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SOME RESULTS ON THE CENTER OF AN ALGEBRA OF OPERATORS ON $VN(G)$ FOR THE HEISENBERG GROUP

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1. Introduction. Let G be an amenable locally compact group. We will use the terminology of [**3**] and denote by $VN(G)$ the Von Neumann algebra of the regular representation and by $A(G)$ its predual, which is the algebra of the coefficients of the regular representation. The Von Neumann algebra $VN(G)$ is, in a natural fashion, a module with respect to $A(G)$ [**3**].

The algebra $\mathscr{A}$ of bounded linear operators on $VN(G)$, which commute with the action of $A(G)$, has been studied in [**6**] and in [**1**]. If $UCB(\hat{G})$ is the space of the elements of $VN(G)$ of the form vT, for some v in $A(G)$ and some T in $VN(G)$ (see for instance [**4**]), in [**6**] and in [**1**] it is proved that, for any amenable locally compact group there exists an isometric bijection between $\mathscr{A}$ and $UCB(\hat{G})^*$. In these papers it is also proved that the algebra $B(G)$ of multipliers of $A(G)$, which is isomorphic to the subalgebra $\mathscr{R}$ of $\mathscr{A}$ of the w^*-continuous operators of $\mathscr{A}$, is contained in the center $\mathscr{Z}_{\mathscr{A}}$ of $\mathscr{A}$.

The following conjecture appears natural: $\mathscr{Z}_{\mathscr{A}}$ is isomorphic to $B(G)$.

The conjecture is motivated, as well as by the previous inclusion, by the result obtained in [**10**] for the case $G = \mathbf{R}$.

In [**10**] the result is obtained making essential use of the usual order of the real line and, therefore, of the total order structure induced by the order of $\mathbf{R}$ in the set of the irreducible representations of $\mathbf{R}$.

In this paper we focus our attention on the Heisenberg group. For this group the set of the irreducible representations U_λ of G, with $\lambda \neq 0$, has a total order structure induced by $\mathbf{R}$ [**9**], as in the case of $\mathbf{R}$; therefore we can apply a non-commutative version of the techniques used in [**10**] to this special group. By doing so, we are able to prove the required statement for a class of operators of $\mathscr{A}$, which includes those corresponding to positive functionals on $UCB(\hat{G})$.

This limitation however appears quite natural, in view of the techniques used in the proof, as will be seen in the conclusion, and might be seen in

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itself as a support to the main conjecture. In fact no additional non-commutative problems arise in the proof.

In Section 2 some preliminaries, true for the general case of locally compact amenable groups, are given. Section 3 is devoted to the necessary applications of the direct integration theory for algebras and groups representations, for the particular case of the Heisenberg group. In Section 4 our main result is given and in Section 5 some concluding remarks and comments are made.

2. Some preliminaries.

LEMMA (2.1). *$UCB(\hat{G})$ is the norm-closure of the set of the compact support operators in $VN(G)$.*

Proof. See [**3**], p. 227.

PROPOSITION (2.1). *$UCB(\hat{G})$ is a C^*-algebra.*

Proof. See [**5**], Proposition 2, p. 65.

Let us recall that we can associate to every positive bounded linear functional Φ on a C^*-algebra A a representation π_Φ of the algebra, in the following way:

$$(2.1)\qquad (\Phi, T) = \langle \pi_\Phi(T)\xi_1 | \xi_2 \rangle, \text{ for all } T \in A,$$

where $\xi_1, \xi_2 \in \mathscr{H}_{\pi_\Phi}$ are totalizing vectors for π_Φ. This follows from [**2**], Theorems 2.4.4, 12.1.3, 12.2.4.

Let $\mathscr{A}$ be the algebra of bounded linear operators on $VN(G)$, which commute with the action of $A(G)$; let also $UCB(\hat{G})^*$ be the dual space of the C^*-algebra $UCB(\hat{G})$. The isometric bijection σ between $\mathscr{A}$ and $UCB(\hat{G})^*$ is defined by

$$(2.2)\qquad (\sigma(\Phi)(T), v) = (\Phi, vT),$$

for all $T \in VN(G)$, $v \in A(G)$, $\Phi \in UCB(\hat{G})^*$ (see [**1**] and [**6**]).

From now on, we shall write $\Phi \in \mathscr{R}$, to denote that $\Phi \in UCB(\hat{G})^*$, $\sigma(\Phi) \in \mathscr{R}$.

PROPOSITION (2.2). *The functionals $\Phi \in \mathscr{R}$ are w^*-dense in $UCB(\hat{G})^*$.*

Proof. Since $A(G) \subset B(G) \sim \mathscr{R}$, we have $UCB(\hat{G}) \subset VN(G) \sim A(G)^* \subset B(G)^*$ and also $UCB(\hat{G}) \subset B(G)^{**}$. Since the unit ball of $B(G)$ is w^*-dense in the unit ball of $B(G)^{**}$, for all Φ in $UCB(\hat{G})^*$ there is $\{\Phi_\alpha\} \subset B(G)$, such that

$$(\Phi_\alpha - \Phi, x) \underset{\alpha}{\longrightarrow} 0, \quad \text{for all } x \in B(G)^*,$$

and therefore in particular for all $x \in UCB(\hat{G})$.

The next proposition characterizes the representations of $UCB(\hat{G})$ corresponding to $\Phi \in \mathscr{R}$.

PROPOSITION (2.3). *Let $\Phi \in UCB(\hat{G})^*$; let π_Φ be the representation of $UCB(\hat{G})$ associated with Φ. Then $\Phi \in \mathscr{R}$ if and only if for all $\{T_\alpha\} \subset VN(G)$, such that $T_\alpha \rightarrow 0$ in the w^*-topology and for all $w \in A(G)$,*

$$\text{(2.3)} \qquad \langle \pi_\Phi(wT_\alpha)\lambda|\eta\rangle \underset{\alpha}{\rightarrow} 0, \quad \forall \lambda, \eta \in \mathscr{H}_{\pi_\Phi}.$$

Proof. If (2.3) holds, then for all $w \in A(G)$

$$(\sigma(\Phi)(T_\alpha), w) = (\Phi, wT_\alpha) = \langle \pi_\Phi(wT_\alpha)\xi_1|\xi_2\rangle \underset{\alpha}{\rightarrow} 0,$$

where $\xi_1, \xi_2 \in \mathscr{H}_{\pi_\Phi}$ are totalizers for π_Φ. Then $\Phi \in \mathscr{R}$.

