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META-CENTRALIZERS OF NON-LOCALLY COMPACT GROUP ALGEBRAS

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Abstract. Meta-centralizers of non-locally compact group algebras are studied. Theorems about their representations with the help of families of generalized measures are proved. Isomorphisms of group algebras are investigated in relation with meta-centralizers.

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1. Introduction. Locally compact group algebras are rather well investigated and play very important role in mathematics [**10**, **12**, **13**, **15**, **16**, **18**, **26**]. Left centralizers of locally compact group algebras were studied in [**29**]. In all those works, Haar measures on locally compact groups were used. Haar measures are invariant or quasi-invariant relative to left or right shifts of the entire locally compact group [**6**, **10**, **13**, **26**]. According to the A.Weil theorem, if a topological group has a non-trivial borelian measure quasi-invariant relative to left or right shifts of the entire group, then it is locally compact. Moreover, it is well-known that the compactification of a topological group may have no group structure so that the theory of non-locally compact groups cannot be reduced to that of compact or locally compact groups.

On the other hand, the theory of non-locally compact groups and their representations differ drastically from that of the locally compact case (see [**2**, **3**, **11**, **20**, **21**, **23**] and references therein). Measures on non-locally compact groups quasi-invariant relative to proper dense subgroups were constructed in [**4**, **7**, **8**, **20**, **21**, **22**, **23**, **25**].

This article continues investigations of non-locally compact group algebras [**19**, **21**, **24**]. The present paper is devoted to centralizers of non-locally compact group algebras, which are substantially different from that of locally compact groups. Their definition in the non-locally compact groups setting is rather specific and they are already called meta-centralizers. Theorems about their representations with the help of families of generalized measures are proved. Isomorphisms of group algebras are investigated in relation with meta-centralizers. The main results of this paper are Theorems 8–10 and 14. They are obtained for the first time.

Henceforth, definitions and notations of [**19**] are used.

2. Group algebra. To avoid misunderstandings, we first present our definitions and notations.

DEFINITION 1. Let Λ be a directed set and let $\{G_\alpha : \alpha \in \Lambda\}$ be a family of topological groups with completely regular (i.e. $T_1 \cap T_{3\frac{1}{2}}$) topologies τ_α satisfying the following restrictions:

(1) $\theta_\alpha^\beta : G_\beta \to G_\alpha$ is a continuous algebraic embedding, $\theta_\alpha^\beta(G_\beta)$ is a proper subgroup in G_α for each $\alpha < \beta \in \Lambda$;
(2) $\tau_\alpha \cap \theta_\alpha^\beta(G_\beta) \subset \theta_\alpha^\beta(\tau_\beta)$ and $\theta_\alpha^\beta(G_\beta)$ is dense in G_α for each $\alpha < \beta \in \Lambda$; then $(\theta_\alpha^\beta)^{-1} : \theta_\alpha^\beta(G_\beta, \tau_\beta) \to (G_\beta, \tau_\beta)$ is considered as the continuous homomorphism;
(3) G_α is complete relative to the left uniformity with entourages of the diagonal of the form $\mathcal{U} = \{(h, g) : h, g \in G_\alpha; h^{-1}g \in U\}$ with neighbourhoods U of the unit element e_α in G_α, $U \in \tau_\alpha$, $e_\alpha \in U$;
(4) for each $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$ the embedding $\theta_\alpha^\beta : (G_\beta, \tau_\beta) \hookrightarrow (G_\alpha, \tau_\alpha)$ is precompact, that is by our definition for every open set U in G_β containing the unit element e_β a neighbourhood $V \in \tau_\beta$ of e_β exists so that $V \subset U$ and $\theta_\alpha^\beta(V)$ is precompact in G_α, i.e. its closure $cl(\theta_\alpha^\beta(V))$ in G_α is compact, where $\phi : \Lambda \to \Lambda$ is an increasing marked mapping.

CONDITIONS 2. Henceforward, it is supposed that Conditions $(1 - 5)$ are satisfied:

(1) $\mu_\alpha : \mathcal{B}(G_\alpha) \to [0, 1]$ is a probability measure on the Borel σ-algebra $\mathcal{B}(G_\alpha)$ of a group G_α from Section 1 with $\mu_\alpha(G_\alpha) = 1$ so that
(2) μ_α is quasi-invariant relative to the right and left shifts on $h \in \theta_\alpha^\beta(G_\beta)$ for each $\alpha < \beta \in \Lambda$, where $\rho_{\mu_\alpha}^r(h, g) = (\mu_\alpha^h)(dg)/\mu(dg)$ and $\rho_{\mu_\alpha}^l(h, g) = (\mu_{\alpha,h})(dg)/\mu(dg)$ denote quasi-invariance μ_α-integrable factors, $\mu_\alpha^h(S) = \mu(Sh^{-1})$ and $\mu_{\alpha,h}(S) = \mu_\alpha(h^{-1}S)$ for each Borel subset S in G_α;
(3) a density $\psi_\alpha(g) = \mu_\alpha(dg^{-1})/\mu_\alpha(dg)$ relative to the inversion exists and it is μ_α-integrable;
(4) a subset $W_\alpha \in \mathcal{A}(G_\alpha)$ exists such that $\rho_{\mu_\alpha}^r(h, g)$ and $\rho_{\mu_\alpha}^l(h, g)$ are continuous on $\theta_\alpha^\beta(G_\beta) \times W_\alpha$ and $\psi_\alpha(g)$ is continuous on W_α with $\mu_\alpha(W_\alpha) = 1$ for each $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$;
(5) each measure μ_α is Borel regular and radonian,
where the completion of $\mathcal{B}(G_\alpha)$ by all μ_α-zero sets is denoted by $\mathcal{A}(G_\alpha)$.

NOTATION 3. *Denote by $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ the subspace in $L^1(G_\alpha, \mu_\alpha, \mathbf{F})$, which is the completion of the linear space $L^0(G_\alpha, \mathbf{F})$ of all (μ_α-measurable) simple functions*

$$f(x) = \sum_{j=1}^n b_j \chi_{B_j}(x),$$

where $b_j \in \mathbf{F}$, $B_j \in \mathcal{A}(G_\alpha)$, $B_j \cap B_k = \emptyset$ for each $j \neq k$, χ_B denotes the characteristic function of a subset B, $\chi_B(x) = 1$ for each $x \in B$ and $\chi_B(x) = 0$ for every $x \in G_\alpha \setminus B$, $n \in \mathbf{N}$, where $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$. A norm on $L^1_{G_\beta}(G_\alpha)$ is by our definition given by the formula:

$$\|f\|_{L^1_{G_\beta}(G_\alpha)} := \sup_{h \in \theta_\alpha^\beta(G_\beta)} \|f_h\|_{L^1(G_\alpha)} < \infty, \tag{1}$$

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where $f_h(g) := f(h^{-1}g)$ *for* $h, g \in G_\alpha$, $L^1(G_\alpha, \mu_\alpha, \mathbf{F})$ *is the usual Banach space of all* μ_α*-measurable functions* $u : G_\alpha \to \mathbf{F}$ *such that*

$$\|u\|_{L^1(G_\alpha)} = \int_{G_\alpha} |u(g)|\mu_\alpha(dg) < \infty. \tag{2}$$

Suppose that

(3) $\phi : \Lambda \to \Lambda$ *is an increasing mapping,* $\alpha < \phi(\alpha)$ *for each* $\alpha \in \Lambda$. *We consider the space,*

(4) $L^\infty(L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) : \alpha < \beta \in \Lambda) := \{f = (f_\alpha : \alpha \in \Lambda);\quad f_\alpha \in L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ *for each* $\alpha \in \Lambda$; $\|f\|_\infty := \sup_{\alpha\in\Lambda} \|f_\alpha\|_{L^1_{G_\beta}(G_\alpha)} < \infty$, *where* $\beta = \phi(\alpha)\}$.

