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ON THE CLOSURE OF THE PRIME RADICAL OF A BANACH ALGEBRA

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The relationship between the prime ideals and the primal ideals of a Banach algebra is investigated. It is shown that the closure of the prime radical of a Banach algebra may be properly contained in the intersection of the closed primal ideals of the algebra.

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We begin with the appropriate definitions. An ideal I ("ideal" means "two-sided ideal") in a ring R is:

(i) *prime* if whenever J and K are ideals of R with $JK\subseteq I$ then either $J\subseteq I$ or $K\subseteq I$;

(ii) *primal* if whenever $J_1,\dots,J_n$ $(1<n<\infty)$ are ideals of R with $J_1\dots J_n=\{0\}$ then $J_i\subseteq I$ for at least one $i\in\{1,\dots,n\}$.

The *prime radical* of a Ring R is the intersection of the prime ideals of R. If the prime radical of R is zero then R is said to be *semiprime*. If A is a Banach algebra let $I(A)$ denote the closure of the prime radical of A. Let $J(A)$ be the intersection of all the closed, primal ideals of A. See [1] and [2] for more information on primal ideals and their relationship to prime ideals in the case when R is a C^*-algebra.

When A is a Banach algebra it is known that the prime radical of A is equal to the sum of all the nilpotent ideals of A [3,4]. It is clear that a primal ideal must contain every nilpotent ideal, from which it follows that every primal ideal in a Banach algebra must contain the prime radical. Hence each closed primal ideal contains $I(A)$. This shows that $I(A)\subseteq J(A)$.

Question 1. Does $I(A)=J(A)$?

The case of particular interest is when A is semiprime, that is, when $I(A)=\{0\}$. Let us say that A is *topologically semiprimal* if $J(A)=\{0\}$. Then as a special case of Question 1, we ask:

Question 2. Is a semiprime Banach algebra necessarily topologically semiprimal?

The purpose of this paper is to show that the answer to Question 1 is, no. We are not able to answer Question 2. Before giving the counter-example to Question 1, we explain the motivation behind the two questions, and we then show that they can be reformulated more simply for commutative Banach algebras.

The question arises in the following way. It is clear that each prime ideal is primal, and that each ideal which contains a primal ideal is primal. Hence each ideal which contains a prime ideal is primal. For commutative rings it is easy to prove the converse of this statement, namely that each primal ideal contains a prime ideal (see below). What about non-commutative rings? The converse is not true here: an example is given in [5] of a primal ideal in an inseparable Banach algebra which does not contain a prime ideal. On the other hand, it is also shown in [5] that in a separable, topologically semiprimal Banach algebra each primal ideal does contain a prime ideal. Since the class of semiprime Banach algebras is reasonably well-known it is natural to enquire whether each semiprime Banach algebra is topologically semiprimal.

Since $I(A)$ and $J(A)$ are both contained in R, the Jacobson radical of A, it appears at first sight that the problem is a problem about radical Banach algebras. But although $I(R)=I(A)$ (because a prime ideal, P, of R is equal to the intersection with R of $\{a \in A: RaR \subseteq P\}$, which is a prime ideal of A), it is not clear what the connection is between $J(R)$ and $J(A)$. It is therefore conceivable that one could have $I(A)=I(R)=J(R)$ but $I(A) \neq J(A)$.

The commutative case

It is well-known that in commutative rings primeness can be described in terms of elements: an ideal I is prime if and only if whenever $a, b \in A \backslash I$ then $ab \notin I$. Similarly it is easily checked that in this case an ideal I is primal if and only if whenever $\{a_1, \ldots, a_n\}$ is a finite subset of $A \backslash I$ then $a_1 \ldots a_n \neq 0$.

One easy consequence of this is that each primal ideal must contain a prime ideal. For if I is a primal ideal let X be the multiplicative closure of the complement of I. Then $0 \notin X$, so there is a maximal ideal disjoint from X which is prime (this is Krull's Prime Ideal Theorem). Hence I contains a prime ideal.

The original reason for asking Questions 1 and 2 was to discover when primal ideals had to contain prime ideals. But although this is already known for commutative Banach algebras the answers to Questions 1 and 2 are not, so this is the natural place to start.

When A is commutative each primal ideal contains a prime ideal, so $J(A)$ can be defined to be the intersection of the closures of the prime ideals of A. Question 1 can therefore be posed without reference to primal ideals.

In the commutative case there are simple characterizations of $I(A)$ and $J(A)$, which are useful.

The prime radical of a commutative ring is well-known to consist of the nilpotent elements, so $I(A)$ is simply the closure of the nilpotent elements of A.