In order to prove the converse implication, let us prove that if $\Phi \in \mathscr{R}$ then (2.3) is satisfied for all $w \in A(G)$ and for all $\{T_\alpha\} \subset VN(G)$, $T_\alpha \rightarrow 0$ in the w^*-topology, such that $\|T_\alpha\| \leqq 1$, $\forall \alpha$. Indeed, if this property is satisfied, then the functional $\Psi \in VN(G)^*$ defined by setting

$$\text{(2.4)} \qquad (\Psi, T) = \langle \pi_\Phi(wT)\lambda|\ \eta\rangle, \text{ for } \lambda, \eta \in \mathscr{H}_{\pi_\Phi},\ w \in A(G) \text{ fixed},$$

is w-continuous on $[VN(G)]_1$ and therefore ultraweakly continuous on $VN(G)$. On the other hand, for $VN(G)$ weak and ultraweak continuity coincide, and therefore Ψ is w-continuous on $VN(G)$ and the thesis is proved. Then let

$$\{T_\alpha\} \subset VN(G),\ \|T_\alpha\| \leqq 1\ \forall \alpha,\ T_\alpha \rightarrow 0 \text{ in the } w^*\text{-topology}.$$

(a). Let $\lambda = \pi_\Phi(A)\xi_1$, $\eta = \pi_\Phi(B)\xi_2$, with A, B with compact support on G. For $\epsilon > 0$, let $u \in A(G)$ with compact support such that $\|u - w\|_{A(G)} < \epsilon$. Then, for all α, the support of uT_α is compact and

$$\|uT_\alpha - wT_\alpha\|_{A(G)} < \epsilon.$$

Then

$$\begin{aligned} |\langle \pi_\Phi(wT_\alpha)\lambda|\ \eta\rangle| &= |\langle \pi_\Phi(B^+(wT_\alpha)A)\xi_1|\xi_2\rangle| \\ &\leqq |\langle \pi_\Phi(B^+(uT_\alpha)A)\xi_1|\xi_2\rangle| \\ &\qquad + |\langle \pi_\Phi(B^+(wT_\alpha - uT_\alpha)A)\xi_1|\xi_2\rangle| \\ &\leqq |\langle \pi_\Phi(B^+(uT_\alpha)A)\xi_1|\xi_2\rangle| + \|\pi_\Phi\|\ \|B\|\ \|A\|\ \epsilon. \end{aligned}$$

Since the support of $B^+(uT_\alpha)A$ is contained, for all α, in a compact K (independent from α), we have

$$B^+(uT_\alpha)A = v(B^+(uT_\alpha)A) \quad \text{for all } \alpha,$$

if $v \in A(G)$ and $v(x) = 1$ for $x \in K$. Since $B^+(uT_\alpha)A \rightarrow 0$ in the

w^*-topology, then by w^*-continuity of $\sigma(\Phi)$,

$$\begin{aligned}\langle \pi_\Phi(B^+(uT_\alpha)A)\xi_1|\xi_2\rangle &= (\Phi, v(B^+(uT_\alpha)A)) \\ &= (\sigma(\Phi)(B^+(uT_\alpha)A), v) \to 0.\end{aligned}$$

(b). Let now $\lambda = \pi_\Phi(A)\xi_1$, $\eta = \pi_\Phi(B)\xi_2$, with A, B in $UCB(\hat{G})$; for $\epsilon > 0$, let A', $B' \in UCB(\hat{G})$ with compact support, such that $\|A - A'\| < \epsilon$, $\|B - B'\| < \epsilon$. Then

$$\begin{aligned}|\langle \pi_\Phi(wT_\alpha)\lambda|\eta\rangle| &= |\langle \pi_\Phi(B^+(wT_\alpha)A)\xi_1|\xi_2\rangle| \\ &\leqq |\langle \pi_\Phi(B'^+(wT_\alpha)A')\xi_1|\xi_2\rangle| \\ &\qquad + |\langle \pi_\Phi((B - B')^+(wT_\alpha)(A - A'))\xi_1|\xi_2\rangle| \\ &\quad + |\langle \pi_\Phi((B - B')^+(wT_\alpha)A')\xi_1|\xi_2\rangle| \\ &\qquad + |\langle \pi_\Phi(B'^+(wT_\alpha)(A - A')\xi_1|\xi_2\rangle| \\ &\leqq |\langle \pi_\Phi(B'^+(wT_\alpha)A')\xi_1|\xi_2\rangle| \\ &\qquad + \|\pi_\Phi\| \, (\|B - B'\| \, \|A - A'\| \\ &\quad + \|B - B'\| \, \|A'\| + \|B'\| \, \|A - A'\| \,) \, \|wT_\alpha\| \\ &\leqq |\langle \pi_\Phi(B'^+(wT_\alpha)A')\xi_1|\xi_2\rangle| \\ &\qquad + \|\pi_\Phi\| \, \|w\|_{A(G)}(\epsilon + \|A'\| + \|B'\| \,) \, \epsilon.\end{aligned}$$

Now, since $\{\pi_\Phi(A)\xi_1, A \in UCB(\hat{G})\}$, $\{\pi_\Phi(B)\xi_2, B \in UCB(\hat{G})\}$ are dense in $\mathscr{H}_{\pi_\Phi}$, from (b) the required property follows.

Let $\Phi \in UBC(\hat{G})^*$ and let π_Φ be the canonically associated representation of $UCB(\hat{G})$; let us denote by $\pi_\Phi{}^G$ the representation of the group G, which is the restriction to $\{U_x, x \in G\}$ of π_Φ. From Proposition (2.3) we have:

COROLLARY (2.1). *Let $\Phi \in UCB(\hat{G})^*$. Then $\Phi \in \mathscr{R}$ if and only if π_Φ is canonically defined by $\pi_\Phi{}^G$, in the sense that, for all $\lambda, \eta \in \mathscr{H}_{\pi_\Phi}$, $w \in A(G)$, the functional $\Psi \in VN(G)^*$ defined by*

$$(\Psi, T) = \langle \pi_\Phi(wT)\lambda| \, \eta\rangle$$

is obtained extending by w^*-continuity and linearity its restriction to $\{U_x, x \in G\}$.

Proof. If the functional Ψ is obtained extending by w^*-continuity and linearity its restriction to $\{U_x, x \in G\}$, then $\Phi \in \mathscr{R}$, by Proposition (2.3). Let us prove the converse implication. The linear space spanned by the set $\{U_x, x \in G\}$ is w^*-dense in $VN(G)$; therefore, for any $T \in VN(G)$, there is $\{T_\alpha\} \subset VN(G)$, such that

$$T_\alpha = \sum_{x \in \Gamma_\alpha} c_x{}^\alpha U_x,$$

where $c_x^\alpha \in \mathbf{C}$ and Γ_α is a finite subset of G and $T_\alpha \to T$ weakly. If $w \in A(G)$, we have

$$wT_\alpha = \sum_{x \in \Gamma_\alpha} c_x^\alpha v(x) U_x.$$

By Proposition (2.3), if $\Phi \in \mathscr{R}$, then for all $\lambda, \eta \in \mathscr{H}_{\pi_\Phi}$

$$\langle \pi_\Phi(wT_\alpha)\lambda|\eta\rangle = \sum_{x \in \Gamma_\alpha} c_x^\alpha v(x) \langle \pi_\Phi(U_x)\lambda|\eta\rangle \to \langle \pi_\Phi(wT)\lambda|\eta\rangle.$$

Hence π_Φ is defined by $\{\pi_\Phi(U_x), x \in G\}$.