When measures μ_α *are specified, spaces are denoted shortly by* $L^1_{G_\beta}(G_\alpha, \mathbf{F})$ *and* $L^\infty(L^1_{G_\beta}(G_\alpha, \mathbf{F}) : \alpha < \beta \in \Lambda)$ *respectively.*

Definition 4. Let the algebra $\mathcal{E} := L^\infty(L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) : \alpha < \beta \in \Lambda)$ be supplied with the multiplication $f\tilde{\star}u = w$ such that

$$w_\alpha(g) = (f_\beta\tilde{\star}u_\alpha)(g) = \int_{G_\beta} f_\beta(h)u_\alpha(\theta^\beta_\alpha(h)g)\mu_\beta(dh), \tag{1}$$

for every $f, u \in \mathcal{E}$ and $g \in G = \prod_{\alpha\in\Lambda} G_\alpha$, where $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$, $\beta = \phi(\alpha)$, $\alpha \in \Lambda$.

If a bounded linear transformation $T : \mathcal{E} \to \mathcal{E}$ satisfies Conditions (2, 3):

(2) $Tf = (T_\alpha f_\alpha : \alpha \in \Lambda)$, $T_\alpha : L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) \to L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ for each $\alpha \in \Lambda$,

(3) $T(f\tilde{\star}u) = f\tilde{\star}(Tu)$,

for each $f, u \in \mathcal{E}$, then T is called a left meta-centralizer.

Definition 5. Let X be a topological space, let $C(X, \mathbf{R})$ be the space of all continuous functions $f : X \to \mathbf{R}$, while $C_b(X, \mathbf{R})$ be the space of all bounded continuous functions with the norm

(1) $\|f\| := \sup_{x\in X} |f(x)| < \infty$.
Suppose that $\mathcal{F}$ is the least σ-algebra on X containing the algebra $\mathcal{Z}$ of all functionally closed subsets $A = f^{-1}(0)$, $f \in C_b(X, \mathbf{R})$. A finitely additive non-negative mapping $m : \mathcal{F} \to [0, \infty)$ such that
(2) $m(A) = \sup\{m(B) : B \in \mathcal{Z}, B \subset A\}$,
for each $A \in \mathcal{F}$ is called (a finitely additive) measure. A generalized measure is the difference of two measures. Denote by $M(X) = M(X, \mathbf{R})$ the family of all generalized (finitely additive) measures.
For short "generalized" may be omitted, when m is considered with values in $\mathbf{R}$.

Theorem 6 (A.D. Alexandroff [**28**]). *$M(X)$ is the topologically dual space to $C_b(X, \mathbf{R})$, that is for each bounded linear functional J on $C_b(X, \mathbf{R})$ there exists a unique generalized (finitely additive) measure $m \in M(X)$ such that*

(1) $J(f) = \int_X f\,dm$ *for each* $f \in C_b(X, \mathbf{R})$,
each measure $m \in M(X)$ defines a unique continuous linear functional by Formula (1). *Moreover,*
(2) $\|J\| = \|m\|$.

Definitions 7. A bounded linear functional J on $C_b(X, \mathbf{R})$ is called σ-smooth, if
(1) $\lim_n J(f_n) = 0$

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for each sequence f_n in $C_b(X, \mathbf{R})$ such that $0 \le f_{n+1}(x) \le f_n(x)$ and $\lim_n f_n(x) = 0$ for each point $x \in X$. The linear space of all σ-smooth linear functionals is denoted by $M_\sigma(X) = M_\sigma(X, \mathbf{R})$.

A bounded linear functional J on $C_b(X, \mathbf{R})$ is called tight, if Formula (1) is fulfilled for each net f_α in $C_b(X, \mathbf{R})$ such that $\|f_\alpha\| \le 1$ for each α and f_α tends to zero uniformly on each compact subset K in X. The space of all tight linear functionals is denoted by $M_t(X) = M_t(X, \mathbf{R})$.

If $m_1, m_2 \in M(X)$, then $m = m_1 + im_2$ is a complex-valued measure, their corresponding spaces are denoted by $M(X, \mathbf{C})$, $M_\sigma(X, \mathbf{C}) = M_\sigma(X) + iM_\sigma(X)$ and $M_t(X, \mathbf{C}) = M_t(X) + iM_t(X)$.

THEOREM 8. *Let $\mathcal{E}$ be a real $\mathbf{F} = \mathbf{R}$ or complex $\mathbf{F} = \mathbf{C}$ algebra (see Section 4), let also T be a left meta-centralizer on $\mathcal{E}$. Then there exists a family $\nu = (\nu_\alpha : \alpha \in \Lambda)$ of generalized* **F**-*valued measures ν_α on G_α of bounded variation such that*

$$Tf = \nu \tilde{\star} f, \tag{1}$$

where

$$(T_\alpha f_\alpha)(g) = (\nu_\beta \tilde{\star} f_\alpha)(g) = \int_{G_\beta} \nu_\beta(dh) f_\alpha(\theta^\beta_\alpha(h)g) \tag{2}$$

for each $\alpha \in \Lambda$ and $g \in G_\alpha$ with $\beta = \phi(\alpha)$.

Proof. For each $\beta \in \Lambda$ and a neutral element $e_\beta \in G_{\beta,}$ we consider a basis of its neighbourhoods $\{V_{a,\beta} : a \in \Psi_\beta\}$ such that $cl_{G_\alpha} \theta^\beta_\alpha(V_{a,\beta})$ is compact in (G_α, τ_α), where Ψ_β is a set, $cl_X A$ denotes the closure of a set A in a topological space X. The set Ψ_β is directed by the inclusion: $a \le b \in \Psi_\beta$ if and only if $V_{b,\beta} \subseteq V_{a,\beta}$.