Lemma 3. *Let A be a commutative Banach algebra and let $a \in A$. The following are equivalent:*

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(i) $a \in J(A)$,

(ii) *for every open neighbourhood U of a there exists a finite set $\{a_1, \ldots, a_n\} \subseteq U$ such that $a_1 \ldots a_n = 0$.*

Proof. Suppose that $a \notin J(A)$ and let J be a closed primal ideal not containing a. Then $U = A \backslash J$ is an open neighbourhood of a and for no finite set $\{a_1, \ldots, a_n\} \subseteq U$ does $a_1 \ldots a_n = 0$. Hence a does not satisfy (ii).

Conversely suppose that a does not satisfy (ii) and let U be an open neighbourhood of a such that for any finite set $\{a_1, \ldots, a_n\} \subseteq U$ we have $a_1 \ldots a_n \neq 0$. Let X be the multiplicative closure of U. Then $0 \notin X$, so there exists a prime ideal P disjoint from X and hence from U. Then $\bar{P}$ is a closed primal ideal disjoint from U, so $a \notin J(A)$. □

For commutative Banach algebras Question 1 can therefore be restated as follows:

Question 4. Let A be a commutative Banach algebra. Let $a \in A$ and suppose that in each open neighbourhood of a it is possible to find a finite number of elements with product equal to zero. Does it follow that each open neighbourhood of a contains a nilpotent element?

We now give an example showing that the answer to Question 4 is, no. This shows that the answer to Question 1 is also, no. The example is based on the following proposition, whose simple proof is left to the reader.

Proposition 5. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces and suppose that there are bilinear maps $\Pi_{ij}: X_i \times X_j \to X_{i+j}$ for each i and j such that*

(1) $\|\Pi_{ij}(a,b)\| \leqq \|a\| \, \|b\|, \quad a \in X_i; \; b \in X_j$;

(2) $\Pi_{ij}(a,b) = \Pi_{ji}(b,a), \quad a \in X_i, \; b \in X_j$; *and*

(3) $\Pi_{i(j+k)}(a, \Pi_{jk}(b,c)) = \Pi_{(i+j)k}(\Pi_{ij}(a,b), c), \quad a \in X_i, \; b \in X_j, \; c \in X_k$.

Then $A \equiv (\bigoplus_{n=1}^{\infty} X_n)_{l^1}$ is a commutative Banach algebra when equipped with the product

$$(\bar{a}\bar{b})_k = \sum_{i+j=k} \Pi_{ij}(a_i, b_j), \quad \bar{a} = \{a_i\}_{i=1}^{\infty}, \; \bar{b} = \{b_j\}_{j=1}^{\infty} \in A.$$

Now let X be any infinite dimensional Banach space and choose $x \in X$ with $\|x\| = 1$ and

$$x_{nq}, \quad n = 2, 3, 4, \ldots, \; q = 1, 2, \ldots, n$$

in X which are linearly independent and satisfy $\|x_{nq}\| = 1$ and $\|x - x_{nq}\| = n^{-1}$ for all n, q.

Denote by S_n the group of permutations of $\{1, 2, \ldots, n\}$. Each $\sigma \in S_n$ defines an operator, L_σ on $X \hat{\otimes} \cdots \hat{\otimes} X$ (n terms) by

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$$L_\sigma(y_1 \otimes y_2 \otimes \cdots \otimes y_n) = y_{\sigma(1)} \otimes y_{\sigma(2)} \otimes \cdots \otimes y_{\sigma(n)}.$$

Let

$$P_n = \frac{1}{n!} \sum_{\sigma \in S_n} L_\sigma.$$

Then P_n is a projection of $X \hat{\otimes} \cdots \hat{\otimes} X$ onto $X \hat{\odot} \cdots \hat{\odot} X$, the closed subspace of $X \hat{\otimes} \cdots \hat{\otimes} X$ consisting of elements invariant under L_σ for all $\sigma \in S_n$. Now put $\tilde{X}_n = X \hat{\odot} \cdots \hat{\odot} X$, ($n$ terms), and define $\tilde{\Pi}_{ij}: \tilde{X}_i \times \tilde{X}_j \to \tilde{X}_{i+j}$ by

$$\tilde{\Pi}_{ij}(a,b) = P_{i+j}(a \otimes b).$$

Then these spaces $\tilde{X}_i$ and maps $\tilde{\Pi}_{ij}$ satisfy (1), (2) and (3) of Proposition 5.

