean curvature hypersurfaces immersed in $\mathbb{H}^{n+1}$ such that the image of the Lorentzian Gauss map lies in a totally umbilical space-like hypersurface of the de Sitter space $\mathbb{S}^{n+1}_1$ must be the totally umbilical ones. The same conclusion holds when the assumption on the Lorentzian Gauss map is replaced by scalar curvature bounded from below and whose angle function f_a, with respect to some fixed vector a of the $(n + 2)$-dimensional Lorentz–Minkowski space $\mathbb{L}^{n+2}$ such that the tangential component $a^\top$ has Lebesgue integrable norm, does not change sign. Afterwards, the first author [4] obtained extensions of these results for the case of complete hypersurfaces of $\mathbb{H}^{n+1}$ having constant scalar curvature.

In this article, our purpose is to study the umbilicity of complete hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$ via their higher order mean curvature functions. Firstly, assuming that some higher order mean curvature is constant, we establish new characterization results concerning totally umbilical hypersurfaces of $\mathbb{H}^{n+1}$, under appropriate constraints on the behavior of the Lorentzian Gauss map (see Theorems 3.1, 3.4 and 3.6 and Corollary 3.5). Afterwards, working with different warped product models for $\mathbb{H}^{n+1}$ and supposing that two consecutive higher order mean curvature functions satisfy certain natural inequalities, we study the rigidity and the nonexistence of complete hypersurfaces of $\mathbb{H}^{n+1}$ (see Theorems 4.2, 4.4 and 4.6).

2. Preliminaries

This section is devoted to recalling some basic facts concerning hypersurfaces immersed in the hyperbolic space. For this, let us consider the Lorentz–Minkowski space $\mathbb{L}^{n+2}$, that is, the Euclidean vector space $\mathbb{R}^{n+2}$ equipped with the metric

$$\langle v, w\rangle = \sum_{j=1}^{n+1} v_j w_j - v_{n+2} w_{n+2}.$$

The $(n+1)$-dimensional hyperbolic space can be regarded as being the following hyperquadric of $\mathbb{L}^{n+2}$:

$$\mathbb{H}^{n+1} = \{p \in \mathbb{L}^{n+2}; \langle p, p\rangle = -1; p_{n+2} \geq 1\}.$$

In this context, we will deal with connected and oriented isometrically immersed hypersurfaces $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$. We recall that the unit normal vector field N of Σ^n can be considered as a map $N : \Sigma^n \to \mathbb{S}_1^{n+1}$, where $\mathbb{S}_1^{n+1}$ stands for the $(n+1)$-dimensional unitary de Sitter space, that is,

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{L}^{n+2}; \langle p, p\rangle = 1\}.$$

In this setting, N is called the *Lorentzian Gauss map* of Σ^n.

Let us denote by $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ the Weingarten endomorphism of Σ^n with respect to the vector field N. Recall that, if ∇^0, $\overline{\nabla}$ and ∇ stand for the Levi-Civita connections in $\mathbb{L}^{n+2}$, $\mathbb{H}^{n+1}$ and Σ^n, respectively, then the Gauss and Weingarten formulas provide

$$\overline{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle N$$

and

$$AX = -\overline{\nabla}_X N = -\nabla^0{}_X N$$

for all tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$.

Since the Weingarten operator A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$, at each $p \in \Sigma^n$,

$$\det(tI - A) = \sum_{r=0}^{n} (-1)^r S_r t^{n-r},$$

where I stands for the identity operator, $S_r(p)$ is the rth elementary symmetric function on the eigenvalues of A_p, for $1 \leq r \leq n$, and $S_0 = 1$ by convention. We define the rth mean curvature H_r of Σ^n, $0 \leq r \leq n$, by

$$\binom{n}{r} H_r = S_r.$$

We observe that $H_0 = 1$, while $H_1 = (1/n)S_1$ is the usual mean curvature H of Σ^n.

For $0 \leq r \leq n$, one defines the rth Newton transformation P_r on Σ^n by setting $P_0 = I$ and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = S_r I - A P_{r-1}.$$

On the other hand, given $f \in C^\infty(\Sigma)$, for each $0 \leq r \leq n$, the second-order differential operator L_r is defined as follows:

$$L_r f = \mathrm{tr}(P_r \nabla^2 f).$$

Here $\nabla^2 f : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f, and it is given by

$$\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X \nabla f, Y\rangle$$

for all $X, Y \in \mathfrak{X}(\Sigma)$. It is important to note that this operator is of divergence type provided that we have a hypersurface $\Sigma^n \to \mathbb{Q}^{n+1}(c)$, where $\mathbb{Q}^{n+1}(c)$ stands for a Riemannian space form of constant sectional curvature c. This fact was proved by Rosenberg in [19] and it reads as follows:

$$L_r f = \operatorname{div}(P_r \nabla f).$$

Fixing a nonzero vector $a \in \mathbb{L}^{n+2}$, we will consider two particular functions naturally attached to a hypersurface $\psi : \Sigma^n \to \mathbb{H}^{n+1}$, namely, the *height* and *angle* functions, which are defined, respectively, by $l_a = \langle \psi, a\rangle$ and $f_a = \langle N, a\rangle$.