There is a natural continuous linear restriction mapping $p^U_V : C_b(U, \mathbf{F}) \to C_b(V, \mathbf{F})$ for each closed subsets U and V in G_β such that $V \subset U$, where $p^U_V(f) = f|_V$ for each $f \in C_b(U, \mathbf{F})$. At the same time, if U is compact, then each continuous bounded function g on V with values in $\mathbf{F}$ has a continuous extension $\pi^V_U(g)$ on U with values in $\mathbf{F}$ such that

$$\|g\|_{C_b(V,\mathbf{F})} \le \|\pi^V_U(g)\|_{C_b(U,\mathbf{F})} \le 2\|g\|_{C_b(V,\mathbf{F})},$$

due to Tietze–Uryson Theorem 2.1.8 [**9**], since G_β is T_0 and hence, completely regular by Theorem 8.4 [**13**] and each Huasdorff compact space is normal by Theorems 5.1.1 and 5.1.5

[**9**]. Thus, there exists a linear continuous embedding $\pi^V_U : C_b(V, \mathbf{F}) \hookrightarrow C_b(U, \mathbf{F})$.

The probability measure μ_β on G_β is Borel regular and radonian. Hence, there exists a σ-compact subset X_β in G_β such that $\mu_\beta(X_\beta) = 1$, i.e. X_β is the countable union of compact subsets $X_{\beta,n}$ in (G_β, τ_β) with $X_{\beta,n} \subset X_{\beta,n+1}$ for each natural number n.

We put

$$q_{a,\beta} := \chi_{V_{a,\beta}} / \mu_\beta(V_{a,\beta}), \tag{3}$$

where χ_A denotes the characteristic function of a subset A in G_β, $\chi_A(x) = 1$ for each $x \in A$, while $\chi_A(x) = 0$ for each $x \notin A$. In view of Proposition 17.7 [**21**] (see also Lemma 13 [**19**]), the net $\{q_{a,\beta} : a \in \Psi_\beta\}$ is an approximation of the identity relative to

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the convolution:

$$\lim_a q_{a,\beta} \tilde{\star} f_\alpha = f_\alpha \tag{4}$$

for each $f_\alpha \in L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$. From Formulas (2, 4) and 4(1–3), it follows that

$$T_\alpha f_\alpha = T_\alpha[\lim_a q_{a,\beta} \tilde{\star} f_\alpha] = \lim_a q_{a,\beta} \tilde{\star}[T_\alpha f_\alpha]. \tag{5}$$

Then $q_{a,\beta} \tilde{\star}[T_\alpha \cdot] : L^1_{G_\beta}(G_\alpha) \to L^1_{G_\beta}(G_\alpha)$ is a continuous linear operator for each $a \in \Psi_\beta$ and $\alpha \in \Lambda$, particularly, for each f_α in the space $C_b(G_\alpha, \mathbf{F})$ of all bounded continuous functions on G_α with values in the field $\mathbf{F}$, where

$$\|f_\alpha\|_{C_b} := \sup_{x \in G_\alpha} |f_\alpha(x)| < \infty, \tag{6}$$

for each $f_\alpha \in C_b(G_\alpha, \mathbf{F})$. The restriction of each $f_\alpha \in C_b(G_\alpha, \mathbf{F})$ on $\theta^\beta_\alpha(G_\beta)$ is bounded and continuous, while $C_b(G_\beta, \mathbf{F})$ is dense in $L^1_{G_\gamma}(G_\beta, \mu_\beta, \mathbf{F})$ with $\gamma = \phi(\beta)$ (see also Lemma 17.8 and Proposition 17.9 [**21**]).

This implies that an adjoint operator $B = T^*$ exists relative to the $\tilde{\star}$ multiplication according to the formula:

$$\begin{aligned}(v_\beta \tilde{\star}[T_\alpha \bar{f}_\alpha])(x) &= \int_{G_\beta} v_\beta(h)[T_\alpha \bar{f}_\alpha](\theta^\beta_\alpha(h)x)\mu_\beta(dh) \\ &=: \int_{G_\beta} (B_\beta v_\beta)(h)\bar{f}_\alpha(\theta^\beta_\alpha(h)x)\mu_\beta(dh),\end{aligned} \tag{7}$$

for each $v, f \in \mathcal{E}$, where $x \in G_\alpha$, $\bar{z}$ denotes the complex conjugated number of $z \in \mathbf{C}$. The operator B_β is bounded and linear from $L^1_{G_\gamma}(G_\beta)$ into itself, since from Formula (7) the estimate follows:

$$\|B_\beta\| \leq \sup_{s \in \theta^\gamma_\beta(G_\gamma),\ t \in \theta^\beta_\alpha(G_\beta),\ 0 \neq v_\beta \in L^1_{G_\gamma}(G_\beta),\ 0 \neq f_\alpha \in L^1_{G_\beta}(G_\alpha)} \frac{|\int_{G_\alpha}\int_{G_\beta} v_\beta(sh)[T_\alpha \bar{f}_\alpha](\theta^\beta_\alpha(h)tx)\mu_\beta(dh)\mu_\alpha(dx)|}{\|v_\beta\|_{L^1_{G_\gamma}(G_\beta)}\|f_\alpha\|_{L^1_{G_\beta}(G_\alpha)}} \leq \|T_\alpha\| < \infty. \tag{8}$$

The family of bounded linear operators $\{(B_\beta q_{a,\beta})\tilde{\star} :\ a \in \Psi_\beta\}$ from $L^1_{G_\beta}(G_\alpha)$ into $L^1_{G_\beta}(G_\alpha)$ is pointwise bounded and hence by the Banach–Steinhaus Theorem (11.6.1) [**27**] it is uniformly bounded:

$$\sup_{a \in \Psi_\beta} \|(B_\beta q_{a,\beta})\tilde{\star}\| < \infty. \tag{$B1$}$$

Therefore, inequality (8) leads to the conclusion that $B_\beta q_{a,\beta} =: h_{a,\beta} \in L^1_{G_\gamma}(G_\beta, \mu_\beta, \mathbf{F})$ for every $a \in \Psi_\beta$ and $\beta \in \Lambda$. Each function $h_{a,\beta}$ induces the linear functional

$$F_{a,\beta}(g_\beta) := \int_{G_\beta} g_\beta(x)\bar{h}_{a,\beta}(x)\mu_\beta(dx). \tag{9}$$

Without loss of generality, we choose $V_{a,\beta}$ such that $cl_{G_\alpha} V_{a,\beta}$ is compact in (G_α, τ_α) for each $a \in \Psi_\beta$. Certainly, if $f \in L^1_{G_\gamma}(G_\beta, \mu_\beta, \mathbf{F})$, then $f \in L^1(G_\beta, \mu_\beta, \mathbf{F})$ and

$$\|f\|_{L^1(G_\beta,\mu_\beta,\mathbf{F})} \leq \|f\|_{L^1_{G_\gamma}(G_\beta,\mu_\beta,\mathbf{F})} < \infty. \tag{10}$$

There is the embedding $C_b(G_\beta, \mathbf{F}) \subset L^1_{G_\gamma}(G_\beta, \mu_\beta, \mathbf{F})$ and

$$\|f\|_{L^1_{G_\gamma}(G_\beta,\mu_\beta,\mathbf{F})} \leq \|f\|_{C_b(G_\beta,\mathbf{F})} < \infty, \tag{11}$$

for each $f \in C_b(G_\beta, \mathbf{F})$, since μ_β is the probability measure on G_β.