3. Some facts on the Heisenberg group. Let G be the Heisenberg group; if we denote an element of G by $[x, y, z]$, where $x, y, z \in \mathbf{R}$, then for $[x, y, z], [x', y', z'] \in G$

$$[x, y, z]\,[x', y', z'] = [x + x', y + y', z + xy' + z'].$$

Let us recall [9] that for all $\lambda \in \mathbf{R}$, $\lambda \neq 0$, the map U_λ of G into $\mathscr{B}(L^2(\mathbf{R}))$ defined by

$$(3.1) \qquad (U_\lambda([x, y, z])f)(t) = e^{i\lambda(z+ty)} f(t + x),$$

with $t \in \mathbf{R}$, $f \in L^2(\mathbf{R})$, $[x, y, z] \in G$, is an unitary continuous irreducible representation of G; furthermore every unitary irreducible representation (of dimension > 1) of G is unitarily equivalent to U_λ, for some $\lambda \in \mathbf{R}$, $\lambda \neq 0$.

LEMMA (3.1). *The following decomposition in direct integral holds:*

(i).

$$(3.2) \qquad L^2(G) = \int^\oplus \mathscr{H}_\lambda d\lambda,$$

where $\lambda \in \mathbf{R}$, $\mathscr{H}_\lambda = L^2(\mathbf{R})$ *for all* $\lambda \in \mathbf{R}$, *and* $d\lambda$ *is the Lebesgue measure on* $\mathbf{R}$;

(ii). *For every* $T \in VN(G)$

$$(3.3) \qquad T = \int^\oplus T_\lambda d\lambda,$$

where for all $\lambda \in \mathbf{R}$, $\lambda \neq 0$, $(VN(G))_\lambda = \mathscr{B}(L^2(\mathbf{R}))$, *that is the set of all bounded operators on* $L^2(\mathbf{R})$;

(iii). *For every* $T \in \mathscr{Z}_{VN(G)}$, *the center of* $VN(G)$,

$$(3.4) \qquad T = \int^\oplus t(\lambda) I_\lambda d\lambda,$$

where $t \in L^\infty(\mathbf{R})$;

(iv).

$$(3.5) \qquad U = \int^\oplus U_\lambda d\lambda.$$

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Proof. By Proposition 18.7.7 of [**2**] and by the characterization of the dual $\hat{G}$ of the Heisenberg group, [**9**], there exists a measure $d\lambda$ on $\mathbf{R}$ such that (3.2), (3.3) and (3.4) hold. We note in particular that, for every $z \in \mathbf{R}$,

$$(3.6) \qquad U_{[0,0,z]} f_\lambda = e^{i\lambda z} f_\lambda,$$

where $f_\lambda \in \mathscr{H}_\lambda$ for every λ. Since the irreducible representation of G for which (3.6) holds is unique, for $\lambda \neq 0, \mathscr{H}_\lambda$ is isomorphic to $L^2(\mathbf{R})$, for all $\lambda \neq 0$, as is well known from the general theory, and (3.5) holds.

Let us prove that $d\lambda$ is the Lebesgue measure on $\mathbf{R}$. Let $v \in A(G)$ and let $f, g \in L^2(G)$ such that

$$v([x, y, z]) = \langle f \mid U_{[x,y,z]} g \rangle$$

for every $[x, y, z] \in G$. Then, by the above decomposition,

$$v([x, y, z]) = \left\langle \int^\oplus f_\lambda d\lambda \,\middle|\, \int^\oplus U_\lambda([x, y, z]) g_\lambda d\lambda \right\rangle$$
$$= \int^\oplus \left\langle f_\lambda \,\middle|\, U_\lambda(x, y, z) g_\lambda \right\rangle d\lambda = \int \int \overline{f_\lambda(t)} e^{i\lambda(z+ty)} g_\lambda(t + x) dt d\lambda.$$

Let us set $x = \bar{x}$, $y = \bar{y}$ and $\psi(z) = v([\bar{x}, \bar{y}, z])$; we have, for almost all $\bar{x}, \bar{y}$, that $\psi(z) \in A(G)$ and

$$\psi(z) = \int \int f_\lambda(t) e^{i\lambda(z+\overline{ty})} g_\lambda(t + \bar{x}) dt d\lambda = \int e^{i\lambda z} \hat{\psi}(\lambda) d\lambda,$$

where

$$\hat{\psi}(\lambda) = \int \overline{f_\lambda(t)} e^{it\bar{y}\lambda} g_\lambda(t + \bar{x}) dt.$$

This relation expresses the ordinary Fourier transform on the real line, and therefore $d\lambda$ is the usual Lebesgue measure on $\mathbf{R}$.

Let us prove finally that $(VN(G))_\lambda = \mathscr{B}(L^2(\mathbf{R}))$, for all $\lambda \in \mathbf{R}$, $\lambda \neq 0$. Indeed it is easy to check that, if $\lambda \neq 0$, $(VN(G))_\lambda$ contains the operators on $L^2(\mathbf{R})$ of the form

$$u(x) f(t) = f(t + x), \; v(k) f(t) = e^{ikt} f(t) \text{ for all } x, k \in \mathbf{R}.$$

It is easy to check that the commutant of $u(x)$ and $v(k)$ is e^{ikx}. By a well known result (see for its most general form [**8**]) $(VN(G))_\lambda$ coincides with $\mathscr{B}(L^2(\mathbf{R}))$.

LEMMA (3.2). (i). *Let $\mathscr{Z}_{UCB(\hat{G})}$, $\mathscr{Z}_{VN(G)}$ denote the center of $UCB(\hat{G})$ and $VN(G)$ respectively. Then*

$$\mathscr{Z}_{UCB(\hat{G})} = \mathscr{Z}_{VN(G)} \cap UCB(\hat{G});$$

(ii). *If* $T \in \mathscr{L}_{VN(G)}$, $v \in A(G)$, *then* $vT \in \mathscr{L}_{UCB(\hat{G})}$;
(iii). *Let* $T \in \mathscr{L}_{VN(G)}$,

$$T = \int^{\oplus} t(\lambda) I_\lambda d\lambda;$$

then $T \in \mathscr{L}_{UCB(\hat{G})}$ *if and only if* $t \in C_u(\mathbf{R})$, *that is the set of the uniformly continuous bounded functions on* $\mathbf{R}$;

(iv). *If* $T \in \mathscr{L}_{UCB(\hat{G})}$, *there are* $v_0 \in A(G)$, $T_0 \in \mathscr{L}_{VN(G)}$ *such that* $T = v_0 T_0$.

Proof. (i). If $T \in \mathscr{L}_{UCB(\hat{G})}$, then T commutes with $\{U_x, x \in G\}$, generating $VN(G)$. The converse is obvious.

(ii). By the above decomposition of $VN(G)$, for all $v \in A(G)$, we can write

$$v = \int^{\oplus} v_\lambda d\lambda,$$

where $\lambda \in \mathbf{R}$, $d\lambda$ is the Lebesgue measure on $\mathbf{R}$ and $v_\lambda \in \mathscr{B}(\mathscr{H}_\lambda)_*$.

