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PROPERTY (G), REGULARITY, AND SEMI-EQUICONTINUITY

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1. This note, motivated by [2], [3], and [4], is devoted to an investigation of properties related to equicontinuity in function spaces of topological spaces. In §2, we study the property (G) defined in [3], and the regularity defined in [4]. A sufficient condition for the simultaneous continuity of a function of two variables, which is analogous to a well known result in equicontinuity, is given at the end of the section. In §3, we relate the regularity with the semi-equicontinuity defined in [2], by localizing the semi-equicontinuity in an obvious way which leads us to weaken some of the hypotheses used in [2]. By the way of constructing an example, we also obtained a sufficient condition for a regular semitopological group to be a topological group.

Throughout this note, X and Y are general topological spaces unless otherwise specified. Y^X will denote the set of all functions on X to Y while (X, Y) will be the set of all continuous functions on X to Y. The reader is referred to [5] for definitions and notations not defined here.

2. **Property (G), and Regularity.**

DEFINITION 1 [3]. $F \subset Y^X$ is said to have the property (G) if for each open set U in Y and each pointwise closed subset G of F, $\bigcap_{f\in G} f^{-1}(U)$ is open in X.

DEFINITION 2 [4]. $F \subset Y^X$ is said to be regular at x in X if for each open set U in Y, and $G \subset F$ such that $\overline{G(x)} \subset U$, there exists an open neighborhood V of x such that $f(V) \subset U$ for each f in G. F is said to be regular if it is regular at each point of X.

REMARK. Members of a regular family $F \subset Y^X$ or members of a family $F \subset Y^X$ having property (G) are not necessary continuous as Example 1 shows. If Y is T_1, or regular, and if $F \subset Y^X$ is regular or has property (G), then each member of F is continuous.

EXAMPLE 1. Let X be the set of all reals with the usual topology, and Y be the set $\{0, 1\}$ endowed with the topology generated by $\{0\}$.

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(a) If $F=Y^X$, it is easy to see that F is regular at each point of X. But F is the pointwise closure of the set $\{f\}$, where $f(x)=0$ for $x \in X$, in Y^X, and F contains the noncontinuous function g, where $g(x)=0$ if $x \leq 0$ and $g(x)=1$ otherwise.

(b) If H is the family $\{g, h\}$, where $h(x)=0$ if $x<0$ and $h(x)=1$ otherwise, then the nonempty pointwise closed subsets of H are $\{h\}$ and H, H has property (G), but g is not continuous.

THEOREM 1. *If $F \subset (X, Y)$ has the property (G), then F is regular.*

Proof. Suppose $F \subset (X, Y)$ has property (G), and x in X. Let U be open in Y, and $G \subset F$ such that $\overline{(Gx)} \subset U$. If $\bar{G}$ is the pointwise closure of G in F, then $\bar{G}(x) \subset \overline{G(x)} \subset U$. Thus $N=\bigcap_{f \in G} f^{-1}(U)$ is open in X and contains x, so $f(N) \subset U$ for each $f \in G$, and F is regular at x.

EXAMPLE 2. Let X be the set of all reals with the usual topology. For each integer n, let $f_n: X \to X$ be defined by $f_n(x)=n+x$, and let $F=\{f_n : n \text{ integers}\}$. It is easy to see that F is equicontinuous at every point of X, but F is not regular at every point of X. To see it is not regular at $p \in X$, let $U=\bigcup_n U_n$, where $U_n=(n+p-(1/n), n+p+(1/n))$ for each n. Then $\overline{F(p)} \subset U$ but no neighborhood N of p exist such that $f_n(N) \subset U$ for each n.

We recall that a family $F \subset Y^X$ is said to be evenly continuous at $x \in X$ if for each y in Y and each neighborhood V of y, there is a neighborhood U of x and a neighborhood W of y such that $f(U) \subset V$ whenever $f(x)$ is in W. A family $F \subset Y^X$ is said to be evenly continuous (on X) if F is evenly continuous at each point of X.

THEOREM 2. *If Y is a regular space, and if $F \subset Y^X$ is regular at x, then F is evenly continuous at x. There is an example of $F \subset (X, Y)$ which is evenly continuous at each point, but F is regular at no point of X.*

Proof. The first half is Lemma (2.5) of [4].

