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Free algebra structure: categorical algebras

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One of the more important concepts in the study of universal algebras is that of a free algebra. It is our purpose in this communication to describe the structure of the free algebra $F_k(\underline{\underline{A}})$ of k generators (k a positive integer) determined by a categorical algebra, and to indicate how this information encompasses results in such diverse areas as the study of Post algebras, boolean rings, p-rings, p^k-rings, finite commutative rings with unity, etc.

A finite algebra $\underline{\underline{A}}$ is called *categorical* if every algebra in its equational class is isomorphic to a sub-direct power of A. If $\underline{\underline{A}}$ has n elements, permutable identities, no non-identical automorphism and exactly m distinct one-element subalgebras, then $F_k(\underline{\underline{A}}) \cong \underline{\underline{A}}^{n^k-m}$.

1. Introduction

One of the more important concepts in the study of universal algebras is that of a free algebra. It is our purpose in this communication to describe the structure of the free algebra of k generators (k a positive integer) determined by a categorical algebra, and to indicate how this information encompasses results in such diverse areas as the study of Post algebras, boolean rings, p-rings, p^k-rings, finite commutative rings with unity, etc.

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In [3] A.L. Foster developed the theory of *primal* universal algebras, subsuming much of the theory of boolean rings, Post algebras, etc. He proved that every primal algebra is categorical. A finite algebra $\underline{\underline{A}}$ is called *categorical* if every algebra in its equational class is isomorphic to a sub-direct power of $\underline{\underline{A}}$. In [4] Foster established the structure of the free algebra of k generators determined by (the identities of) a primal algebra. The primal theory is itself contained in the study of the wider class of *semi-primal* algebras. In [7] we have considered the structure of a large class of algebras which are semi-primal, but not primal, namely, *subprimals*; and in [8] the structure of the free algebra with respect to this type of algebra is established along with a generalization of Foster's results to clusters of primal algebras.

Since the primal theory is also subsumed in the study of categorical algebras, we now direct our attention to them. In particular we prove

THEOREM 1. *Let* $\underline{\underline{A}}$ *be a categorical algebra with* n *elements. Suppose that* $\underline{\underline{A}}$ *has permutable identities and no non-identical automorphisms. Then, if* $\underline{\underline{A}}$ *has exactly* m *distinct one-element sub-algebras, the free algebra* $F_k(\underline{\underline{A}})$ *of* k *generators (*k *a positive integer), determined by the identities of* $\underline{\underline{A}}$*, is isomorphic with the* (n^k-m)*-th direct power of* $\underline{\underline{A}}$ *:*

$$F_k(\underline{\underline{A}}) \cong \underline{\underline{A}}^{n^k-m} .$$

2. Fundamental concepts

We recall some basic definitions of [3] (see also [2], [6]). Let $\underline{\underline{A}} = (A, \Omega)$ be a universal algebra of *species* (similarity class) $S = (n_1, n_2, \ldots)$, where the n_i are non-negative integers, A is a set, and $\Omega = (o_1, o_2, \ldots)$ is a collection of finitary operations on A - each o_i being an n_i-ary operation on A. When two algebras $\underline{\underline{A}}$ and $\underline{\underline{A}}'$ are of the same species we, as is customary, identify the two sets of operations Ω and Ω' and use the same operation symbols for both. An *S-expression* $\Phi(\xi_1, \xi_2, \ldots)$ is any primitive composition of free symbols $\xi_1, \ldots$ via the primitive operations o_i of Ω. A (set theoretic)

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function $F(\xi_1, \ldots, \xi_k)$, called an $\underline{\underline{A}}$-*function*, from the k-th cartesian product (power) of A into A , is said to be *expressible* if there exists an S-expression Φ such that $F = \Phi$ on A ; that is, Φ yields F when members of A are substituted for the free symbols in Φ . When, for two S-expressions Φ and Ψ , we write

$$\Phi = \Psi(\underline{\underline{A}}) ,$$

we indicate an *identity* of $\underline{\underline{A}}$. A finite algebra $\underline{\underline{A}}$, different from the one element algebra $\overset{\circ}{|}$ (non-trivial) is called *primal* if every $\underline{\underline{A}}$-function is expressible.