Let us set $A_v(\lambda) = (v_\lambda, I_\lambda)$ for a.e. $\lambda \in \mathbf{R}$; it is easy to see that $A_v \in L^1(\mathbf{R})$; moreover, if $w \in A(G)$, then

(3.7) $\quad A_{vw} = A_v * A_w.$

Indeed, for $z \in \mathbf{R}$, we have

$$\begin{aligned}(U_{[0,0,z]}, vw) &= (U_{[0,0,z}, v)(U_{[0,0,z]}, w) \\ &= \int\int e^{i(\mu+\nu)z} A_v(\mu) A_w(\nu) d\mu d\nu = \int\int e^{i\lambda z} A_v(\mu) A_w(\lambda - \mu) d\lambda d\mu \\ &= \int e^{i\lambda z} (A_v * A_w)(\lambda) d\lambda,\end{aligned}$$

where $\lambda = \mu + \nu$. On the other hand, by definition,

$$(U_{[0,0,z]}, vw) = \int e^{i\lambda z} A_{vw}(\lambda) d\lambda,$$

and since z is arbitrary, (3.7) follows.

If $T \in \mathscr{L}_{VN(G)}$,

$$T = \int^{\oplus} t(\lambda) I_\lambda d\lambda,$$

$v \in A(G)$, let us consider $vT \in UCB(\hat{G})$. For all $w \in A(G)$ we have

$$\begin{aligned}(vT, w) &= (T, vw) = \int t(\lambda) A_{vw}(\lambda) d\lambda \\ &= \int t(\lambda)(A_v * A_w)(\lambda) d\lambda = \int\int t(\lambda) A_v(\mu - \lambda) A_w(\mu) d\lambda d\mu \\ &= \int (t * \tilde{A}_v)(\mu) A_w(\mu) d\mu = (S, w),\end{aligned}$$

where $\tilde{A}_v(\mu) = A_v(-\mu)$ for all $\mu \in \mathbf{R}$, and

$$S = \int^{\oplus} (t * \tilde{A}_v)(\mu) I_\mu d\mu.$$

From this equality, true for every $w \in A(G)$, it follows that $vT = S$. Since $S \in \mathcal{L}_{VN(G)}$, we conclude that $vT \in \mathcal{L}_{UCB(\hat{G})}$.

(iii). Let $T \in \mathcal{L}_{UCB(\hat{G})}$,

$$T = \int^{\oplus} t(\lambda) I_\lambda d\lambda;$$

let us prove that $t \in C_u(\mathbf{R})$. Let $v_0 \in A(G)$ be an approximate identity on $A(G)$. For all α, $v_\alpha T \in \mathcal{L}_{UCB(\hat{G})}$, by (ii), and hence

$$v_\alpha T = \int^{\oplus} t_\alpha(\lambda) I_\lambda d\lambda.$$

Let us prove that $t_\alpha \in C_u(\mathbf{R})$. For all $w \in A(G)$, we have, with the notation in (ii),

$$(v_\alpha T, w) = (T, v_\alpha w) = \int t(\lambda) A_{v_\alpha w}(\lambda) d\lambda = \int (t * \tilde{A}_{v_\alpha})(\mu) A_w(\mu) d\mu.$$

On the other hand

$$(v_\alpha T, w) = \int t_\alpha(\mu) A_w(\mu) d\mu.$$

Then, since w is arbitrary in $A(G)$, we have $t_\alpha(\mu) = (t * \tilde{A}_{v_\alpha})(\mu)$, for a.e. $\mu \in \mathbf{R}$. Since $t \in L^\infty(\mathbf{R})$, $A_{v_\alpha} \in L^1(\mathbf{R})$, it follows that $t_\alpha \in C_u(\mathbf{R})$. This implies that $t \in C_u(\mathbf{R})$. Indeed we have

$$\|v_\alpha T - T\| = \|v_\alpha v_0 T_0 - v_0 T_0\| = \|(v_\alpha v_0 - v_0) T_0\| \leqq \|v_\alpha v_0 - v_0\|_A \|T_0\| \to 0,$$

and therefore $\|t_\alpha - t\|_\infty \to 0$.

Conversely, let us suppose that $t \in C_u(\mathbf{R})$. If $\phi \in L^1(\mathbf{R})$, $\psi \in L^\infty(\mathbf{R})$ satisfy $t = \phi * \tilde{\psi}$, we define $v_0 \in A(G)$ and $T_0 \in VN(G)$ in the following way:

$$v_0 = \int^{\oplus} v_{0\lambda} d\lambda, \quad \text{where } (v_{0\lambda}, I_\lambda) = \phi(\lambda) \text{ for a.e. } \lambda \in \mathbf{R},$$

$$T_0 = \int^{\oplus} \psi(\lambda) I_\lambda d\lambda;$$

then $T = v_0 T_0$.

(iv). This follows immediately from (iii).

We note that it is possible to choose $v_0 \in A(G)$ such that

$$\|v_0\|_{A(G)} = \|A_{v_0}\|_{L^1(\mathbf{R})}.$$

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From Theorem 2.9 in [**7**] it follows immediately that every unitary representation of G can be written as

$$\pi = \int^{\oplus} U_\lambda dm(\lambda). \tag{3.8}$$

Let us consider $\Phi \in UCB(\hat{G})^*$ and the canonically associated representation of $UCB(\hat{G})$, π_Φ; from Theorem 8.5.1 of [**2**], it is possible to write π_Φ as a direct integral of irreducible representations of $UCB(\hat{G})$:

$$\pi_\Phi = \int^{\oplus} \pi^\tau dm(\tau). \tag{3.9}$$

From Corollary (2.1) and (3.8), (3.9) it follows immediately that:

COROLLARY (3.1). *Let $\Phi \in UCB(\hat{G})$, π_Φ the canonically associated representation of $UCB(\hat{G})$ and $\pi_\Phi{}^G$ its restriction to G. Then $\Phi \in \mathscr{R}$ if and only if*

$$\pi_\Phi = \int^{\oplus} \pi^\tau dm(\tau) \tag{3.10}$$

where $dm(\tau)$ is supported on the set of irreducible representations π^λ of the algebra obtained extending the group representation U_λ, for some $\lambda \neq 0$.

Proof. This is immediate if we note that π_Φ is irreducible if and only if $\pi_\Phi{}^G$ is also.

Remark (3.1). Let us denote by $\mathscr{S}$ the set of the functionals $\Phi \notin \mathscr{R}$, such that the support of the measure $dm(\tau)$ in (3.9) has void intersection with the set of the irreducible representations π^λ of the algebra $UCB(\hat{G})$, obtained by extending the group representations U_λ, for some $\lambda \neq 0$.

If $\Phi \notin \mathscr{R}$, it is possible to write

$$\Phi = \Phi' + \Phi'', \tag{3.11}$$

where $\Phi' \in \mathscr{S}$, $\Phi'' \in \mathscr{R}$.

Remark (3.2). For every $a \in \mathbf{R}$, let us denote by $\mathscr{M}_a$ (respectively $\mathscr{L}_a$) the set of the functionals $\Phi \in \mathscr{R}$ such that the measure $dm(\lambda)$ in (3.8) is supported on the interval $[a, +\infty)$ (respectively $(-\infty, a]$).

PROPOSITION (3.1). *Let $\Phi \in UCB(\hat{G})^*$. For every $\epsilon > 0$, there exist Φ_M, $\Phi_L \in UCB(\hat{G})^*$ such that*

(i) $\Phi_M \in \bar{\mathscr{M}}^{(w)}_{\lambda_0+\delta}$, $\Phi_L \in \bar{\mathscr{L}}^{(w)}_{\lambda_0-\delta}$, *for some* $\lambda_0 \in \mathbf{R}$, $\delta > 0$;

(ii) $\|\Phi - (\Phi_M + \Phi_L)\| < \epsilon$.