EXAMPLE 3. Let X be the set of all reals with the topology having all intervals of the form $[a, b)$, $a<b$, as a base. For each a in X, let $f_a(x)=x+a$, for x in X. Then it is not hard to see that the family $\{f_a : a \text{ in } X\}$ is evenly continuous, but is regular at no point of X. To see this, for each positive integer n, let $f_n: X \to X$ be defined by $f_n(x)=x+n$. If p is in X, and $U=\bigcup_n [n+p, n+p+(1/n))$, then U is open in X, and $\overline{F(p)} \subset U$ since the family $\{[n+p, n+p+(1/n)) : n \text{ positive integers}\}$ is locally finite, where $F=\{f_n : n \text{ positive integers}\}$. In order that the family $\{f_a : a \text{ in } X\}$ be regular at p, we would have to have a neighborhood $V=[p, p+b)$, $b>0$, of p such that $f_n([p, p+b))=[n+p, n+p+b) \subset [n+p, n+p+(1/n))$ for each positive integer n, but it is impossible. Thus the family $\{f_a : a \text{ in } X\}$ is not regular at p. Note that $\overline{F(p)}$ is not compact for each p in X.

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REMARK. If Y is not regular, $F \subset Y^X$ may be regular at a point p in X without being evenly continuous at p, as Example 1 has shown. If Y is a regular space, and $F \subset Y^X$ is regular, then the pointwise closure $\bar{F}$ of F in Y^X is contained in (X, Y). As pointed out in [5, p. 237], even if $F \subset (X, Y)$ is evenly continuous and $F(x)$ is a totally bounded subset of a uniform space Y, F need not be equicontinuous at x. The following theorem reflects the fact that the regularity is much stronger than the even continuity in some sense.

THEOREM 3. *If Y is a uniform space, $F \subset (X, Y)$ is regular at x, and $F(x)$ is a totally bounded subset of Y, then F is equicontinuous at x. Conversely, if F is equicontinuous at x and every two-element open cover for $\overline{F(x)}$ is uniform, then F is regular at x.*

Proof. Let U be an entourage of Y, V an open symmetric entourage of Y, and W a closed entourage of Y such that $V^2 \subset U$ and $W \subset V$. For y in $\overline{F(x)}$, if $G_y = \{f \in F: (y, f(x)) \in W\}$, then G_y is a nonempty subset of F and $\overline{G_y(x)} \subset W[y] \subset V[y]$. Thus there is a neighborhood N_y of x such that $f(N) \subset V[y]$ for each f in G_y. By totally boundedness of $F(x)$ there is a finite subset $\{y_1, y_2, \ldots, y_n\}$ of $F(x)$ such that $F(x) \subset \bigcap_{i=1}^n W[y_i]$. For each y_i, define G_i and N_i as above, and let $N = \bigcap_{i=1}^n N_i$. Then N is a neighborhood of x. If $f \in F$, then $f(x) \in W[y_i]$ for some i, hence $f(N) \subset V[y_i]$. Thus, if z is in N, then $(f(x), f(z)) \in V^2 \subset U$. Hence F is equicontinuous at x.

For the second part, let U be an open subset of Y, and $G \subset F$ such that $\overline{G(x)} \subset U$. If $U = \{U, Y - \overline{G(x)}\}$, U is a two-element open cover for $\overline{F(x)}$, so there is an entourage $\bar{V}$ of Y such that $V[f(x)]$ is contained in one of the member of U whenever $f \in G$. Hence, for each f in G, $V[f(x)] \subset U$. By the equicontinuity of F at x, there is a neighborhood N of x such that $f(N) \subset V[f(x)]$ for each f in G. This shows that F is regular at X.

COROLLARY. *If Y is a uniform space, and $F \subset (X, Y)$ such that $\overline{F(x)}$ is compact, then F is equicontinuous at x if and only if F is regular at x.*

THEOREM 4. *If a family F of functions on a topological space X to a Hausdorff or regular space Y is compact relative to a jointly continuous topology τ, then F has the property (G).*

Proof. If Y is Hausdorff, the pointwise topology for F is Hausdorff and is smaller than τ, thus it coincides with τ. If Y is regular, F is regular by Theorem (2.1) of [4], thus F is evenly continuous by Theorem 2 above, the pointwise topology for F is jointly continuous Theorem 7.19 [5], and F is compact relative to the pointwise topology. Hence, if either Y is Hausdorff or regular, F is compact relative to the jointly continuous pointwise topology.