If $f \in L^1_{G_\gamma}(G_\beta)$, then $s \mapsto f\tilde{\star}s$ is a continuous linear operator from $C_b(G_\beta, \mathbf{F})$ into $C_b(G_\beta, \mathbf{F})$. This follows from the formulas:

$$(f\tilde{\star}s)(g) = \int_{G_\beta} f(h)s(hg)\mu_\beta(dh), \tag{12}$$

where $g \in G_\beta$ and

$$\sup_g |(f\tilde{\star}s)(g)| \leq \|s\|_{C_b} \int_{G_\beta} |f(h)|\mu_\beta(dh) \leq \|s\|_{C_b}\|f\|_{L^1(G_\beta)} \leq \|s\|_{C_b}\|f\|_{L^1_{G_\gamma}(G_\beta)}.$$

It remains to verify that the function $(f\tilde{\star}s)(g)$ is continuous for each f and s as just above. For the proof consider the term

$$|(f\tilde{\star}s)(g_1) - (f\tilde{\star}s)(g_2)| = \left| \int_{G_\beta} f(h)[s(hg_1) - s(hg_2)]\mu_\beta(dh) \right|. \tag{13}$$

From $f \in L^1_{G_\gamma}(G_\beta)$ and $s \in C_b(G_\beta, \mathbf{F})$, it follows that for each $\epsilon > 0$ there exists a compact subset V in G_β such that $\int_{G_\beta \setminus V} |f(h)|\mu_\beta(dh) < \epsilon$ and hence $\int_{G_\beta \setminus V} |f(h)[s(hg_1) - s(hg_2)]|\mu_\beta(dh) < \delta$, where $0 < \delta = \epsilon 2\|s\|_{C_b}$. Indeed, for each $\delta > 0$, there exists a simple function $q \in L^1_{G_\gamma}(G_\beta)$ such that $\|f - q\|_{L^1_{G_\gamma}(G_\beta)} < \delta$ and hence the measure $|f(h)|\mu_\beta(dh)$ is radonian together with $|q(h)|\mu_\beta(dh)$. At the same time, certainly, $\int_V |f(h)|\mu_\beta(dh) \leq \|f\|_{L^1(G_\beta)}$.

On the other hand, $[s(hg_1) - s(hg_2)]$ is uniformly continuous on V by the variable h, since V is compact and s is the continuous function. For each symmetric open neighbourhood $U = U^{-1}$ of the neutral element e_β in G_β, there exists a finite family of elements $p_1, \dots, p_n \in G_\beta$ such that $V \subset p_1 U \cup \dots \cup p_n U$, since V is compact. Thus $VU \subset p_1 U^2 \cup \dots \cup p_n U^2$. Consider a family of symmetric open neighbourhoods $U_k = U_k^{-1}$ of e_β such that $\{p_k U_k : k \in \omega\}$ is a covering of V and $|s(hg_1) - s(hg_2)| < \epsilon$ for each $h \in p_k U_k$ and $g_1, g_2 \in U_k$, where $p_k \in G_\beta$ for each k, whilst ω is an ordinal. The covering $p_k U_k$ of V has a finite subcovering for $k \in M$, where M is a finite subset in ω. Thus for each $\epsilon > 0$ there exists a symmetric neighbourhood $U \subseteq \bigcap_{k \in M} U_k$ of e_β such that $|s(hg_1) - s(hg_2)| < \epsilon$ for each $h \in V$ and $g_1, g_2 \in U$. Therefore,

$$|(f\tilde{\star}s)(g_1) - (f\tilde{\star}s)(g_2)| \leq \delta + \epsilon\|f\|_{L^1} = \epsilon(\|f\|_{L^1} + 2\|s\|_{C_b}),$$

for each $g_1, g_2 \in U$. Thus

$$f \tilde{\star} s \in C_b(G_\beta, \mathbf{F}) \tag{14}$$

for each $f \in L^1_{G_\gamma}(G_\beta, \mathbf{F})$ and $s \in C_b(G_\beta, \mathbf{F})$.

This implies that

$$C_b(G_\beta, \mathbf{F}) \ni s \mapsto (f \tilde{\star} s)(e_\beta) \in \mathbf{F}, \tag{15}$$

is the continuous linear functional on $C_b(G_\beta, \mathbf{F})$. In particular each operator $(B_\beta q_{a,\beta})\tilde{\star}$ indices the continuous linear functional

$$J_{a,\beta}(s) = [(B_\beta q_{a,\beta})\tilde{\star} s](e_\beta) \text{on} C_b(G_\beta, \mathbf{F}). \tag{16}$$

There are the inclusions $M_t(X) \subset M_\sigma(X) \subset M(X)$ (see Section 1.4 [**28**] and Definitions 5, 7 and Theorem 6 above) and for $X = G_\beta$ in particular. On the other hand, each $w_{a,\beta}(dx) := (B_\beta q_{a,\beta})(x)\mu_\beta(dx)$ is the radonian measure on G_β, i.e. belongs to the space $M_t(G_\beta, \mathbf{F})$ of radonian measures on G_β.

Let Φ_β be a family of all left-invariant pseudo-metrics on (G_β, τ_β) providing its left uniformity denoted by $\mathcal{L}_\beta$ (see Section 8.1.7 [**9**] and Condition 1(3)). This means that each $\kappa \in \Phi_\beta$ satisfies the restrictions:

$(P1)$ $\kappa(x, y) \geq 0$,
$(P2)$ $\kappa(x, x) = 0$,
$(P3)$ $\kappa(x, y) = \kappa(y, x)$,
$(P4)$ $\kappa(x, y) \leq \kappa(x, z) + \kappa(z, y)$,
$(P5)$ $\kappa(zx, zy) = \kappa(x, y)$ for each $x, y, z \in G_\beta$.

The family Φ_β is directed: $\kappa_1 \leq \kappa \in \Phi_\beta$ if and only if $\kappa_1(x, y) \leq \kappa(x, y)$ for each $x, y \in G_\beta$; without loss of generality for each $\kappa, \kappa_1 \in \Phi_{\beta,}$ there exists $\kappa_2 \in \Phi_\beta$ such that $\kappa \leq \kappa_2$ and $\kappa_1 \leq \kappa_2$, since $\kappa + \kappa_1 \in \Phi_\beta$. Each pseudo-metric $\kappa \in \Phi_\beta$ defines the equivalence relation: $x \Xi_\kappa y$ if and only if $\kappa(x, y) = 0$. Then as the uniform space $(G_\beta, \mathcal{L}_\beta)$ has the projective limit decomposition (i.e. the limit of the inverse mapping system)

$$G_\beta = \lim \left\{G_{\beta,\kappa}, \pi^\kappa_\omega, \Phi_\beta\right\},$$