Proof. Let $\epsilon > 0$; let also $\{a_i\}$ be a increasing sequence of $\mathbf{R}$, such that

$a_i - a_{i-1} > 2\delta$. For every i, we define $r_i \in C_u(\mathbf{R})$ by

$$r_i(\lambda) = \begin{cases} 1 & , \text{ for } a_i - \delta/2 < \lambda < a_i + \delta/2 \\ 0 & , \text{ for } \lambda < a_i - \delta, \lambda > a_i + \delta \\ \text{linear}, & \text{for } a_i - \delta \leqq \lambda \leqq a_i - \delta/2, \\ & \qquad a_i + \delta/2 \leqq \lambda \leqq a_i + \delta \end{cases}$$

and $R_i \in UCB(\hat{G})$ by

$$R_i = \int^{\oplus} r_i(\lambda) I_\lambda d\lambda$$

and also $\Psi_i, \Psi \in UCB(\hat{G})^*$ by

$$(\Psi_i, T) = (\Phi, R_i T)$$
$$(\Psi, T) = (\Phi, \sum R_i T), \text{ for all } T \in UCB(\hat{G}).$$

Since $\sum \|\Psi_i\| = \|\Psi\| \leqq \|\Phi\|$, then, for every $\epsilon > 0$ and for every $\delta > 0$, there exists some $\bar{i}$, such that $\|\Psi_{\bar{i}}\| < \epsilon$; let $\lambda_0 = a_{\bar{i}}$.

Let us define $q_M, q_L \in C_u(\mathbf{R})$ by

$$q_M(\lambda) = \begin{cases} 1 & , \text{ for } \lambda > \lambda_0 + \delta \\ 0 & , \text{ for } \lambda_0 + \delta/2 > \lambda \\ \text{linear}, & \text{for } \lambda_0 + \delta/2 \leqq \lambda \leqq \lambda_0 + \delta \end{cases}$$

$$q_L(\lambda) = \begin{cases} 1 & , \text{ for } \lambda < \lambda_0 - \delta \\ 0 & , \text{ for } \lambda > \lambda_0 - \delta/2 \\ \text{linear}, & \text{for } \lambda_0 - \delta \leqq \lambda \leqq \lambda_0 - \delta/2 \end{cases}$$

and $Q_M, Q_L \in UCB(\hat{G})$ by

$$Q_M = \int^{\oplus} q_M(\lambda) I_\lambda d\lambda, \quad Q_L = \int^{\oplus} q_L(\lambda) I_\lambda d\lambda.$$

The functionals $\Phi_M, \Phi_L \in UCB(\hat{G})^*$ such that

$$(\Phi_M, T) = (\Phi, Q_M T), \ (\Phi_L, T) = (\Phi, Q_L T) \text{ for all } T \in UCB(\hat{G})$$

are the required functionals. Indeed (i) is an immediate consequence of the definition and (ii) follows from the identity

$$\Phi - (\Phi_M + \Phi_L) = \Psi_{\bar{i}}.$$

4. The main results.

LEMMA (4.1). *Let $\Phi \in \mathscr{R}$, $\Phi \neq 0$. If $\Phi \in \mathscr{M}_{\lambda_0}$, for some $\lambda_0 \in \mathbf{R}$, then for every $T \in UCB(\hat{G})$ such that $T_\lambda = 0$ for $\lambda \geqq \lambda_0$, we have $(\Phi, T) = 0$.*

Proof. If ξ_1, ξ_2 are totalizing vectors for π_Φ, then, since $\Phi \in \mathscr{R}$,

$$(\Phi, T) = \langle \pi_\Phi(T)\xi_1 | \xi_2 \rangle = \int \langle \pi_\lambda(T)\xi_{1\lambda} | \xi_{2\lambda} \rangle dm(\lambda),$$

where the integral is given by the direct decomposition of π_Φ, as in [**2**], $\lambda \in \mathbf{R}$, $\pi_\lambda(T)$ are the irreducible direct integrands of $\pi_\Phi(T)$ and $\xi_{i\lambda}$ $(i = 1, 2)$ are the vectors in $\mathscr{H}_\lambda$ such that

$$\xi_i = \int^{\oplus} \xi_{i\lambda} dm(\lambda), \quad i = 1, 2.$$

Since, for every λ, we have $\pi_\lambda(T) = T_\lambda$, then

$$(\Phi, T) = \int \langle T_\lambda \xi_{1\lambda} | \xi_{2\lambda} \rangle dm(\lambda)$$

and since $\langle T_\lambda \xi_{1\lambda} | \xi_{2\lambda} \rangle$ is supported on $\lambda \leqq \lambda_0$, the above integral is null.

LEMMA (4.2). *Let* $\lambda_0 \in \mathbf{R}$, $\lambda_0 \neq 0$, $f_{\lambda_0} \in L^2(\mathbf{R})$, $\| f_{\lambda_0} \|_2 = 1$. *For every* $v \in A(G)$ *and for all* $[x, y, z] \in G$, *let*

$$(4.1) \qquad v_{\lambda_0}([x, y, z]) = v([x, y, z]) e^{i\lambda_0 z} \int f_{\lambda_0}(t) e^{i\lambda_0 t y} f_{\lambda_0}(t + x) dt.$$

Then $v_{\lambda_0} \in A(G)$.

Proof. Let $v \in A(G)$, and $v^{(1)}, v^{(2)} \in L^2(\mathbf{R})$ such that

$$v([x, y, z]) = \langle v^{(1)} | U_{[x,y,z]} v^{(2)} \rangle, \quad \text{for all } [x, y, z] \in G.$$

Moreover, for $i = 1, 2$, $\lambda \in \mathbf{R}$, let $v_\lambda{}^{(i)} \in L^2(\mathbf{R})$ such that

$$v^{(i)} = \int v_\lambda{}^{(i)} d\lambda.$$

We have, by definition, for all $[x, y, z] \in G$,

$$v_{\lambda_0}([x, y, z]) = \langle v^{(1)} | U_{[x,y,z]} v^{(2)} \rangle \, \langle f_{\lambda_0} | U_{\lambda_0}([x, y, z]) f_{\lambda_0} \rangle.$$

If $U \otimes U_{\lambda_0}$ is the tensor product of the regular representation U and U_{λ_0}, then

$$\begin{aligned} v_{\lambda_0}([x, y, z]) &= \langle v^{(1)} \otimes f_{\lambda_0} | (U \otimes U_{\lambda_0})([x, y, z]) (v^{(2)} \otimes f_{\lambda_0}) \rangle \\ &= \langle v^{(1)} \otimes f_{\lambda_0} | U([x, y, z]) v^{(2)} \otimes U_{\lambda_0}([x, y, z]) f_{\lambda_0} \rangle \\ &= \int \langle v_\lambda{}^{(1)} \otimes f_{\lambda_0} | U_\lambda([x, y, z]) v_\lambda{}^{(2)} \otimes U_{\lambda_0}([x, y, z]) f_{\lambda_0} \rangle d\lambda \\ &= \int d\lambda \int \int \overline{v_\lambda{}^{(1)}(t) f_{\lambda_0}(s)} e^{i\lambda(z + ty)} e^{i\lambda_0(z + sy)} v_\lambda{}^{(2)}(t + x) f_{\lambda_0}(s + x) \\ &\qquad \times dsdt. \end{aligned}$$