If $\underline{\underline{A}}$ is any algebra, then the *equational class* of $\underline{\underline{A}}$ is the largest class K of algebras such that the identities of $\underline{\underline{A}}$, which we denote by $|\underline{\underline{A}}|$, are satisfied by the algebras in K . Birkhoff [2; 149] has demonstrated that the equational class of $\underline{\underline{A}}$ is the closure of $\{\underline{\underline{A}}\}$ under the constructions: direct product, sub-algebra, and homomorphic image. Algebras in the equational class of $\underline{\underline{A}}$ are referred to as $\underline{\underline{A}}$-*algebras*. If $\underline{\underline{A}}$ is a finite non-trivial algebra, and if every $\underline{\underline{A}}$-algebra is isomorphic to a subdirect power of $\underline{\underline{A}}$, then $\underline{\underline{A}}$ is called *categorical*.

We note the following facts concerning categorical algebras.

LEMMA 1. *Let* $\underline{\underline{A}}$ *be a categorical algebra. Then*

(i) $\underline{\underline{A}}$ *is simple; that is, has no non-trivial congruences;*

(ii) $\underline{\underline{A}}$ *has essentially no sub-algebras; that is, every sub-algebra of* $\underline{\underline{A}}$ *is either* $\underline{\underline{A}}$ *or has only one element;*

(iii) if $\underline{\underline{B}}$ *is a subdirect power of* $\underline{\underline{A}}$ *and* Θ *is any congruence relation on* $\underline{\underline{B}}$ *, then* Θ *is maximal, if and only if* $\underline{\underline{B}}/\Theta \cong \underline{\underline{A}}$ *or* $\underline{\underline{B}}/\Theta \cong \overset{\circ}{|}$ *;*

(iv) if $\underline{\underline{B}}$ *is a subdirect power of* $\underline{\underline{A}}$ *, then every congruence* Θ *of* $\underline{\underline{B}}$ *is the intersection of maximal congruences containing it;*

(v) every pair of elements $x, y \in A$ *generates* $\underline{\underline{A}}$.

Conversely, if $\underline{\underline{A}}$ *is any finite algebra with more than one element* ,

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satisfying (ii), (iii) and (iv), then $\underline{\underline{A}}$ *is categorical.*

Proof. For *(i)* we note that $\underline{\underline{A}}/\Theta$ is a homomorphic image of $\underline{\underline{A}}$, hence an $\underline{\underline{A}}$-algebra, so a subdirect power of $\underline{\underline{A}}$. Therefore, it must be $\underline{\underline{A}}$ itself. Parts *(ii)*, *(iii)* and *(iv)* and the converse are due to Astromoff [1; Theorem 1], and *(v)* follows from *(ii)*.

Congruence relations Θ_1 and Θ_2 of an algebra $\underline{\underline{A}}$ are (pairwise) *permutable* if $x \equiv y(\Theta_1)$ and $y \equiv z(\Theta_2)$ implies that there exists $w \in A$ such that $x \equiv w(\Theta_2)$, and $w \equiv z(\Theta_1)$. We write $\Theta_1\Theta_2 = \Theta_2\Theta_1$. The importance to us of permutable congruences is seen in the following result of Foster and Pixley [5; Theorem 2.4].

LEMMA 2. *If* $\underline{\underline{A}}$ *is an algebra with permutable congruences and* $\underline{\underline{A}}$ *is isomorphic with a subdirect product of finitely many simple algebras* $\underline{\underline{A}}_1, \underline{\underline{A}}_2, \ldots, \underline{\underline{A}}_n$, *then* $\underline{\underline{A}}$ *is isomorphic with a direct product of a subset of the* $\underline{\underline{A}}_i$.

If every algebra in the equational class of an algebra $\underline{\underline{A}}$ has permutable congruences, then the permutability is clearly dependent upon the identities of $\underline{\underline{A}}$. When this is true, we say that the *identities* $|\underline{\underline{A}}|$ of $\underline{\underline{A}}$ *are permutable.*

3. The free algebra $F_k(\underline{\underline{A}})$

We turn our attention to the free algebra $F_k(\underline{\underline{A}})$ of k generators determined by the categorical algebra $\underline{\underline{A}}$ with n elements, no non-identical automorphisms, and permutable identities. That this algebra, which we shall henceforth denote by F_k, exists, belongs to the equational class of $\underline{\underline{A}}$, and is a sub-algebra of $\underline{\underline{A}}^{n^k}$ (the n^k-th direct power of $\underline{\underline{A}}$) follows easily from results of Birkhoff [2; p. 144].