Now for each n let B_n be the closed subspace of $X \hat{\odot} \cdots \hat{\odot} X$ generated by $P_n(x_{p1} \otimes x_{p2} \otimes \cdots \otimes x_{pp} \otimes X \otimes \cdots \otimes X)$, $p = 2, 3, \ldots, n$. Put $X_n = (X \hat{\odot} \cdots \hat{\odot} X)/B_n$ for $n = 2, 3, \ldots$ and set $X_1 = X$. Then, since $\tilde{\Pi}_{ij}(B_i, X \hat{\odot} \cdots \hat{\odot} X) \subseteq B_{i+j}$ and $\tilde{\Pi}_{ij}(X \hat{\odot} \cdots \hat{\odot} X, B_j) \subseteq B_{i+j}$ the maps $\tilde{\Pi}_{ij}$, $i, j = 1, 2, \ldots$, induce maps $\Pi_{ij}: X_i \times X_j \to X_{i+j}$ satisfying (1), (2) and (3) of Proposition 5. The algebra A constructed from these spaces and maps is the required example, as we now show.

For each $y \in X$ let $[\bar{y}]$ denote the sequence $(y, 0, 0, \ldots)$ in A. Then $\|[\bar{y}]\| = \|y\|$ and $[\bar{y}_1][\bar{y}_2]\ldots[\bar{y}_k] = (0, \ldots, 0, z, 0, \ldots)$, where $z = P_k(y_1 \otimes y_2 \otimes \cdots \otimes y_k) + B_k$ and occurs in the kth place.

The next proposition is now obvious.

Proposition 6. (i) $\|[\bar{x}] - [\bar{x}_{nq}]\| = n^{-1}$; $n = 2, 3, \ldots$, $q = 1, 2, \ldots, n$; *and*

(ii) $\Pi_{q=1}^{n} [\bar{x}_{nq}] = 0$ *for each* $n = 2, 3, \ldots$.

In order to show that there are no nilpotents near $[\bar{x}]$ we need the following lemma.

Lemma 7. *Let $a \neq 0$ be in X. Then there are $q_2, q_3, \ldots, q_n, \ldots$ with $1 \leqq q_n \leqq n$ such that $\{a, x_{2q_2}, x_{3q_3}, \ldots, x_{nq_n}, \ldots\}$ is linearly independent.*

Proof. We use induction. Suppose that $q_1, \ldots, q_n$ have been found such that $\{a, x_{2q_2}, \ldots, x_{nq_n}\}$ are linearly independent. Then they span an n-dimensional subspace of X. Since $x_{n+1\,1}$, $x_{n+1\,2}, \ldots, x_{n+1\,n+1}$ are $n+1$ linearly independent vectors there is a $1 \leqq q_{n+1} \leqq n+1$ such that $x_{n+1\,q_{n+1}}$ does not belong to the span of $\{a, x_{2q_2}, \ldots, x_{nq_n}\}$. Then $\{a, x_{2q_2}, \ldots, x_{nq_n}, x_{n+1\,q_{n+1}}\}$ are linearly independent. □

Proposition 8. *If $\bar{a} \in A$ is such that $\|\bar{a} - [\bar{x}]\| < 1$*, then ā is not nilpotent.

Proof. Suppose that $\bar{a} = (a_1, a_2, \ldots)$. Then $(\bar{a})^n = (0, 0, \ldots, 0, y, \ldots)$ where $y = P_n(a_1 \otimes a_1 \otimes \cdots \otimes a_1) + B_n$ and occurs in the nth place. To show that $\bar{a}$ is not nilpotent it suffices to show, for each n, that $P_n(a_1 \otimes a_1 \otimes \cdots \otimes a_1) \notin B_n$. (Note that, since $\|\bar{a} - [\bar{x}]\| < 1$, $\|a_1 - x\| < 1$ and so $a_1 \neq 0$.)

Choose $q_2, \ldots, q_n$ such that $\{a_1, x_{2q_2}, \ldots, x_{nq_n}\}$ are linearly independent, using Lemma 7. Then there is a continuous linear functional ϕ on X such that $\phi(a_1) = 1$ and $\phi(x_{kq_k}) = 0$, $k = 2, 3, \ldots, n$. Now define a linear functional Φ on $X \hat{\otimes} \cdots \hat{\otimes} X$ (n terms) by

$$\Phi(y_1 \otimes \cdots \otimes y_n) = \phi(y_1)\phi(y_2)\ldots\phi(y_n).$$

Then $\Phi(a_1 \otimes \cdots \otimes a_1) = 1$ whereas $\Phi(B_n) = \{0\}$. Therefore $a_1 \otimes \cdots \otimes a_1 = P_n(a_1 \otimes \cdots \otimes a_1)$ is not in B_n. □

Propositions 6 and 8, together with Lemma 3, show that $[\bar{x}]$ belongs to $J(A)$ but not to $I(A)$, giving a negative answer to Questions 1 and 4. We do not know whether $I(A)$ is equal to $\{0\}$, and we are not able to answer Question 2.

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