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By setting $r = (\lambda t + \lambda_0 s)/(\lambda + \lambda_0)$, $w = s - t$, we have

$$v_{\lambda_0}([x, y, z]) = \int d\lambda \int \int v_\lambda^{(1)}\left(r - \frac{\lambda_0}{(\lambda + \lambda_0)} w\right) f_{\lambda_0}\left(r + \frac{\lambda}{(\lambda + \lambda_0)} w\right) \times e^{i(\lambda+\lambda_0)(z+ry)} v_\lambda^{(2)}\left(r - \frac{\lambda_0}{(\lambda + \lambda_0)} w + x\right) \times f_{\lambda_0}\left(r + \frac{\lambda}{(\lambda + \lambda_0)} w + x\right) dr dw.$$

For a.e. $\lambda, w \in \mathbf{R}$ (namely for $\lambda \neq -\lambda_0$ and $\lambda \neq 0$) let $g^{(1)}_{\lambda+\lambda_0,w}$, $g^{(2)}_{\lambda+\lambda_0,w}$ be the functions of $L^2(\mathbf{R})$ defined by

$$g^{(i)}_{\lambda+\lambda_0,w}(r) = v_\lambda^{(i)}\left(r - \frac{\lambda_0}{(\lambda + \lambda_0)} w\right) f_{\lambda_0}\left(r + \frac{\lambda}{(\lambda + \lambda_0)} w\right), \quad i = 1, 2$$

and let $g_{\lambda+\lambda_0,w}$ be the linear continuous functionals on $\mathscr{B}(L^2(\mathbf{R}))$ such that

$$(g_{\lambda+\lambda_0,w}, T) = \langle g^{(1)}_{\lambda+\lambda_0,w} | T g^{(2)}_{\lambda+\lambda_0,w} \rangle, \quad \text{for all } T \in \mathscr{B}(L^2(\mathbf{R})).$$

Let us show that for a.e. $\lambda \in \mathbf{R}$

$$\int ||g_{\lambda+\lambda_0,w}|| dw < ||v_\lambda||_{(A(G))_\lambda}.$$

Indeed, for a.e. $w \in \mathbf{R}$ and all $T \in \mathscr{B}(L^2(\mathbf{R}))$,

$$|(g_{\lambda+\lambda_0,w}, T)| \leqq ||g^{(1)}_{\lambda+\lambda_0,w}||_2 \, ||T g^{(2)}_{\lambda+\lambda_0,w}||_2 \leqq ||T|| \, ||g^{(1)}_{\lambda+\lambda_0,w}||_2 \, ||g^{(2)}_{\lambda+\lambda_0,w}||_2$$

and hence

$$\begin{aligned}
\int ||g_{\lambda+\lambda_0,w}|| dw
&= \int \sup \{ |(g_{\lambda+\lambda_0,w}, T)|, \quad \text{for } T \in \mathscr{B}(L^2(\mathbf{R})), \, ||T|| \leqq 1 \} dw \\
&\leqq \int ||g^{(1)}_{\lambda+\lambda_0,w}||_2 \, ||g^{(2)}_{\lambda+\lambda_0,w}||_2 \, dw \\
&\leqq \left(\int ||g^{(1)}_{\lambda+\lambda_0,w}||_2^2 \, dw \right)^{1/2} \left(\int ||g^{(2)}_{\lambda+\lambda_0,w}||_2^2 \, dw \right)^{1/2} \\
&= \left(\int \int (g^{(1)}_{\lambda+\lambda_0,w}(r))^2 dw dr \right)^{1/2} \left(\int \int (g^{(2)}_{\lambda+\lambda_0,w}(r))^2 dw dr \right)^{1/2} \\
&= \left(\int \int (v_\lambda^{(1)}(t) f_{\lambda_0}(s))^2 dt ds \right)^{1/2} \left(\int \int (v_\lambda^{(2)}(t) f_{\lambda_0}(s))^2 dt ds \right)^{1/2} \\
&= ||v_\lambda^{(1)}||_2 \, ||v_\lambda^{(2)}||_2.
\end{aligned}$$

By Lemma (3.1) (ii), $\|v_\lambda^{(1)}\|_2\|v_\lambda^{(2)}\|_2 = \|v_\lambda\|$ and the inequality follows. Then, for a.e. $\lambda \in \mathbf{R}$, the integral

$$\int g_{\lambda+\lambda_0,w}dw$$

is finite and has value in $\mathscr{B}(L^2(\mathbf{R}))$. Set

$$g_{\lambda+\lambda_0} = \int g_{\lambda+\lambda_0,w}dw.$$

By Lemma (3.1) (ii), for a.e. $\lambda \in \mathbf{R}$, $g_{\lambda+\lambda_0} \in (VN(G))_\lambda$; then there exists $g \in VN(G)^*$ such that, for all $T \in VN(G)$,

$$(g, T) = \int (g_\lambda, T_\lambda)d\lambda, \text{ if } T = \int^\oplus T_\lambda d\lambda.$$

Let us prove that $g \in A(G)$. Let us notice that, for a.e. $\lambda \in \mathbf{R}$, $\|g_{\lambda+\lambda_0}\| \leqq \|v_\lambda\|$. Hence

$$\int \|g_{\lambda+\lambda_0}\|d\lambda \leqq \int \|v_\lambda\|d\lambda = \|v\|_{A(G)}.$$

On the other hand, for $[x, y, z] \in G$, $\lambda \neq -\lambda_0$,

$$\begin{aligned} v_{\lambda_0}([x, y, z]) &= \int d\lambda \int \langle g^{(1)}_{\lambda+\lambda_0,w}|U_{\lambda+\lambda_0}([x, y, z])g^{(2)}_{\lambda+\lambda_0,w}\rangle dw \\ &= \int d\lambda \int (g_{\lambda+\lambda_0,w}, U_{\lambda+\lambda_0}([x, y, z])dw \\ &= \int d\lambda (g_{\lambda+\lambda_0}, U_{\lambda+\lambda_0}([x, y, z)]) = \int (g_\lambda, U_\lambda([x, y, z])d\lambda \\ &= (g, U([x, y, z])) = g([x, y, z]). \end{aligned}$$

Therefore $v_{\lambda_0} \in A(G)$.