A direct computation allows us to conclude that the gradients of such functions are given by $\nabla l_a = a^\top$ and $\nabla f_a = -A(a^\top)$, where $a^\top$ is the orthogonal projection of a onto the tangent bundle $T\Sigma$, that is,

$$a^\top = a - f_a N + l_a \psi. \tag{2.1}$$

Based on the ideas of the classical paper of Reilly [18], Rosenberg [19] obtained suitable formulas for the operator L_r acting on the height and angle functions of a hypersurface of a Riemannian space form. For the case of the hyperbolic space, these formulas read as follows (compare [1, Section 3] or [19, Section 5]):

$$L_r l_a = c_r (H_{r+1} f_a + H_r l_a) \tag{2.2}$$

and

$$L_r f_a = -\Big(\frac{n}{r+1} c_r H H_{r+1} - c_{r+1} H_{r+2}\Big) f_a - c_r H_{r+1} l_a - \frac{c_r}{r+1}\langle \nabla H_{r+1}, a^\top\rangle, \tag{2.3}$$

where $c_r = (r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$.

Now, we observe that for $r = 0$, (2.2) particularizes to $\Delta l_a = nHf_a + nl_a$. Then, combining this formula with (2.3) for the case H_{r+1} constant,

$$\operatorname{div}\Big(P_r \nabla f_a + \frac{c_r}{n} H_{r+1} \nabla l_a\Big) = c_{r+1}(H_{r+2} - HH_{r+1}) f_a. \tag{2.4}$$

Our next auxiliary result establishes an analytical tool to detect the umbilicity of a hypersurface immersed in $\mathbb{H}^{n+1}$. For this, we recall that a point p_0 in a hypersurface $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ is said to be *elliptic* when all principal curvatures $\lambda_i(p_0)$ are positive with respect to an appropriate choice of the Lorentzian Gauss map of Σ^n.

Lemma 2.1. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be a hypersurface immersed in the hyperbolic space with H_{r+1} positive. Assume that there exist an elliptic point in Σ^n. Then*

$$H_{r+2} - HH_{r+1} \le 0$$

and equality holds if, and only if, Σ^n is totally umbilical.

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Proof. Making use of [17, Lemma 1], we have that any H_i, $i \leq r$, is also positive. Moreover,

$$H_{r+2} \leq H_{r+1}^{(r+2)/(r+1)} = H_{r+1}H_{r+1}^{1/(r+1)} \leq H_{r+1}H_r^{1/r} \leq HH_{r+1},$$

with equality at any stage only at umbilical points, which completes the proof (alternatively, see [1, page 204]). □

We close this section by obtaining the following result.

Lemma 2.2. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be a hypersurface immersed in the hyperbolic space with H_{r+1} positive. Then*

$$\left|\frac{c_r}{n}H_{r+1}I - AP_r\right| \leq \binom{n-1}{r}|H_{r+1}| + \binom{n}{r}|H_r||A| + |A^2P_{r-1}|.$$

In particular, if Σ^n has an elliptic point and H is bounded, then

$$\left|\frac{c_r}{n}H_{r+1}I - AP_r\right|$$

is bounded on Σ^n.

Proof. Using the definition of P_r,

$$\frac{c_r}{n}H_{r+1}I - AP_r = \binom{n-1}{r}H_{r+1}I - \binom{n}{r}H_rA + A^2P_{r-1}.$$

On the other hand, we observe that $|A| \leq n^2H^2 - n(n-1)H_2$ and

$$|A^2P_{r-1}| \leq \operatorname{tr}(A^2P_{r-1}) = \left(c_rH_{r+1} - \frac{n}{r}c_{r-1}HH_r\right).$$

If Σ^n has an elliptic point, since we are assuming that H_{r+1} is positive, then we can reason as in the proof of Lemma 2.1 to get that $H_j > 0$ for any $j \leq r$. Hence, the result follows on observing that the hypothesis that H is bounded implies, by [17, Equation (11)], that H_j is also bounded for all $j \leq r+1$. □

3. Main results

In order to establish our first results, we recall the description of totally umbilical space-like hypersurfaces of the de Sitter space $\mathbb{S}_1^{n+1}$ due to Montiel in [15] and its dual relation with totally umbilical hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$ described by López and Montiel in [14]. For this, we note that $\mathbb{S}_1^{n+1}$ admits a foliation by means of totally umbilical space-like hypersurfaces

$$L(\tau) = \{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle = \tau\},$$

where $a \in \mathbb{L}^{n+2}$, $\langle a, a\rangle = 1, 0, -1$ and $\tau^2 > \langle a, a\rangle$ (cf. [15, Example 1]). Consequently, we have that there exists a natural duality between the foliations of $\mathbb{S}_1^{n+1}$ and $\mathbb{H}^{n+1}$ through totally umbilical hypersurfaces. This duality follows from the fact that the