where, $G_{\beta,\kappa} := G_\beta / \Xi_\kappa$ denotes the quotient uniform space with the quotient uniformly, π_κ is a uniformly continuous mapping from G_β onto $G_{\beta,\kappa}$, π^κ_ω are uniformly continuous mappings from $G_{\beta,\kappa}$ onto $G_{\beta,\omega}$ for each $\omega \leq \kappa \in \Psi_\beta$ such that $\pi^\omega_\xi \circ \pi^\kappa_\omega = \pi^\kappa_\xi$ and $\pi_\omega = \pi^\kappa_\omega \circ \pi_\kappa$ for each $\xi \leq \omega \leq \kappa \in \Phi_\beta$ (see Sections 8.2.B, 2.5.F and Proposition 2.4.2 [**9**] or [**14**]). Moreover, the equality is satisfied: $\{y \in G_\beta :\ x \Xi_\kappa y\} = x\Omega_{\beta,\kappa}$ with $\Omega_{\beta,\kappa} := \{y \in G_\beta :\ e_\beta \Xi_\kappa y\}$, since $\kappa(x, y) = 0$ if and only if $\kappa(e_\beta, x^{-1}y) = 0$ by Property $(P5)$, where e_β denotes the neutral element in the group G_β. That is, $G_{\beta,\kappa}$ is called the homogeneous quotient uniform space.

At the same time the σ-compact subset X_β is dense in G_β, since $\mu_\beta(U) > 0$ for each open subset U in G_β, but $\mu_\beta(X_\beta) = \mu_\beta(G_\beta) = 1$ (see the proof above). Therefore, $\pi_\kappa(X_\beta)$ is dense in $G_{\beta,\kappa}$. Then $\pi_\kappa(X_{\beta,n})$ is compact for each $\kappa \in \Phi_\beta$ as the continuous image of the compact space according to Theorem 3.1.10 [**9**], consequently, $\pi_\kappa(X_\beta) = \bigcup_{n=1}^\infty \pi_\kappa(X_{\beta,n})$ is σ-compact. On the other hand, $G_{\beta,\kappa}$ is metrizable and complete, since $(G_\beta, \mathcal{L}_\beta)$ is complete. Therefore, the topological space $\pi_\kappa(X_\beta)$ is separable, since each $\pi_\kappa(X_{\beta,n})$ is separable by Theorems 4.3.5 and 4.3.27 [**9**] and $\pi_\kappa(X_\beta) = \bigcup_{n=1}^\infty \pi_\kappa(X_{\beta,n})$. This implies that each metrizable space $G_{\beta,\kappa}$ is separable and complete.

The spaces $C_b(G_\beta, \mathbf{F})$ and $C^*_b(G_\beta, \mathbf{F})$ form the dual pair (see Sections 9.1 and 9.2 [**27**]). Then we get that the space of bounded continuous functions $C_b(G_\beta, \mathbf{F})$ has the inductive limit representation $C_b(G_\beta, \mathbf{F}) = ind - \lim_{\Phi_\beta} C_b(G_{\beta,\kappa}, \mathbf{F})$, while its topologically dual space has the projective limit decomposition $C^*_b(G_\beta, \mathbf{F}) = pr - \lim_{\Phi_\beta} C^*_b(G_{\beta,\kappa}, \mathbf{F})$ (see Sections 9.4, 9.9, 12.2, 12.202 [**27**] and also the note after Theorem 2.5.14 in [**9**]). This implies that $\nu_\beta \in M(G_\beta, \mathbf{F})$ if and only if

$$\nu_\beta = \lim\{\nu_{\beta,\kappa}, \pi^\kappa_\omega, \Phi_\beta\}, \tag{M1}$$

where, $\nu_{\beta,\kappa} \in M(G_{\beta,\kappa}, \mathbf{F})$ for each $\kappa \in \Phi_\beta$ so that

$$\nu_\beta(\pi^{-1}_\omega(C)) = \nu_{\beta,\omega}(C) \text{ and } \nu_{\beta,\kappa}((\pi^\kappa_\omega)^{-1}(C)) = \nu_{\beta,\omega}(C) \tag{M2}$$

for every $C \in \mathcal{B}(G_{\beta,\omega})$ and $\omega \le \kappa \in \Phi_\beta$.

Then we consider the measure net $\{w_{a,\beta,\kappa} : a \in \Psi_\beta\}$ for each $\kappa \in \Phi_\beta$ corresponding to measures $w_{a,\beta}(dx) = (B_\beta q_{a,\beta})(x)\mu_\beta(dx)$ according to Formula $(M2)$, where $x \in G_\beta$. Since the measure $w_{a,\beta}(dx)$ is absolutely continuous relative to the radonian measure μ_β, then $w_{a,\beta}$ is also radonian. Therefore, there is the inclusion $\{w_{a,\beta,\kappa} : a \in \Psi_\beta\} \subset M_t(G_{\beta,\kappa}, \mathbf{F})$ and it is known that $M_t(Y, \mathbf{F}) \subset M_\sigma(Y, \mathbf{F}) \subset M(Y, \mathbf{F})$ for a completely regular topological space Y. Thus the measure net $\{w_{a,\beta} : a \in \Psi_\beta\}$ weakly converges to some measure ν_β in $M(G_\beta, \mathbf{F})$ if and only if the net $\{w_{a,\beta,\kappa} : a \in \Psi_\beta\}$ weakly converges in $M(G_{\beta,\kappa}, \mathbf{F})$ for each $\kappa \in \Phi_\beta$ according to Theorem 2.5.6 and Corollary 2.5.7 [**9**]. The net $\{w_{a,\beta} : a \in \Psi_\beta\}$ is norm bounded, since

$$\|B_\beta q_{a,\beta}\|_{L^1(G_\beta)} \le \sup\{\|(B_\beta q_{a,\beta})\tilde{\star} f_\alpha\|_{L^1_{G_\beta}(G_\alpha)} : f_\alpha \in L^1_{G_\beta}(G_\alpha),\ \|f_\alpha\|_{L^1_{G_\beta}(G_\alpha)} \le 1\}$$

$$= \sup\{\|q_{a,\beta}\tilde{\star}(T_\alpha f_\alpha)\|_{L^1_{G_\beta}(G_\alpha)} : f_\alpha \in L^1_{G_\beta}(G_\alpha),\ \|f_\alpha\|_{L^1_{G_\beta}(G_\alpha)} \le 1\} \le$$

$$\|T_\alpha\| \sup\{\|q_{a,\beta}\tilde{\star} g_\alpha\|_{L^1_{G_\beta}(G_\alpha)} : g_\alpha \in L^1_{G_\beta}(G_\alpha),\ \|g_\alpha\|_{L^1_{G_\beta}(G_\alpha)} \le 1\}$$

$$\le \|T_\alpha\| < \infty, \text{ since}$$

$$\|u_\beta \tilde{\star} g_\alpha\|_{L^1_{G_\beta}(G_\alpha)} \le \|u\|_{L^1(G_\beta)} \|g_\alpha\|_{L^1_{G_\beta}(G_\alpha)},$$

for each $u \in L^1(G_\beta)$ and $g_\alpha \in L^1_{G_\beta}(G_\alpha)$ (see Lemma 17.2 [**21**]). This implies that for each $\epsilon > 0$ and $\kappa \in \Phi_\beta$ there exists a compact set $K_{\epsilon,\kappa}$ in $G_{\beta,\kappa}$ such that $w_{a,\beta,\kappa}(G_{\beta,\kappa} \setminus K_{\epsilon,\kappa}) < \epsilon$ for each $a \in \Psi_\beta$, since $\mu_{\beta,\kappa}$ as the image of μ_β is the radonian measure on the complete separable metric space $G_{\beta,\kappa}$ and each measure $w_{a,\beta,\kappa}$ is absolutely continuous relative to $\mu_{\beta,\kappa}$ (see also Theorem 1.2 [**7**] and Formulas $(M1, M2)$).