PROPOSITION (4.1). *Let $\lambda_0 \in \mathbf{R}$, $\lambda_0 \neq 0$ and $f_{\lambda_0} \in L^2(\mathbf{R})$, $\|f_{\lambda_0}\|_2 = 1$. Let $\sigma(\Psi_{\lambda_0})$ be the map in $VN(G)$ defined by*

(4.2) $\quad (\sigma(\Psi_{\lambda_0})(T), v) = (T, v_{\lambda_0})$, *for all $T \in VN(G)$.*

Then

(i) $\sigma(\Psi_\lambda) \in \mathscr{A}$;

(ii) *If $T \in \mathscr{Z}_{VN(G)}$, then $\sigma(\Psi_{\lambda_0})(T) \in \mathscr{Z}_{VN(G)}$ and if*

$$T = \int^\oplus t(\lambda)I_\lambda d\lambda$$

is the direct decomposition of T, then

(4.3) $$\sigma(\Psi_{\lambda_0})(T) = \int^\oplus t(\lambda + \lambda_0)I_\lambda d\lambda.$$

Proof. By definition, Ψ_{λ_0} is bounded and linear and $\|\Psi_{\lambda_0}\| \leqq 1$. If $u, v \in A(G)$ then

$$(u\sigma(\Psi_{\lambda_0})(T), v)\ (\sigma(\Psi_{\lambda_0})(T), uv) = (T, (uv)_{\lambda_0})$$
$$= (T, u(v_{\lambda_0})) = (uT, v_{\lambda_0}) = (\sigma(\Psi_{\lambda_0})(uT), v).$$

Therefore $\sigma(\Psi_{\lambda_0}) \in \mathscr{A}$.

(ii). Let $T \in \mathscr{Z}_{VN(G)}$ and

$$T = \int^{\oplus} t(\lambda) I_\lambda d\lambda,$$

with $t \in L^\infty(\mathbf{R})$. Then, for all $v \in A(G)$,

$$(\sigma(\Psi_{\lambda_0})(T), v) = (T, v_{\lambda_0}) = \int (g_{\lambda+\lambda_0}, t(\lambda + \lambda_0) I_{\lambda+\lambda_0}) d\lambda.$$

On the other hand

$$(g_{\lambda+\lambda_0}, I_{\lambda+\lambda_0}) = \int dw (g_{\lambda+\lambda_0,w}, I_{\lambda+\lambda_0}) = \int \langle g^{(1)}_{\lambda+\lambda_0,w} | g^{(2)}_{\lambda+\lambda_0,w} \rangle dw$$
$$= \int\int \overline{v_\lambda^{(1)}(t) f_{\lambda_0}(s)} v_\lambda^{(2)}(t) f_{\lambda_0}(s) dt ds = \langle v_\lambda^{(1)} | v_\lambda^{(2)} \rangle = (v_\lambda, I_\lambda)$$

and so

$$(\sigma(\Psi_{\lambda_0})(T), v) = \int t(\lambda + \lambda_0)(v_\lambda, I_\lambda) d\lambda = (S, v),$$

where $S \in \mathscr{Z}_{VN(G)}$ and $S_\lambda = t(\lambda + \lambda_0) I_\lambda$, for a.e. $\lambda \in \mathbf{R}$.

THEOREM (4.1). *Let $\Phi \in UCB(\hat{G})^*$, $\Phi \notin \mathscr{R}$. If $\Phi = \Phi' + \Phi''$, where $\Phi' \in \mathscr{S}$, $\Phi'' \in \mathscr{R}$, and Φ' is not zero on $\mathscr{Z}_{UCB(\hat{G})}$, then there exist $S_0 \in VN(G)$, $\Psi \in UCB(\hat{G})^*$ such that*

(4.4) $\quad \sigma(\Psi)\,\sigma(\Phi)\,(S_0) \neq \sigma(\Phi)\cdot\sigma(\Psi)\,(S_0).$

Proof. Let us prove the theorem for $\Phi \in \mathscr{S}$. Indeed, if $\Phi = \Phi' + \Phi''$, where $\Phi'' \neq 0$, then Φ is central if and only if Φ' is also.

(a). Let us suppose $\Phi \in \mathscr{M}_a^{(w)}$, for some $a \in \mathbf{R}$. We choose, for example, $a = 0$ (if $a \neq 0$ the proof is the same).

Let $T \in \mathscr{Z}_{UCB(\hat{G})}$ such that $(\Phi, T) \neq 0$; by Lemma (3.2) there are $v_0 \in A(G)$ and $T_0 \in \mathscr{Z}_{VN(G)}$ such that $T = v_0 T_0$. Let $t_0 \in L^\infty(\mathbf{R})$ such that

$$T_0 = \int^{\oplus} t_0(\lambda) I_\lambda d\lambda.$$

Take a sequence $\{\nu_n\}$ in $\mathbf{R}^+$ such that $\nu_n \to +\infty$ and define $\{\mu_n\}$ by setting

$$\begin{cases} \mu_1 = \nu_1 \\ \mu_n = \mu_{n-1} + \nu_{n-1} + \nu_n, \text{ for } n = 2, 3 \ldots . \end{cases}$$

Let us define $S_0 \in \mathscr{L}_{VN(G)}$ by

$$S_0 = \int^{\oplus} s_0(\lambda) I_\lambda d\lambda,$$

where $s_0 \in L^\infty(\mathbf{R})$ is the function

$$\begin{cases} s_0(\lambda) = 0, & \text{for } \lambda \geqq 0 \\ s_0(\lambda) = t_0(\lambda + \mu_n), & \text{for } -\mu_n - \nu_n < \lambda < -\mu_n + \nu_n. \end{cases}$$

Let us prove that $(\Phi, vS_0) = 0$, for all $v \in A(G)$.

If $\{\Phi_\alpha\}$ is a sequence of $\mathscr{M}_0$ such that

$$\Phi_\alpha \xrightarrow[\alpha]{(w)} \Phi,$$

we denote by λ_α the maximum positive number such that $\Phi_\alpha \in \mathscr{M}_{\lambda_0}$. We have $\lambda_\alpha \to +\infty$. Otherwise there would be a subnet $\{\lambda_\beta\}$ converging to some $\bar{\lambda} \in \mathbf{R}$ and therefore for all β the support of the measure dm_β would contain $\bar{\lambda}$ and therefore the support of the measure $dm(\tau)$ associated with π_Φ by (3.7) would contain the representation U_λ, against our hypothesis.

Let now $v \in A(G)$. By Lemma (3.2),

$$S = vS_0 \in \mathscr{L}_{UCB(\hat{G})} \quad \text{and} \quad S = \int^{\oplus} s(\lambda) I_\lambda d\lambda,$$

where $s(\lambda) = (s_0 * \tilde{A}_v)(\lambda)$, for a.e. $\lambda \in \mathbf{R}$.