From Lemma 2 we can conclude that F_k is not only a subdirect power of $\underline{\underline{A}}$, but is, in fact, a direct power.

$$F_k \cong \underline{\underline{A}}^t .$$

We shall demonstrate that $t = n^k - m$, where m is the number

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of distinct one-element sub-algebras of $\underline{\underline{A}}$. Although the restrictions on $\underline{\underline{A}}$ - that it have permutable identities and no non-identical automorphism - seem rather severe, it is to be noted that all of the well-known examples of these algebras satisfy both conditions. We proceed to the proof of Theorem 1.

Proof. As in [5; Lemma 4.2] we let $G = \{\xi_1, \xi_2, \ldots, \xi_k\}$ be a set of free symbols, k a fixed positive integer. Consider the n^k maps f_i from G into $\underline{\underline{A}}$. The range of each f_i generates a sub-algebra of $\underline{\underline{A}}$.

For each one-element sub-algebra $\underline{\underline{A}}_i = \{a_i\}$ of $\underline{\underline{A}}$ there is exactly one map f_i from G to $\underline{\underline{A}}_i$, namely the map $f_i(\xi_j) = \alpha_i$, all $j = 1, 2, \ldots, k$, since any two distinct elements of $\underline{\underline{A}}$ generate all of $\underline{\underline{A}}$. We select our indices $i = 1, 2, \ldots, n^k$ in such a way that the last m f_i's are of this type. Thus the range of each of the first $n^k - m$ maps generates all of $\underline{\underline{A}}$. Using G as a generating set, now construct the free algebra $F_k(\underline{\underline{A}}) = F_k$.

For each map f_i there is a homomorphism h_i of F_k to $\underline{\underline{A}}$ defined as follows: for each $\rho \in F_k$ set

$$(3.1) \qquad h_i(\rho) = r\left[f_i(\xi_1), \ldots, f_i(\xi_k)\right] , \quad i = 1, 2, \ldots, n^k ,$$

where r is any S-expression (word) in ρ . Since r_1, r_2 belong to the same ρ if, and only if, $r_1 = r_2$ is an identity of $\underline{\underline{A}}$, h_i is well defined by (3.1). To see that h_i is indeed a homomorphism, for each $i = 1, 2, \ldots, n^k$, let $\rho_1, \rho_2, \ldots, \rho_n$ be elements of F_k , and let o be any q-ary operation on F_k . Then set $o(\rho_1, \ldots, \rho_q) = \rho$. It is clear that, for $r_i \in \rho_j$, $(j = 1, 2, \ldots, q)$, $o(r_1, \ldots, r_q) \in \rho$. Conversely, if $r \in \rho$, one can always find $r_j \in \rho_j$ such that $o(r_1, \ldots, r_q) = r$. Thus,

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$$h_i\left[o\left(\rho_1, \ldots, \rho_q\right)\right] = h_i(\rho) ,$$

and

$$\begin{aligned} h_i(\rho) &= r\left[f_i(\xi_1), \ldots, f_i(\xi_k)\right] = o(r_1, \ldots, r_q)\left[f_i(\xi_1), \ldots, f_i(\xi_k)\right] \\ &= o\left[r_1\left[f_i(\xi), \ldots, f_i(\xi_k)\right], \ldots, r_q\left[f_i(\xi_1), \ldots, f_i(\xi_r)\right]\right] \\ &= o\left[h_i(\rho_1), \ldots, h_i(\rho_q)\right] . \end{aligned}$$

Let Θ_i be the congruence in F_k induced by h_i ; that is, $\rho_1 = \rho_2(\Theta_i)$ iff $h_i(\rho_1) = h_i(\rho_2)$. Thus, $F_k/\Theta_i \cong \underline{\underline{A}}$, for $i = 1, 2, \ldots, n^k - m$, and $F_k/\Theta_i \cong \mathfrak{l}$, for $i = n^k-m+1, \ldots, n^k$. Suppose ρ_1 and $\rho_2 \in F_k$. We have $\rho_1 \equiv \rho_2\left(\bigwedge_{i=1}^{n^k} \Theta_i\right)$ if, and only if, for any choice of $r_1 \in \rho_1$ and $r_2 \in \rho_2$,

$$(3.2) \qquad r_1\left[f_i(\xi_1), \ldots, f_i(\xi_k)\right] = r_2\left[f_i(\xi_1), \ldots, f_i(\xi_k)\right] ,$$