Applying theorems either 2.24 and 2.27 or 2.30 [**28**], we get that a measure $\nu_{\beta,\kappa} \in M_\sigma(G_{\beta,\kappa}, \mathbf{F})$ exists such that the net $w_{a,\beta,\kappa}$ weakly converges to $\nu_{\beta,\kappa}$ for each $\beta \in \Lambda$ and $\kappa \in \Phi_\beta$. Thus, using Formulas $(M1, M2)$ we have deduced that

$$\lim_a J_{a,\beta}(f) = \int_{G_\beta} f\, d\nu_\beta, \tag{17}$$

for each $f \in C_b(G_\beta, \mathbf{F})$. The variation of ν_β is finite and $M(G_\beta, \mathbf{F})$ is the Banach space relative to the variation norm according to Theorems 1.2 and 1.3 [**28**].

Let $x \in C_b(G_\beta, \mathbf{F})$ and $y \in C_b(G_\gamma, \mathbf{F})$, we consider the function

$$z(g) = \int_{G_\gamma} y(h)x(\theta^\gamma_\beta(h)g)\mu_\gamma(dh). \tag{18}$$

It evidently exists and is μ_β-measurable, since $\mu_\gamma(G_\gamma) = 1$, consequently,

$$\sup_{g\in G_\beta}\left|\int_{G_\gamma} y(h)x(\theta^\gamma_\beta(h)g)\mu_\gamma(dh)\right| \leq \|y\|_{C_b(G_\gamma,\mathbf{F})}\|x\|_{C_b(G_\beta,\mathbf{F})}.$$

Moreover, $z \in C_b(G_\beta, \mathbf{F}) \subset L^1_{G_\gamma}(G_\beta)$ due to the latter inequality and Properties (11, 14) (see above). Since ν_β is the weak limit of the net $J_{a,\beta}$, then for each $\epsilon > 0$, there exists $b \in \Psi_\beta$ such that

$$\left|\int_{G_\beta} z(g)\nu_\beta(dg) - \int_{G_\beta} z(g)(B_\beta q_{a,\beta})(g)\mu_\beta(dg)\right| < \epsilon, \tag{19}$$

for each $a > b$. In view of the Fubini theorem the latter inequality implies that

$$\left|\int_{G_\gamma} y(h)\mu_\gamma(dh)\int_{G_\beta} x(\theta^\gamma_\beta(h)g)\nu_\beta(dg)\right.$$
$$\left.-\int_{G_\gamma} y(h)\mu_\gamma(dh)\int_{G_\beta} x(\theta^\gamma_\beta(h)g)(B_\beta q_{a,\beta})(g)\mu_\beta(dg)\right| \leq \epsilon \tag{20}$$

for each $a > b$. Therefore, $T_\alpha x(g) = (\nu_\beta\tilde{\star}x)(g)$ for each $x \in C_b(G_\beta, \mathbf{F}) \cap [(\theta^\beta_\alpha)^{-1}(C_b(G_\alpha, \mathbf{F}))]$ and $g \in G_\beta$. If $f_\alpha \in C_b(G_\alpha, \mathbf{F})$, then its restriction $f_\alpha|_{\theta^\beta_\alpha(G_\beta)}$ is continuous and bounded, that is $f_\alpha \circ (\theta^\beta_\alpha)^{-1}$ is continuous and bounded on (G_β, τ_β) due to 1(2). Moreover, the function $\psi_g(h) := f_\alpha(\theta^\beta_\alpha(h)g)$ is continuous and bounded by $h \in G_\beta$ for each $g \in G_\alpha$. Hence,

$$(\nu_\beta\tilde{\star}\psi_g)(s) = \int_{G_\beta} f_\alpha(\theta^\beta_\alpha(hs)g)\nu_\beta(dh) = [\nu_\beta\tilde{\star}f_\alpha](\theta^\beta_\alpha(s)g), \tag{21}$$

is defined for each $s \in G_\beta$ and $g \in G_\alpha$, particularly for $s = e_\beta$.

By the conditions of this theorem $T_\alpha : L^1_{G_\beta}(G_\alpha) \to L^1_{G_\beta}(G_\alpha)$ is a continuous linear operator. There is also the inclusion $C_b(G_\alpha, \mathbf{F}) \subset L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ so that $C_b(G_\alpha, \mathbf{F})$ is dense in $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$, since $\mu_\alpha(X_\alpha) = \mu_\alpha(G_\alpha) = 1$ with the σ-compact subset X_α in G_α (see also Lemma 17.8 and Proposition 17.9 **[21]** and Property (14) above). Let $f_\alpha \in L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ and we take any sequence of bounded continuous functions $f_{\alpha,n} \in C_b(G_\alpha, \mathbf{F})$ converging to f_α in $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$. We have

$$\lim_a (B_\beta q_{a,\beta})\tilde{\star}f_{\alpha,n} = f_\alpha \text{ and } \lim_n f_{\alpha,n} = f_\alpha, \tag{22}$$

in $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$. Then

$$\|(B_\beta q_{a,\beta})\tilde{\star}f_{\alpha,n} - (B_\beta q_{b,\beta})\tilde{\star}f_{\alpha,m}\|_{L^1_{G_\beta}(G_\alpha)}$$
$$\leq \|(B_\beta q_{a,\beta} - B_\beta q_{b,\beta})\tilde{\star}f_{\alpha,n}\|_{L^1_{G_\beta}(G_\alpha)} + \|(B_\beta q_{b,\beta})\tilde{\star}\|\,\|f_{\alpha,n} - f_{\alpha,m}\|_{L^1_{G_\beta}(G_\alpha)}, \tag{23}$$

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consequently, for each $\epsilon > 0$ there exist $a_0 \in \Psi_\beta$ and $n_0 \in \mathbf{N}$ such that

$$\|(B_\beta q_{a,\beta})\tilde{\star} f_{\alpha,n} - (B_\beta q_{b,\beta})\tilde{\star} f_{\alpha,m}\|_{L^1_{G_\beta}(G_\alpha)} < \epsilon, \tag{24}$$

for each $a, b > a_0$ and $n, m > n_0$ (see Lemma 17.2 and Proposition 17.7 [**21**] and Formula $(B1)$ above). That is the net $\{(B_\beta q_{a,\beta})\tilde{\star} f_{\alpha,n} : (a, n)\}$ is fundamental (i.e. of the Cauchy type) in the Banach space $L^1_{G_\beta}(G_\alpha)$, where $(a, n) \le (b, m)$ if $a \le b$ and $n \le m$. Therefore the limit exists

$$T_\alpha f_\alpha = \lim_{a,n}(B_\beta q_{a,\beta})\tilde{\star} f_{\alpha,n} = \lim_n \lim_a (B_\beta q_{a,\beta})\tilde{\star} f_{\alpha,n} = \lim_n \nu_\beta \tilde{\star} f_{\alpha,n} = \nu_\beta \tilde{\star} f_\alpha. \tag{25}$$

Thus

$$T_\alpha f_\alpha = \nu_\beta \tilde{\star} f_\alpha,$$

for each $f_\alpha \in L^1_{G_\beta}(G_\alpha)$ as well, that is, Formulas (1, 2) are fulfilled.