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totally umbilical hypersurfaces of $\mathbb{H}^{n+1}$ can be realized in the Lorentz–Minkowski model in the following way:

$$\mathcal{L}(\varrho) = \{p \in \mathbb{H}^{n+1}; \langle p, a\rangle = \varrho\},$$

where $a \in \mathbb{L}^{n+2}$ is a nonzero fixed vector and $\varrho^2 + \langle a, a\rangle > 0$ (cf. [14]). Furthermore, with a straightforward computation it is not difficult to verify that the Lorentzian Gauss mapping $N : \mathcal{L}(\varrho) \to \mathbb{S}_1^{n+1}$ of such a hypersurface is given by

$$N(p) = \frac{1}{\sqrt{\varrho^2 + \langle a, a\rangle}}(a + \varrho p). \tag{3.1}$$

Hence, from (3.1), we have that the angle function f_a of a totally umbilical hypersurface of $\mathbb{H}^{n+1}$ satisfies

$$f_a = \langle N, a\rangle = \sqrt{\varrho^2 + \langle a, a\rangle} = \tau = constant.$$

Therefore, Montiel's result [15] allows us to conclude that one of the following situations holds:

(i) if a is a unit space-like vector, then $N(\mathcal{L}(\varrho))$ is isometric to an n-dimensional hyperbolic space of constant sectional curvature $-1/(\tau^2 - 1)$;
(ii) if a is a nonzero null vector, then $N(\mathcal{L}(\varrho))$ is isometric to the Euclidean space $\mathbb{R}^n$;
(iii) if a is a unit time-like vector, then $N(\mathcal{L}(\varrho))$ is isometric to an n-dimensional sphere of constant sectional curvature $1/(\tau^2 + 1)$.

This description of the totally umbilical space-like hypersurfaces of $\mathbb{S}_1^{n+1}$ enables us to characterize some particular regions in $\mathbb{S}_1^{n+1}$. In the case that $a \in \mathbb{L}^{n+2}$ is a unit time-like vector, the level set $L(0) = \{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle = 0\}$ defines a round sphere of radius one, which is a totally geodesic hypersurface in $\mathbb{S}_1^{n+1}$. According to the terminology established by Aledo *et al.* [1], we refer to this round sphere as being the equator of $\mathbb{S}_1^{n+1}$ determined by a and we observe that it divides $\mathbb{S}_1^{n+1}$ into two connected components, the *chronological future*, which is given by

$$\{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle < 0\},$$

and the *chronological past*, given by

$$\{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle > 0\}.$$

We note that the first and third authors obtained a characterization of totally umbilical geodesic round spheres as the only closed constant mean curvature hypersurfaces immersed in the hyperbolic space $\mathbb{H}^{n+1}$ having its image by the Lorentzian Gauss map contained in the closure of a chronological future (or past) of an equator of $\mathbb{S}_1^{n+1}$ (cf. [6, Theorem 3.4]). Here, working with a new approach, we are able to give an extension of this result. More precisely, we have the following theorem.

THEOREM 3.1. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be a closed hypersurface immersed in the hyperbolic space with some constant $(r+1)$th mean curvature. If the image of the Lorentzian Gauss map of Σ^n is contained in the closure of the chronological future (or past) of an equator of $\mathbb{S}^{n+1}_1$, then Σ^n must be a totally umbilical geodesic sphere of $\mathbb{H}^{n+1}$.*

PROOF. Initially, we observe that, since Σ^n is closed, [8, Proposition 3.2] (see also [3, Lemma 8]) guarantees that there exists an elliptic point in Σ^n. Consequently, taking into account its constancy, it follows that $H_{r+1} > 0$ on Σ^n. Thus, from Lemma 2.1,

$$H_{r+2} - HH_{r+1} \leq 0 \quad \text{on } \Sigma^n. \tag{3.2}$$

Moreover, our hypothesis on the Lorentzian Gauss image $N(\Sigma)$ assures that there exists a time-like vector $a \in \mathbb{L}^{n+2}$ such that the corresponding angle function f_a does not change sign on Σ^n. Hence, from (2.4) and (3.2), we conclude that the following divergence:

$$\operatorname{div}\Big(P_r\nabla f_a + \frac{c_r}{n}H_{r+1}\nabla l_a\Big) = c_{r+1}(H_{r+2} - HH_{r+1})f_a \tag{3.3}$$

does not change sign on Σ^n. Consequently, using the divergence theorem in (3.3), we conclude that the following equation holds on Σ^n:

$$(H_{r+2} - HH_{r+1})f_a = 0.$$

We claim that, in fact, the function $h = H_{r+2} - HH_{r+1}$ vanishes identically on Σ^n. Indeed, if there exists $p_0 \in \Sigma^n$ such that $h(p_0) \neq 0$, then there exists a neighborhood $\mathcal{U}$ of p_0 in Σ^n in which $h \neq 0$ and $f_a = 0$ in $\mathcal{U}$. Thus, taking into account (2.3), this will give that f_a and l_a are simultaneously zero in $\mathcal{U}$. But such a situation cannot occur since (2.1) implies that $|\nabla l_a|^2 + f_a^2 - l_a^2 = -1$. Therefore, we must have $h = H_{r+2} - HH_{r+1} = 0$ on Σ^n and, hence, Lemma 2.1 assures that Σ^n is a totally umbilical geodesic sphere of $\mathbb{H}^{n+1}$. □

REMARK 3.2. In the previous theorem, the compactness of Σ^n cannot be dropped. In fact, it is possible to show that the hyperbolic cylinder

$$\Sigma^n = \mathbb{S}^k(\rho) \times \mathbb{H}^{n-k}\Big(\sqrt{1+\rho^2}\Big) \to \mathbb{H}^{n+1}$$

has the following Lorentzian Gauss map:

$$N(p) = -\frac{1}{\rho\sqrt{1+\rho^2}}(\nu(p) + \rho^2 p), \tag{3.4}$$

where $\nu : \Sigma^n \to \mathbb{L}^{n+2}$ is given by $\nu(p) = (p_1, \ldots, p_{k+1}, 0, \ldots, 0)$. The hyperbolic cylinders are examples of complete isoparametric hypersurfaces of $\mathbb{H}^{n+1}$. Now, let us consider the time-like vector $a = (0, \ldots, 0, 1)$. After a simple computation, we have from the expression in (3.4) that the corresponding angle and height functions satisfy the following linear dependence relation:

$$f_a = -\frac{\rho}{\sqrt{1+\rho^2}}l_a.$$

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Hence, using the reverse Cauchy–Schwarz inequality, we obtain that the angle function satisfies $|f_a| \geq \rho/\sqrt{1+\rho^2} > 0$ and, therefore, this means that the image of the Lorentzian Gauss map of Σ^n is contained in the chronological future (or past) of the equator of $\mathbb{S}_1^{n+1}$ determined by a.

In order to present our next theorems, we will quote another auxiliary lemma, which is a consequence of the version of Stokes' theorem given by Karp in [13] (see also [10, Proposition 2.1]). In what follows, $\mathcal{L}^1(\Sigma)$ denotes the space of Lebesgue integrable functions on Σ^n.

Lemma 3.3. *Let X be a smooth vector field on the n-dimensional complete noncompact oriented Riemannian manifold Σ^n, such that* $\mathrm{div}X$ *does not change sign on Σ^n. If $|X| \in \mathcal{L}^1(\Sigma)$, then* $\mathrm{div}X = 0$.

Motivated by the fact that the Lorentzian Gauss map N of a totally umbilical hypersurface of $\mathbb{H}^{n+1}$ satisfies $f_a = \langle N, a\rangle = \tau$ for some nonzero vector $a \in \mathbb{L}^{n+2}$ and some constant $\tau \in \mathbb{R}$, we obtain the following result.

Theorem 3.4. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be a complete hypersurface immersed in the hyperbolic space with some constant $(r+1)$th mean curvature. Suppose that Σ^n has an elliptic point and the image of the Lorentzian Gauss map of Σ^n is contained in a totally umbilical space-like hypersurface of $\mathbb{S}_1^{n+1}$ orthogonal to some nonzero vector $a \in \mathbb{L}^{n+2}$. If $|a^\top| \in \mathcal{L}^1(\Sigma)$, then Σ^n must be a totally umbilical hypersurface of $\mathbb{H}^{n+1}$ orthogonal to a.*

Proof. We note that our constraint on the Lorentzian Gauss map of Σ^n implies that the angle function $f_a = \tau$ for some constant τ. We claim that $\tau \neq 0$. Indeed, if $\tau = 0$, from (2.3) we also get that $l_a = 0$ on Σ^n. Thus, since $|\nabla l_a|^2 + f_a^2 - l_a^2 = \langle a, a\rangle$, a must be a nonzero null vector. But, by completeness, Σ^n should be a horosphere of $\mathbb{H}^{n+1}$, which contradicts the fact that $l_a = 0$.

Now, from (2.4),

$$\Delta l_a = \frac{n\tau c_{r+1}}{c_r H_{r+1}}(H_{r+2} - HH_{r+1}). \tag{3.5}$$

Hence, from (3.5) and Lemma 2.1, we have that Δl_a does not change sign on Σ^n and, since we are supposing that $|a^\top| \in \mathcal{L}^1(\Sigma)$, we can apply Lemma 3.3 to conclude that l_a is, in fact, a harmonic function on Σ^n.