for each $i = 1, 2, \ldots, n^k$. But since the f_i are all the possible maps of G into $\underline{\underline{A}}$, (3.2) holds if, and only if, $r_1 = r_2$ is an identity of $\underline{\underline{A}}$; whence, if, and only if, $\rho_1 = \rho_2$. Thus $\bigwedge_{i=1}^{n_k} \Theta_i = 0$. But since the last m of the algebras F_k/Θ_i are each one element, the last m congruences $\Theta_i = 1, = (F_k)$. Hence, we conclude

$$(3.3) \qquad \bigwedge_{i=1}^{n^k-m} \Theta_i = 0$$

when these are deleted. By Birkhoff's theorem [2; p. 140], then, F_k is isomorphic with a subdirect power of $\underline{\underline{A}} \cong F_k/\Theta_i$:

$$F_k \subseteq \underline{\underline{A}}^{n^k-m} ,$$

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where the isomorphism is given by the correspondence

$$(3.5)\quad \rho \to \left[r[f_1(\xi_1), \ldots, f_1(\xi_k)], \ldots, r[f_{n^k-m}(\xi_1), \ldots, f_{n^k-m}(\xi_k)]\right].$$

When $i \neq j$, h_i and h_j are distinct homomorphisms, hence if $F_k/\Theta_i = F_k/\Theta_j$ we would have $\underline{\underline{A}} \cong F_k/\Theta_i = F_k/\Theta_j \cong \underline{\underline{A}}$. This induces a non-identical automorphism of $\underline{\underline{A}}$, contrary to hypothesis. Each F_k/Θ_i is subdirectly irreducible, each Θ_i is meet irreducible, and (3.3) is a representation of the zero congruence (diagonal of F_k) as the meet of distinct meet irreducible elements; it therefore cannot be shortened. The maximality of each Θ_i implies that, for $i = 2, \ldots, n^k-m$, $(\Theta_1 \wedge \ldots \wedge \Theta_{i-1}) \vee \Theta_i = 1$. Birkhoff's theorem [2; p. 164] yields the desired result that F_k is the *direct* product of the algebras F_k/Θ_i.

4. Applications and problems

The wide applicability of this theorem is seen in the following examples of categorical algebras.

(1) *Every basic Post algebra of order* n *is categorical* (Wade [9]).

We recall that the basic Post algebra of order $n \geq 2$ is the algebra $(A, \cdot, \frown)$, where $A = \{1, 2, \ldots, n\}$, with usual ordering, 1 being the least element and n the largest. The operations $\cdot$ and $\frown$ are defined by

$$x \cdot y = \max\{x, y\},$$

$$x^\frown = \begin{cases} x+1 & \text{if } x \neq n \\ 1 & \text{if } x = n. \end{cases}$$

(2) *Every primal algebra is categorical* (Foster [3], Astromoff [1]).

This includes the classical results for boolean rings, Post algebras, p-rings, and p^k rings. (Some modifications of the latter two are necessary before they are primal.)

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(3) *Every singular subprimal algebra is categorical* (Yaqub [10]), where *singular subprimals* are semi-primal algebras with exactly one subalgebra $\underline{\underline{C}}$, which consists of a single element. In particular, Yaqub has shown that if $(R, \times, +)$ is *any* finite commutative ring with unity 1 $(1 \neq 0)$, a permutation $\frown$ (unary operation) is constructable such that $\hat{0} = 0$ and $(R, \times, +, \frown)$ is a singular subprimal.

Each of the above examples of categorical algebras satisfies the conditions of Theorem 1. Each has permutable identities and no non-identical automorphisms [3], [5]. Thus, Theorem 1 gives, in each case, the structure of the free algebra $F_k(\underline{\underline{A}})$ in k generators determined by the identities of the respective algebras $\underline{\underline{A}}$. In particular for the finite commutative ring example we have:

COROLLARY. *Let* $(R, \times, +)$ *be a finite commutative ring with unity* 1, $(1 \neq 0)$. *Let* $\frown$ *be the permutation of* R *such that* $\underline{\underline{R}} = (R, \times, +, \frown)$ *is a singular subprimal. Then the free algebra* $F_k(\underline{\underline{R}})$ *determined by* $\underline{\underline{R}}$ *has* n^{n^k-1} *elements where* n *is the order of* R, *and* $F_k(\underline{\underline{R}}) \cong \underline{\underline{R}}^{n^k-1}$.

Two interesting questions arise in this context.

(1) Since the existence of permutable identities is a fairly restrictive requirement, do categorical algebras exist whose congruences are not permutable?

(2) Even though each of the above examples has no non-trivial automorphisms, are there categorical algebras with non-trivial automorphisms? How are these automorphisms related to the congruences of F_k?

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