THEOREM 9. *Let the assumptions of Theorem 8 be satisfied. Then the statement of Theorem 8 is equivalent to the following:*

(1) *relative to the strong operator topology the set of all convolution operators of the form* $8(1, 2)$ *on* $\mathcal{E} := L^\infty(L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) : \alpha < \beta \in \Lambda)$ *with values in* $\mathcal{E}$ *is a closed subset of the ring of all bounded linear operators from* $\mathcal{E}$ *into* $\mathcal{E}$.

Proof. $(8 \Rightarrow 9)$. Let $\nu_{a,\beta}\tilde{\star}$ be a net of convolution operators converging to an operator $T_\alpha : L^1_{G_\beta}(G_\alpha) \to L^1_{G_\beta}(G_\alpha)$ in the strong operator topology for each $\alpha \in \Lambda$, hence T is the left meta-centralizer on $\mathcal{E}$, since each operator $\{\nu_{a,\beta}\tilde{\star} : \alpha \in \Lambda, \beta = \phi(\alpha)\}$ is the left meta-centralizer.

$(9 \Rightarrow 8)$. From the proof of Theorem 8, we analogously get

$$T_\alpha f_\alpha = \lim_a \nu_{a,\beta}\tilde{\star} f_\alpha,$$

for each $\alpha \in \Lambda$ and $f_\alpha \in L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ with $\beta = \phi(\alpha)$, where $\nu_{a,\beta} \in M(G_\beta, \mathbf{F})$ for each $\beta \in \Lambda$ and $a \in \Psi_\beta$ consequently, $T = (T_\alpha : \alpha)$ is the convolution operator.

THEOREM 10. *Let S be a bounded linear mapping of $\mathcal{E}$ (see Section 4) into itself such that $Sf = (S_\alpha f_\alpha : \alpha \in \Lambda)$ with $S_\alpha : L^1_{G_\beta}(G_\alpha) \to L^1_{G_\beta}(G_\alpha)$ for each $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$. Then the following statements (i) and (ii) are equivalent:*

(i) *an operator S has the form*

(1) *$S = p\hat{U}_a$ for some marked elements $a \in G := \prod_{\alpha\in\Lambda} G_\alpha$ and $p = \{p_\alpha : |p_\alpha| = 1\ \forall\alpha \in \Lambda\} \in \mathbf{F}^\Lambda$, that is*

(2) *$S_\alpha f_\alpha(x) = p_\alpha \hat{U}_{a_\beta} f_\alpha(x)$ for any $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$ and each $x \in G_\alpha$, where*

(3) *$\hat{U}_{g_\beta} f_\alpha(x) = f_\alpha(\theta^\beta_\alpha(g_\beta)x)$ for each $g_\beta \in G_\beta$ and $x \in G_\alpha$;*

(ii) *(4) S is a left meta-centralizer and*

(4) *$\|S_\alpha f_\alpha\| = \|f_\alpha\|$ for every $f_\alpha \in L^1_{G_\beta}(G_\alpha)$ and $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$.*

Proof. The $\mathbf{F}$-linear span of the set of all non-negative functions $f \in L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ is dense in $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$. Therefore, each bounded linear operator S_α can be written in the form $S_\alpha = S_{1,\alpha} + iS_{2,\alpha} = S^+_{1,\alpha} - S^-_{1,\alpha} + iS^+_{2,\alpha} - iS^-_{2,\alpha}$, where $S^+_{k,\alpha} f \ge 0$ and $S^-_{k,\alpha} f \ge 0$ for $k = 1, 2$ and each $f \in P_\alpha$, $S_{k,\alpha} = S^+_{k,\alpha} - S^-_{k,\alpha}$, where P_α denotes the cone of functions in $L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$ non-negative μ_α-almost everywhere on G_α.

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Certainly over the real field additives $S^{\pm}_{2,\alpha}$ vanish. In view of Theorem 11 [**19**], there exist $a^+_k \in G$ and $p^+_k = \{p^+_{k,\alpha} : p^+_{k,\alpha} > 0 \ \forall \alpha \in \Lambda\} \in \mathbf{R}^\Lambda$ such that $S^+_{k,\alpha} f_\alpha(x) = p^+_{k,\alpha} \hat{U}_{a^+_{k,\beta}} f_\alpha(x)$ and analogously for $S^-_{k,\alpha}$ for each $k = 1, 2$.

Suppose that $a^t_k \neq a^s_l$ for some $t, s \in \{+, -\}$ and $k, l \in \{1, 2\}$, then there exists $\alpha \in \Lambda$ such that $a^t_{k,\beta} \neq a^s_{l,\beta}$ with $\beta = \phi(\alpha)$. On the other hand, we have $S_{k,\alpha} f_\alpha = S^+_{k,\alpha} f_\alpha - S^-_{k,\alpha} f_\alpha = p^+_{k,\alpha} f_\alpha(\theta^\beta_\alpha(a^+_{k,\beta})x) - p^-_{k,\alpha} f_\alpha(\theta^\beta_\alpha(a^-_{k,\beta})x)$ for each $f_\alpha \in L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F})$, since $f_\alpha = [f^+_{1,\alpha} - f^-_{1\alpha}] + i[f^+_{2,\alpha} - f^-_{2,\alpha}]$, where $f^+_{k,\alpha}(x) = \max(f_{k,\alpha}(x), 0)$ for every $k = 1, 2$ and $x \in G_\alpha$, $f^+_{k,\alpha}, f^-_{k,\alpha} \in P_\alpha$. Then if U is an open subset in G_α such that $\theta^\beta_\alpha(a^s_{k,\beta})U \cap \theta^\beta_\alpha(a^t_{l,\beta})U = \emptyset$ for every $k, l = 1, 2$ and $t, s \in \{+, -\}$, then $\|S_\alpha \chi_U\| = \sum_{k=1}^2 \sum_{t\in\{+,-\}} (|p^t_{k,\alpha}| \|\hat{U}_{a^t_{k,\beta}} \chi_U\|)$. If the interior of the intersection $\cap_{k=1}^2 \cap_{t\in\{+,-\}} (\theta^\beta_\alpha(a^t_{k,\beta})U)$ is non-void, then $\|S_\alpha \chi_U\| < \sum_{k=1}^2 \sum_{t\in\{+,-\}} (|p^t_{k,\alpha}| \|\hat{U}_{a^t_{k,\beta}} \chi_U\|)$, since $\mu_\alpha(V) > 0$ for each open subset V in G_α, consequently, S_α is not an isometry.