Returning to (3.5), we infer that $H_{r+2} - HH_{r+1} = 0$ on Σ^n and, consequently, Σ^n must be totally umbilical. Moreover, considering $r = 0$ in (2.2), we get that l_a is also constant on Σ^n and, therefore, Σ^n is orthogonal to a. □

Taking into account once more the existence of an elliptic point for closed hypersurfaces of the hyperbolic space, from Theorem 3.4 we get the following consequence.

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COROLLARY 3.5. *The only closed hypersurfaces immersed in $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ with some constant $(r+1)$th mean curvature and whose image of the Lorentzian Gauss map is contained in a totally umbilical space-like hypersurface of $\mathbb{S}_1^{n+1}$ are the totally umbilical geodesic spheres.*

Adopting the terminology due to López and Montiel in [14], when $a \in \mathbb{L}^{n+2}$ is either a nonzero null or a space-like vector, we will refer to the *interior domain* enclosed by $L(\tau)$ as being the set

$$\{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle \geq \tau\},$$

while the *exterior domain* enclosed by $L(\tau)$ is given by

$$\{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle \leq \tau\}.$$

In this setting, we get the following result. Compare with [5, Theorem 1.2].

THEOREM 3.6. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be a complete noncompact hypersurface immersed in the hyperbolic space with some constant $(r+1)$th mean curvature. Suppose that Σ^n has an elliptic point, its mean curvature H is bounded and the image of its Lorentzian Gauss map is contained in a domain enclosed by a totally umbilical space-like hypersurface of $\mathbb{S}_1^{n+1}$ determined by either a nonzero null or a space-like vector $a \in \mathbb{L}^{n+2}$. If $|a^\top| \in \mathcal{L}^1(\Sigma)$, then Σ^n must be a totally umbilical hypersurface of $\mathbb{H}^{n+1}$.*

PROOF. Proceeding as in the proof of Theorem 3.1,

$$\operatorname{div}\left(P_r \nabla f_a + \frac{c_r}{n} H_{r+1} \nabla l_a\right) = c_{r+1}(H_{r+2} - HH_{r+1}) f_a$$

does not change sign on Σ^n for some nonzero null (space-like) vector $a \in \mathbb{L}^{n+2}$. On the other hand, since the mean curvature H of Σ^n is bounded, it follows from Lemma 2.2 that

$$\left|P_r \nabla f_a + \frac{c_r}{n} H_{r+1} \nabla l_a\right| \leq \left|AP_r + \frac{c_r}{n} H_{r+1}\right| |a^\top| \in \mathcal{L}^1(\Sigma).$$

Thus, from Lemma 3.3, we get $(H_{r+2} - HH_{r+1}) f_a = 0$ on Σ^n. Therefore, observing that the angle function f_a has strict sign on Σ^n, we conclude that $H_{r+2} - HH_{r+1}$ vanishes identically on Σ^n and, hence, Lemma 2.1 assures that Σ^n is a totally umbilical hypersurface of $\mathbb{H}^{n+1}$. □

4. Other rigidity and nonexistence results

In this last section, we will use different warped product models for the hyperbolic space $\mathbb{H}^{n+1}$ to obtain rigidity and nonexistence results concerning complete hypersurfaces in $\mathbb{H}^{n+1}$ having two consecutive higher order mean curvatures obeying a suitable inequality. For this, we will need the following generalized maximum principle due to Yau (cf. [22, Theorem 3]).

LEMMA 4.1. *Let Σ^n be a complete Riemannian manifold. If f is a nonnegative subharmonic function on Σ^n such that $f \in \mathcal{L}^q(\Sigma)$ for some $q > 1$, then f must be constant.*

Based on this fact, we get the following result.

THEOREM 4.2. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be a complete hypersurface immersed in the hyperbolic space. Assume that, for some $0 \le r \le n-1$, P_r is bounded from above (in the sense of quadratic forms). Suppose that, for some fixed time-like vector $a \in \mathbb{L}^{n+1}$, the following inequality is satisfied:*

$$0 \le H_{r+1} \le \mathcal{H} H_r, \tag{4.1}$$

where $\mathcal{H}(p)$ stands for the mean curvature of the totally umbilical geodesic sphere $\mathcal{L}(\varrho)$ of $\mathbb{H}^{n+1}$ which is orthogonal to a and passes through $p \in \psi(\Sigma)$. If $l_a \in \mathcal{L}^q(\Sigma)$ for some $q > 1$, then Σ^n must be isometric to a totally umbilical geodesic sphere $\mathcal{L}(\varrho_0)$ for some $\varrho_0 > 1$.