Let G be any pointwise closed subset of F and let U be any open subset of Y. We need to show that $\bigcap_{f\in G} f^{-1}(U)$ is open in X. For this purpose, let $x \in \bigcap_{f\in G} f^{-1}(U)$. The compact set $G\times\{x\}$ of $F\times X$ is contained in $p^{-1}(U)$, where P is the function from $E\times X$ to Y such that $P(f, x)=f(x)$, and $p^{-1}(U)$ is open since the pointwise topology for F is jointly continuous. Therefore, there exists an open neighborhood V of x such that $P(G\times V) \subset U$, i.e. $f(V) \subset U$ for all f in G. Hence $\bigcap_{f\in G} f^{-1}$ (U) is open and the family F has property (G).

THEOREM 5. *A family F of continuous functions on a k-space X to a regular space Y has a compact closure $\bar{F}$ in (X, Y) relative to the compact-open topology if and only if* (1) *$\overline{F(x)}$ is compact for every x in X, and* (2) *$\bar{F}$ has the property* (G).

Proof. If (2) is satisfied, F is evenly continuous by Theorem 1 and Theorem 2 above, thus F has the same closure $\bar{F}$ in Y^X relative to the compact-open and pointwise topologies by the Lemma of [6, p. 20], $\bar{F} \subset (X, Y)$ and two topologies for $\bar{F}$ coincide. Since $\bar{F}$ is a closed subset of the compact space $X\{\overline{F(x)}:x \in X\}$, $\bar{F}$ is compact in the compact-open topology.

Conversely, suppose F has a compact closure $\bar{F}$ in (X, Y) relative to the compact-open topology. By Theorem A of [1], the compact-open topology for $\bar{F}$ is jointly continuous on compacta. But $\bar{F}\times X$ is a k-space, the compact-open topology for $\bar{F}$ is jointly continuous, thus $\bar{F}$ has the property (G) by Theorem 4. Thus the compact-open and the pointwise topologies for $\bar{F}$ coincide, and that $\overline{F(x)}$ is compact follows easily.

We recall that a topological space X is called a P-space if every G_δ set in X is open.

THEOREM 6. *Assume Y is a regular space, and $F \subset (X, Y)$ is evenly continuous at x in X. If either* (a) *$\overline{F(x)}$ is compact, or* (b) *X is a P-space and $\overline{F(x)}$ is Lindelof, then F is regular at x.*

Proof. Part (a) is a part of Theorem A [4].

For the second part, assume X is a P-space, and $\overline{F(x)}$ is Lindelof. Let U be an open subset of Y, and $G \subset F$ such that $\overline{G(x)} \subset U$. For each y in $\overline{G(x)}$, there is a neighborhood V_y of x and an open neighborhood W_y of y, $\overline{W_y} \subset U$, such that $f(V_y) \subset U$ whenever $f\in F$ with $f(x) \in W_y$. The family $\{W_y:y \in \overline{G(x)}\}$ forms an open cover for $\overline{G(x)}$, so there is a countable subcover $\{W_1, W_2, \ldots W_n \ldots\}$ corresponding to a countable subset $\{y_1, y_2, \ldots y_n\}$ of $\overline{G(x)}$. For each i, $i=1, 2, \ldots n, \ldots$ let V_i be the neighborhood of x associated with W_i as stated above, and let $V=\bigcap_{i=1}^{\infty} V_i$. Then V is a neighborhood of x, and $f(V) \subset U$ for each $f \in G$. Thus F is regular at x.

Example 3 shows that if either $\overline{F(x)}$ is not compact, or X is not a P-space, Theorem 6 is false.

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COROLLARY. *If Y is regular, and if $\overline{F(x)}$ is compact for each $x \in X$, then the property (G), regularity, and even continuity of $F \subset (X, Y)$ are equivalent.*

Proof. The equivalence of regularity and even continuity follows from Theorem 6, and the equivalence of the property (G) and even continuity follows from Theorem B of [6] and Theorems 1 and 4 above.