If $A_v \in C_c(\mathbf{R})$ and supp $(A_v) \subset (-\infty . K]$, then supp $(s) \subset (-\infty, K]$; therefore by Lemma (4.1) it follows that $(\Phi_\alpha, vS_0) = 0$, for all $\alpha \geqq \alpha_K$, where α_K is such that $\lambda_{\alpha_K} \geqq \lambda_K$; we conclude, by taking the limit, that $(\Phi, vS_0) = 0$. Let now supp (A_v) be not necessarily compact. For $\epsilon > 0$, let $W \in C_c(\mathbf{R})$ such that $\|A_v - W\|_1 < \epsilon$ and $w \in A(G)$ such that $A_w = W$. Then

$$\begin{aligned} |(\Phi, vS_0)| &= |(\Phi, (v - w)S_0) + (\Phi, wS_0)| \\ &= |(\Phi, (v - w)S_0)| \leqq \|\Phi\| \, \|S_0\| \, \|v - w\|_A. \end{aligned}$$

On the other hand $\|v - w\|_{A(G)} = \|A_v - W\|_1$ and therefore

$$|(\Phi, vS_0)| < \epsilon.$$

Since ϵ is arbitrary, $(\Phi, vS_0) = 0$ and therefore $\sigma(\Phi)(S_0) = 0$. Let us consider, for all n, $\Psi_{-\mu_n} \in UCB(\hat{G})^*$, as defined in Proposition (4.1). Since the unit ball of $UCB(\hat{G})^*$ is compact in the w^*-topology, there is $\Psi \in UCB(\hat{G})^*$ such that

$$(\Psi_{-\mu_n}, vT) \xrightarrow[n]{} (\Psi, vT), \quad \text{for all } vT \in UCB(\hat{G}),$$

by taking a subnet of $\{\mu_n\}$. If $\sigma(\Psi) \in \mathscr{A}$ is the operator associated to Ψ, let us prove that

(4.5) $\sigma(\Psi)\ (S_0) = T_0.$

Indeed, we notice that, for all $\lambda \in \mathbf{R}$,

$$s_0(\lambda - \mu_n) \underset{n}{\rightarrow} t_0(\lambda),$$

because, for $\lambda \in [-\nu_n, \nu_n)$, $s_0(\lambda - \mu_n) = t_0(\lambda)$ and, for all $\lambda \in \mathbf{R}$, it is possible to choose $\bar{n}$ such that, for $n \geqq \bar{n}$, we can choose $\lambda \in [-\nu_n, \nu_n)$. On the other hand

$$(\sigma(\Psi_{-\mu_n})(S_0), v) = \int s_0(\lambda - \mu_n)(I_\lambda, v_\lambda)d\lambda \rightarrow ((\sigma(\Psi)(S_0))_\lambda, v_\lambda)d\lambda$$
$$= (\sigma(\Psi)(S_0), v).$$

It follows then, for a.e. $\lambda \in \mathbf{R}$,

$$((\sigma(\Psi)\ (S_0))_\lambda, \nu_\lambda) = t_0(\lambda)\ (I_\lambda, \nu_\lambda)$$

and therefore

$$(\sigma(\Psi)\ (S_0), v) = (T_0, v).$$

Since $v \in A(G)$ is arbitrary, (4.5) follows.

Briefly, if S_0 and ψ are defined as above, we have

(4.6) $\sigma(\Phi)\sigma(\Psi)(S_0) = \sigma(\Phi)(T_0) \neq 0,\ \sigma(\Psi)\sigma(\Phi)(S_0) = 0.$

(b). If $\Phi \in \bar{\mathscr{L}}_a{}^{(w)}$, for some $a \in \mathbf{R}$, the proof is the same as in (a).

(c). In the general case let, for $\epsilon > 0$, $T \in \mathscr{X}_{UCB(\hat{G})}$, $\|T\| \leqq 1$ such that

$$|(\Phi, T)| \geqq \sup\{|(\Phi, S)|, S \in \mathscr{X}_{UCB(\hat{G})}\} - \epsilon.$$

We can suppose that $T = v_0 T_0$, where $v_0 \in A(G)$, $\|v_0\| \leqq 1$, $T_0 \in \mathscr{X}_{UCB(\hat{G})}$, $\|T_0\| \leqq 1$. For $\epsilon > 0$, let Φ_M, Φ_L as in Proposition (3.1), then

$$\Phi_M = \Phi_M' + \Phi_M'',\ \Phi_L = \Phi_L' + \Phi_L'',$$

where $\Phi_M', \Phi_L' \in \mathscr{S}$, $\Phi_M'', \Phi_L'' \in \mathscr{R}$ and $\Phi_M'' \in \bar{\mathscr{M}}^{(w)}_{\lambda_0+\delta}$, $\Phi_L'' \in \bar{\mathscr{L}}_{\lambda_0-\delta}$, $\Phi_M' \in \bar{\mathscr{M}}^{(w)}_{\lambda_0+\delta}$, $\Phi_L' \in \bar{\mathscr{L}}_{\lambda_0-\delta}$.

From property (i) of Proposition (3.1) and since $\mathscr{R}$ is norm closed, from the hypothesis on Φ it follows that (as necessary replacing Φ_M with Φ_L)

$$(\Phi_M', T) \neq 0,\ \|\Phi_M''\| < \omega(\epsilon),$$

where $\omega(\epsilon) \rightarrow 0$ for $\epsilon \rightarrow 0$. Indeed if $\Phi \in \mathscr{S}$, then the norm of the functionals $(\Phi - \Phi_M' - \Phi_L')$, Φ_M'', Φ_L'' are small.

Then if we construct $S_0 \in VN(G)$, $\Psi \in UCB(\hat{G})^*$ for Φ_M', as in the case (a), we have

$$|(\sigma(\Phi)\sigma(\Psi)\, S_0), v_0)| = |(\Phi, T)| \geqq \|\Phi_{|\mathscr{Z}}\| - \epsilon$$
$$= \|\Phi_{M|\mathscr{Z}}\| + \|\Phi_{L|\mathscr{Z}}\| - \epsilon;$$
$$|(\sigma(\Psi)\sigma(\Phi)(S_0), v_0)| \leqq |(\sigma(\Psi)\sigma(\Phi_M + \Phi_L)(S_0), v_0)| + \epsilon$$
$$= |(\sigma(\Psi)\sigma(\Phi_M'' + \Phi_L)(S_0), v_0)| + \epsilon$$
$$\leqq \|\Phi_{L|\mathscr{Z}}\| + \epsilon + \omega(\epsilon).$$

Since ϵ is arbitrary, the result follows.

5. A concluding remark. In the abelian case of $\mathbf{R}$, the total order of $\mathbf{R}$ induces a total order structure again in the set of the functions

$$\{ f_\lambda(x) = e^{i\lambda x}, \lambda \in \mathbf{R}\},$$

seen as coefficients of the (unidimensional) irreducible representations of $\mathbf{R}$.

On the other hand, in the case of the Heisenberg group, the total order structure of the set of the irreducible representations U_λ of G, with $\lambda \neq 0$, induces only a partial order in the set of their coefficients, while the order induced in the set of the restrictions to the center of G of the same coefficients (not zero on $\mathscr{Z}_G$) is total. Indeed, for all $\lambda \in \mathbf{R}$, $\lambda \neq 0$, the restriction to $\mathscr{Z}_G$ of a coefficient of the irreducible representation U_λ is given by

$$\langle U_\lambda([0, 0, z])\, f \mid g\rangle = e^{i\lambda z}\langle f \mid g\rangle,$$

for $f, g \in L^2(\mathbf{R})$; therefore all the coefficients of an irreducible representation restricted to $\mathscr{Z}_G$ are functions on $\mathscr{Z}_G$ which differ only by a scalar factor (not zero in the above hypothesis).

In view of our techniques, it is easy to note that the difference of our result from the one obtained for $G = \mathbf{R}$ reflects these structural differences between the dual objects of the Heisenberg group and the real line.

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