PROOF. Considering the vector field $X(p) = a + \langle p, a\rangle p$ defined on $\mathbb{H}^{n+1}$ and since $a \in \mathbb{L}^{n+2}$ is a time-like vector, we can apply item (a) of [16, Proposition 2] to see that $\mathbb{H}^{n+1}$ is isometric to the warped product space $\mathbb{R}^+ \times_{\sinh t} \mathbb{S}^n$, where each slice $\{t\} \times \mathbb{S}^n$ corresponds to a totally umbilical geodesic sphere $\mathcal{L}(\varrho) \subset \mathbb{H}^{n+1}$ which is orthogonal to a. In this setting, up to isometry, we have that $X = \sinh t\partial_t$ and $l_a = \cosh h$, where $h = \pi_{\mathbb{R}^+}|_\Sigma$ stands for the vertical height function of Σ^n.

Consequently, subtending the isometry between the quadric and this warped product model of $\mathbb{H}^{n+1}$ and assuming that there exists a positive constant β such that $P_r \le \beta$, it is not difficult to see that from (2.2),

$$\beta\Delta l_a \ge L_r l_a = c_r(\cosh h\, H_r + \sinh h\, H_{r+1}\langle N, \partial_t\rangle). \tag{4.2}$$

On the other hand, from [16, Proposition 1], we have that the mean curvature of a slice $\{t\} \times \mathbb{S}^n$ oriented by $-\partial_t$ is equal to $\coth t$. Thus, inequality (4.1) amounts to

$$0 \le H_{r+1} \le \coth h H_r. \tag{4.3}$$

From (4.2) and (4.3),

$$\Delta l_a \ge \frac{c_r \sinh h}{\beta}(\coth h H_r - H_{r+1}) \ge 0.$$

Therefore, since $l_a \ge 1$ and using our hypothesis that $l_a \in \mathcal{L}^q(\Sigma)$ for some $q > 1$, we can apply Lemma 4.1 to conclude that l_a is constant on Σ^n and, hence, Σ^n must be isometric to $\mathcal{L}(\varrho_0)$ for some $\varrho_0 > 1$. □

REMARK 4.3. Fix a point $p \in \mathbb{H}^3 \subset \mathbb{L}^4$; let us consider an orthonormal basis $\{e_1, e_2, e_3\}$ of $T_p\mathbb{H}^3$. According to [3, Example 10], we can define a revolution torus $\psi : [0, 2\pi] \times [0, 2\pi] \to \mathbb{H}^3$ as follows:

$$\begin{aligned}\psi(\theta, \phi) = {} & \cosh r(\cosh R\, p + \sinh R(\cos\theta e_1 + \sin\theta e_2) \\ & + \sinh r(\cos\phi(\sinh R\, p + \cosh R(\cos\theta e_1 + \sin\theta e_2))) + \sin\theta e_3),\end{aligned}$$

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where $R > r > 0$. With a straightforward computation we can verify that the principal curvatures λ_1 and λ_2 of this immersion are given by

$$\lambda_1 = \frac{\sinh r \sinh R + \cosh r \cosh R \cos\phi}{\cosh r \sinh R + \sinh r \cosh R \cos\phi}$$

and

$$\lambda_2 = -\coth r.$$

In particular, through such a torus, we see that hypothesis (4.1) in Theorem 4.2 is, in fact, necessary to conclude that the hypersurface be isometric to a totally umbilical geodesic sphere of the hyperbolic space.

Fixing a nonzero space-like vector $a \in \mathbb{L}^{n+1}$, in analogy with the terminology used in the Euclidean sphere, we will call the totally geodesic hyperbolic hyperplane $\mathcal{L}(0) = \{p \in \mathbb{H}^{n+1}; \langle p, a\rangle = 0\}$ the *equator* of $\mathbb{H}^{n+1}$ determined by a. So, such an equator naturally divides $\mathbb{H}^{n+1}$ into two *closed hemispheres*, which are given by

$$\mathbb{H}_a^+ = \{p \in \mathbb{H}^{n+1}; \langle p, a\rangle \geq 0\}$$

and

$$\mathbb{H}_a^- = \{p \in \mathbb{H}^{n+1}; \langle p, a\rangle \leq 0\}.$$

In this setting, reasoning in a similar way as that in the proof of Theorem 4.2, we obtain the following result.

Theorem 4.4. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be a complete hypersurface immersed in the hyperbolic space. Assume that, for some $0 \leq r \leq n-1$, P_r is bounded from above (in the sense of quadratic forms). Suppose that, for some fixed nonzero space-like vector $a \in \mathbb{L}^{n+1}$, $\psi(\Sigma) \subset \mathbb{H}_a^+$ and that the following inequality is satisfied:*

$$0 \leq H_{r+1} \leq \mathcal{H} H_r, \tag{4.4}$$

where $\mathcal{H}(p)$ stands for the mean curvature of the totally umbilical hyperbolic hyperplane $\mathcal{L}(\varrho)$ of $\mathbb{H}^{n+1}$ which is orthogonal to a and passes through $p \in \psi(\Sigma)$. If $l_a \in \mathcal{L}^q(\Sigma)$ for some $q > 1$, then Σ^n must be isometric to the totally geodesic hyperbolic hyperplane $\mathcal{L}(0)$.