If X, Y, and Z are sets, and if f is a function from $X \times Y$ to Z, we define functions f^a and f_b for each $a \in X$ and $b \in Y$ as follows: $f^a(y)=f(a, y)$, for y in Y, and $f_b(x)=f(x, b)$, for x in X. If $A \subset Y$ then f_A denotes the family $\{f_y : y \in A\}$.

THEOREM 7. *Let X, Y, and Z be topological spaces, a and b be points of X and Y respectively, and suppose that f is a function from $X \times Y$ to Z satisfying the following conditions:*

(1) *The function f^a is continuous at b.*

(2) *The family of functions f_Y is evenly continuous at a. Then f is continuous at (a, b).*

Proof. Let U be an open neighborhood of $f(a, b)$. By even continuity, there is a neighborhood V of $f(a, b)$ with $f(a, b) \in V \subset U$, and a neighborhood U_a of a in X such that $f_y(U_a) \subset U$ whenever $f(a, y) \in V$. There is a neighborhood U_b of b such that $f^a(U_b) \subset V$. Note that $y \in U_b$, $f(a, y) \in V$, thus $f(x, y) \in U$ for each $x \in U_a$. Hence if $x \in U_a$ and $y \in U_b$, then $f(x, y) \in U$, i.e. f is continuous at (a, b).

COROLLARY. *If the function f^a is continuous at b, Y is regular, and f_Y is regular at a, then f is continuous at (a, b).*

3. **Semi-equicontinuity vs regularity.**

DEFINITION 3. [2] A collection $\mathscr{V}$ of two-element open covers for a topological space X is said to be a semi-uniformity for X is for each point x in X, and each neighborhood U of x, there is $\{V_1, V_2\}$ in $\mathscr{V}$ such that $x \in V_1 \subset U$ and $X - V_2$ is a neighborhood of x.

It is remarked in [2] that a topological space has a semi-uniformity if and only if it is regular, and that every uniform space $(X, \mathscr{U})$ has a semi-uniformity consisting of all two-element uniform open covers of X, called the uniform semi-uniformity for X.

The following definition is a localization of the one given in [2].

DEFINITION 4. Let F be a family of functions from a topological space X to a semi-uniform space $(Y, \mathscr{V})$. F is said to be semi-equicontinuous at x in X if for each $\{V_1, V_2\}$ in $\mathscr{V}$ there is a neighborhood U of x such that $f(U) \subset V_1$ or $f(U) \subset V_2$ for each $f \in F$. F is said to be semi-equicontinuous if F is semi-equicontinuous at each point of X.

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REMARK. F is semi-equicontinuous at x in X if and only if for each $\{V_1, V_2\}$ in $\mathscr{V}$ and each pointwise closed subset G of F, there is a neighborhood U of x such that $f(U) \subset V_1$ or $f(U) \subset V_2$ for each $f \in G$.

REMARK. If a family F of functions from a topological space to a semi-uniform space $(Y, \mathscr{V})$ is semi-equicontinuous at x, then each $f \in F$ is continuous at x.

REMARK. It is easy to see that if a family of functions F from a topological space X to a uniform space $(Y, \mathscr{U})$ is equicontinuous at $x \in X$, then F is semi-equicontinuous at x relative to the uniform semi-uniformity of Y. Therefore Example 2 is an example of a family F of functions which is semi-equicontinuous at x but is not regular at x.

THEOREM 8. *If a family of functions F from a topological space X to a semi-uniform space $(Y, \mathscr{V})$ is semi-equicontinuous at x, then F is evenly continuous at x.*

Proof. Let y be a point in Y, and U a neighborhood of y in Y. If $y \notin \overline{F(x)}$, then there is a neighborhood W of y such that $W \cap F(x) = \varnothing$, and the conclusion is vacuously satisfied in this case. If $y \in \overline{F(x)}$, let $\{V_1, V_2\}$ in $\mathscr{V}$ such that $y \in V_1 \subset U$ and $X - V_2$ is a closed neighborhood of y. If $W = X - V_2$, then $W \cap F(x) \neq \varnothing$. Let N be a neighborhood of x such that $f(N) \subset V_1 \subset U$ or $f(N) \subset V_2$. But if $f \in F$ with $f(x) \in W$, then $f(x) \notin V_2$, thus $f(N) \subset U$. Hence F is evenly continuous at x.