Proof. Considering the vector field $X(p) = a + \langle p, a\rangle p$ defined on $\mathbb{H}^{n+1}$ and since $a \in \mathbb{L}^{n+2}$ is a nonzero space-like vector, we can apply item (c) of [16, Proposition 2] to see that $\mathbb{H}^{n+1}$ is isometric to the warped product space $\mathbb{R} \times_{\cosh t} \mathbb{H}^n$, where each slice $\{t\} \times \mathbb{H}^n$ corresponds to a totally umbilical hyperbolic hyperplane $\mathcal{L}(\varrho) \subset \mathbb{H}^{n+1}$ which is orthogonal to a. In this setting, up to isometry, we have that $X = \cosh t\, \partial_t$ and $l_a = \sinh h$, where $h = \pi_{\mathbb{R}}|_\Sigma$ stands for the vertical height function of Σ^n.

Consequently, subtending the isometry between the quadric and this warped product model of $\mathbb{H}^{n+1}$ and assuming that there exists a positive constant β such that $P_r \leq \beta$, it is not difficult to see that from (2.2),

$$\beta \Delta l_a \geq L_r l_a = c_r(\sinh h\, H_r + \cosh h\, H_{r+1}\langle N, \partial_t\rangle). \tag{4.5}$$

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On the other hand, from [16, Proposition 1], we have that the mean curvature of a slice $\{t\} \times \mathbb{H}^n$ oriented by $-\partial_t$ is equal to $\tanh t$. Thus, inequality (4.4) amounts to

$$0 \le H_{r+1} \le \tanh h H_r. \tag{4.6}$$

From (4.5) and (4.6),

$$\Delta l_a \ge \frac{c_r \cosh h}{\beta}(\tanh h H_r - H_{r+1}) \ge 0.$$

Moreover, we note that our hypothesis that $\psi(\Sigma) \subset \mathbb{H}^+_a$ implies that $l_a \ge 0$. Therefore, since we are also assuming that $l_a \in \mathcal{L}^q(\Sigma)$ for some $q > 1$, we can apply Lemma 4.1 to conclude that l_a is constant on Σ^n and, hence, Σ^n must be isometric to $\mathcal{L}(\varrho)$ for some $\varrho \ge 0$. Finally, taking into account once more that $l_a \in \mathcal{L}^q(\Sigma)$ for some $q > 1$, we see that, in fact, $\varrho = 0$. □

Remark 4.5. Considering the totally umbilical geodesic spheres of $\mathbb{H}^{n+1}$ which are contained in $\mathbb{H}^+_a$, we see that hypothesis (4.4) in Theorem 4.4 is necessary to conclude that the hypersurface Σ^n is isometric to the totally geodesic hyperbolic hyperplane $\mathcal{L}(0)$.

To close our paper, we will reason once more as in the proof of Theorem 4.2 to establish the following nonexistence result.

Theorem 4.6. *There exists no complete hypersurface $\psi : \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ such that, for some $0 \le r \le n-1$, P_r is bounded from above (in the sense of quadratic forms), $0 \le H_{r+1} \le H_r$ and, for some nonzero null vector $a \in \mathbb{L}^{n+2}$, $l_a \in \mathcal{L}^q(\Sigma)$ with $q > 1$.*

Proof. Suppose, by contradiction, that there exists such a hypersurface. Considering the vector field $X(p) = a + \langle p, a\rangle p$ defined on $\mathbb{H}^{n+1}$ and since $a \in \mathbb{L}^{n+2}$ is a nonzero null vector, we can apply item (b) of [16, Proposition 2] to see that $\mathbb{H}^{n+1}$ is isometric to the warped product space $\mathbb{R} \times_{e^t} \mathbb{R}^n$, where each slice $\{t\} \times \mathbb{R}^n$ corresponds to a totally umbilical Euclidean hyperplane $\mathcal{L}(\varrho) \subset \mathbb{H}^{n+1}$ which is orthogonal to a. In this setting, since $l_{-a} = -l_a$, we have (up to isometry) that $X = e^t\, \partial_t$ and $l_a = e^h$, where $h = \pi_{\mathbb{R}}|_\Sigma$ stands for the vertical height function of Σ^n.

Consequently, subtending the isometry between the quadric and this warped product model of $\mathbb{H}^{n+1}$, it is not difficult to see that from (2.2),

$$\beta \Delta l_a \ge L_r l_a = c_r e^h (H_r + H_{r+1}\langle N, \partial_t\rangle). \tag{4.7}$$

Thus, since we are assuming that $0 \le H_{r+1} \le H_r$, from (4.7),

$$\Delta l_a \ge \frac{c_r e^h}{\beta}(H_r - H_{r+1}) \ge 0.$$

Hence, from $l_a \in \mathcal{L}^q(\Sigma)$ for some $q > 1$, we can apply Lemma 4.1 to conclude that l_a is a positive constant on Σ^n and, consequently, Σ^n must be a horosphere of $\mathbb{H}^{n+1}$. In particular, we get that Σ^n is isometric to $\mathbb{R}^n$. Therefore, since the hypothesis $l_a \in \mathcal{L}^q(\Sigma)$ also implies that Σ^n must have finite volume, we have reached a contradiction. □

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References

[1] J. Aledo, L. J. Alías and A. Romero, 'Integral formulas for compact space-like hypersurfaces in de Sitter space: applications to the case of constant higher order mean curvature', *J. Geom. Phys.* **31** (1999), 195–208.

[2] L. J. Alías and M. Dajczer, 'Uniqueness of constant mean curvature surfaces properly immersed in a slab', *Comment. Math. Helv.* **81** (2006), 653–663.

[3] L. J. Alías, T. Kurose and G. Solanes, 'Hadamard-type theorems for hypersurfaces in hyperbolic spaces', *Differential Geom. Appl.* **24** (2006), 492–502.

[4] C. P. Aquino, 'On the Gauss mapping of hypersurfaces with constant scalar curvature in $\mathbb{H}^{n+1}$', *Bull. Braz. Math. Soc.* **45** (2014), 117–131.

[5] C. P. Aquino, A. Barros and H. F. de Lima, 'Complete CMC hypersurfaces in the hyperbolic space with prescribed Gauss mapping', *Proc. Amer. Math. Soc.* **142** (2014), 3597–3604.

[6] C. P. Aquino and H. F. de Lima, 'On the Gauss map of complete CMC hypersurfaces in the hyperbolic space', *J. Math. Anal. Appl.* **386** (2012), 862–869.

[7] C. P. Aquino and H. F. de Lima, 'On the geometry of horospheres', *Comment. Math. Helv.* **89** (2014), 617–629.

[8] J. L. Barbosa and A. G. Colares, 'Stability of hypersurfaces with constant r-mean curvature', *Ann. Global Anal. Geom.* **15** (1997), 277–297.

[9] F. Camargo, A. Caminha and H. F. de Lima, 'Bernstein-type theorems in semi-Riemannian warped products', *Proc. Amer. Math. Soc.* **139** (2011), 1841–1850.

[10] A. Caminha, 'The geometry of closed conformal vector fields on Riemannian spaces', *Bull. Braz. Math. Soc.* **42** (2011), 277–300.

[11] A. Caminha and H. F. de Lima, 'Complete vertical graphs with constant mean curvature in semi-Riemannian warped products', *Bull. Belg. Math. Soc.* **16** (2009), 91–105.

[12] M. do Carmo and B. Lawson, 'The Alexandrov–Bernstein theorems in hyperbolic space', *Duke Math. J.* **50** (1983), 995–1003.

[13] L. Karp, 'On Stokes' theorem for noncompact manifolds', *Proc. Amer. Math. Soc.* **82** (1981), 487–490.

[14] R. López and S. Montiel, 'Existence of constant mean curvature graphs in hyperbolic space', *Calc. Var.* **8** (1999), 177–190.

[15] S. Montiel, 'An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature', *Indiana Univ. Math. J.* **37** (1988), 909–917.

[16] S. Montiel, 'Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds', *Indiana Univ. Math. J.* **48** (1999), 711–748.

[17] S. Montiel and A. Ros, 'Compact hypersurfaces: the Alexandrov theorem for higher order mean curvatures', in: *Differential Geometry*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 52 (eds. B. Lawson and K. Tenenblat) (Longman Scientific & Technical, Harlow, 1991), 279–296.

[18] R. Reilly, 'Variational properties of functions of the mean curvature for hypersurfaces in space form', *J. Differential Geom.* **8** (1973), 447–453.

[19] H. Rosenberg, 'Hypersurfaces of constant curvature in space forms', *Bull. Sci. Math.* **117** (1993), 217–239.

[20] S. Shu, 'Complete hypersurfaces with constant scalar curvature in a hyperbolic space', *Balkan J. Geom. Appl.* **12** (2007), 107–115.

[21] Q. Wang and C. Xia, 'Topological and metric rigidity theorems for hypersurfaces in a hyperbolic space', *Czechoslovak Math. J.* **57** (2007), 435–445.

[22] S. T. Yau, 'Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry', *Indiana Univ. Math. J.* **25** (1976), 659–670.

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