COROLLARY. [2] *If a family F of functions from a topological space to a semi-uniform space is semi-equicontinuous, then F is evenly continuous.*

REMARK. If Y is a regular space, the set $\mathscr{V}_N$ of all two-element open covers for Y is a semi-uniformity for Y, called the natural semi-uniformity for Y. It is easy to see that if a family F of functions from a topological space X to a regular space Y is semi-equicontinuous at x in X relative to the natural semi-uniformity $\mathscr{V}_N$ for Y, then F is regular at x.

The following generalizes Theorem 2 of [2].

THEOREM 9. *If a family F of continuous functions from a topological space X to a semi-uniform space $(Y, \mathscr{V})$ is compact relative to a jointly continuous topology, then F is semi-equicontinuous.*

Proof. It follows from Theorem (2.1) of [4] that F is regular, and thus is evenly continuous and the pointwise topology for F is jointly continuous since Y is regular. Note also that F is compact relative to the pointwise toplogy.

Now let $\{V_1, V_2\} \in \mathscr{V}$, and let $x \in X$ and $y \in Y$. If $f \in F$ and if $f(x) \in V_1$, we can find open sets U_f in F with the pointwise topology and U_x in X such that $f \in U_f$ and $x \in U_x$ and $P(U_f \times U_x) \subset V_1$ where again $P(f, x)=f(x)$; if $f \in F$ with $f(x) \notin V_1$, then $f(x) \in V_2$ and we also can find open neighborhoods U_f and U_x of f and x respectively such that $P(U_f \times U_x) \subset V_2$. The family $\{U_f : f \in F\}$ forms an open cover for F in the pointwise topology, thus there are $f_1, f_2, \ldots f_n$ in F and corresponding U_{f_i}, $i=1, 2, \ldots n$, such that $F \subset \bigcup_{i=1}^n U_{f_i}$. If N is the intersection of the open sets U_{x_i} which are associated with the open sets U_{f_i}, then N is a neighborhood of x. For each $f \in F$, $f \in U_{f_i}$ for some i, thus $f(N) \subset V_1$ or $f(N) \subset V_2$, and F is semi-equicontinuous at x.

REMARK. Using Theorem 9 we may also obtain an Ascoli type theorem similar to Theorem 5.

Recall that a semitopological group is a group endowed with a topology under which the group multiplication is continuous separately.

EXAMPLE 4. Let X be a regular semitopological group in which every open cover of X by left translates of neighborhoods of the identity has a refinement by left translates of a neighborhood of the identity, and let Y be any regular space, and suppose f is a continuous function of X into Y. For each a in X, let f_a be the function on X defined by $f_a(x)=f(ax)$, and let $F=\{f_a : a \in X\}$. Then F is semi-equicontinuous relative to every semi-uniformity of Y. To see this let $\mathscr{V}$ be a semi-uniformity of Y, let $p \in X$, and let $\{V_1, V_2\} \in \mathscr{V}$. If $f_a \in F$ such that $f_a(p) \in V_1$, then, by the continuity of f at ap, there is a neighborhood U_a of the identity e such that $f(aU_ap) \subset V_1$; if $f_a \in F$ such that $f_a(p) \notin V_1$, then $f_a(p) \in V_2$, so there is a neighborhood U_a of the identity e such that $f(aU_ap) \subset V_2$. The family $\{aU_a : a \in X\}$ forms an open cover for X, thus there is a neighborhood U of e such that, for each $a \in X$, aU is contained in bU_b for some $b \in X$. Now if $a \in X$, $f_a(Up)=f(aUp) \subset f(bU_bp) \subset V_1$ or V_2. Thus F is semi-equicontinuous at $p \in X$.

THEOREM 10. *If X is a regular semitopological group in which each open cover of X by left translates of neighborhoods of the identity has a refinement by left translates of a neighborhood of the identity, then X is a topological group.*

Proof. In the above Example 4 take Y to be X, and take the continuous function f to be the identity map. Then each f_a will then be a left translation of X, and the conclusion then follows from Theorem 7 of [2].

REMARK. The property stated in Theorem 10 implies paracompactness of X, but, however, Theorem 10 is false if we simply assume X to be paracompact as Example 3 shows. The group of all reals with usual addition endowed with the

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topology having all intervals of the form $[a, b)$, $a<b$, as a base is a semitopological group but is not a topological group since inversion is not continuous.

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