Therefore, if S satisfies Conditions *ii*(4, 5), then $a^t_{k,\beta} = a^s_{l,\beta}$ for each $t, s \in \{+, -\}$ and $k, l \in \{1, 2\}$. Thus $(S_\alpha f_\alpha) = p_\alpha \hat{U}_{a_\beta} f_\alpha(x)$ for any $\alpha \in \Lambda$ and each $x \in G_\alpha$, where $p_\alpha = p^+_{1,\alpha} - p^-_{1,\alpha} + ip^+_{2,\alpha} - ip^-_{2,\alpha}$. Naturally, in the case $\mathbf{F} = \mathbf{R}$ the terms $p^{\pm}_2$ vanish. In view of Lemma 7 [**19**] $\hat{U}_a$ is the isometry. Since S preserves norms, then $|p_\alpha| = 1$ for each α.

Vice versa Conditions i(1–3) imply *ii*(4, 5) due to Lemma 7 [**19**].

LEMMA 11. *Let $\hat{U}_c$ be a left translation on $\mathcal{E}$ as in Section 10, let also $T : \mathcal{E} \to \mathcal{F}$ be an isomorphism of normed algebras such that $Tf = (T_\alpha f_\alpha : \alpha \in \Lambda)$, $T_\alpha : L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) \to L^1_{H_\beta}(H_\alpha, \lambda_\alpha, \mathbf{F})$ and $\|T_\alpha\| \leq 1$ for each α, where $\mathcal{F} = L^\infty(L^1_{H_\beta}(H_\alpha, \lambda_\alpha, \mathbf{F}) : \alpha < \beta \in \Lambda)$. If $\hat{K}_c = T\hat{U}_c T^{-1}$, then there exist mappings of groups $\xi : G \to H$ and $p : G \to \mathbf{F}^\Lambda$ such that*

(1) $\hat{K}_c = p_c \hat{V}_t$ for $t = \xi(c)$ and
(2) $p_c = \{p_{c,\alpha} : |p_{c,\alpha}| = 1 \ \forall \alpha \in \Lambda\} \in \mathbf{F}^\Lambda$, where $\hat{V}_d$ denotes the left translation operator on $\mathcal{F}$, $c \in G$.

Proof. We have $T(f \tilde{\star} u) = (Tf)\tilde{\star}(Tu)$ for each $u, f \in \mathcal{E}$ and $T^{-1}(g\tilde{\star}v) = (T^{-1}g)\tilde{\star}(T^{-1}v)$ for each $v, g \in \mathcal{F}$. One can take the approximate identity $\{q_{a,\beta} : a \in \Psi_\beta\}$ as in Section 8 and consider functions $s_{a,\beta} = T_\beta q_{a,\beta}$. The operator T is bijective and continuous from $\mathcal{E}$ onto $\mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ as linear normed spaces are complete. According to the Banach theorem 4.5.4.3 [**17**] (or see [**1**]) the inverse operator T^{-1} is also bounded. Due to Formulas 8(7, 8) there exists the adjoint operator $(\hat{K}_{c_\gamma})^*$ relative to the $\tilde{\star}$ multiplication for each $c \in G$ and $\gamma \in \Lambda$. For each $f, g \in \mathcal{F}$, $\gamma = \phi(\beta)$ and $\beta = \phi(\alpha)$ the limit exists

$$(\hat{K}_{c_\gamma} f_\beta)\tilde{\star} g_\alpha = f_\beta \tilde{\star}[(\hat{K}_{c_\gamma})^* g_\alpha] = \lim_a f_\beta \tilde{\star}\{s_{a,\beta}\tilde{\star}[(\hat{K}_{c_\gamma})^* g_\alpha]\}$$

$$= f_\beta \tilde{\star}\{\lim_a (\hat{K}_{c_\gamma} s_{a,\beta})\tilde{\star} g_\alpha\} = f_\beta \tilde{\star}\{\lim_a (T_\beta \hat{U}_{c_\gamma} T_\beta^{-1} T_\beta q_{a,\beta})\tilde{\star} g_\alpha\}$$

$$= f_\beta \tilde{\star}\{\lim_a (T_\beta \hat{U}_{c_\gamma} q_{a,\beta})\tilde{\star} g_\alpha\} \text{ and hence}$$

$$\|(\hat{K}_{c_\gamma} f_\beta)\tilde{\star} g_\alpha\| \leq$$

$$\overline{\lim}_a \|f_\beta \tilde{\star}([T_\beta \hat{U}_{c_\gamma} q_{a,\beta}]\tilde{\star} g_\alpha)\| \leq \|f_\beta\| \|T_\beta\| \|g_\alpha\| \ \overline{\lim}_a \|[\hat{U}_{c_\gamma} q_{a,\beta}]\tilde{\star}\| \leq \|f_\beta\| \|g_\alpha\|,$$

for each $f, g \in \mathcal{E}$, since $\|T\| \leq 1$. On the other hand, $\hat{K}_{c_\gamma^{-1}} = (\hat{K}_{c_\gamma})^{-1}$. Thus the inequalities $\|\hat{K}_{c_\gamma}\| \leq 1$ and $\|(\hat{K}_{c_\gamma})^{-1}\| \leq 1$ are satisfied for each $\gamma \in \Lambda$ and $c \in G$, consequently, $\hat{K}_c$ is the isometry for each $c \in G$.

Applying Theorem 10 we get the statement of this lemma.

LEMMA 12. *The mappings $(G, \tau_G^b) \ni c \to p_c \in (B^\Lambda, \tau_B^b)$ for each β and $(G, \tau_G^b) \ni c \mapsto \xi(c) \in (H, \tau_H^b)$ of Lemma 11 are continuous homomorphisms, where $B = \{x \in \mathbf{F} : |x| = 1\}$ is the multiplicative group, the product B^Λ is in the box topology τ_B^b, where τ_G^b denotes the box topology on G (see Section 9 [**19**]).*

Proof. These mappings are homomorphisms, since

$$p_{ch,\gamma} \hat{V}_{\xi_\gamma(c_\gamma h_\gamma)} = T_\beta \hat{U}_{c_\gamma h_\gamma} T_\beta^{-1} = T_\beta \hat{U}_{c_\gamma} T_\beta^{-1} T_\beta \hat{U}_{h_\gamma} T_\beta^{-1} = p_{c,\gamma} \hat{V}_{\xi_\gamma(c_\gamma)} p_{h,\gamma} \hat{V}_{\xi_\gamma(h_\gamma)},$$

for each $c, h \in G$, $\beta \in \Lambda$ with $\gamma = \phi(\beta)$, where $\xi(c) = \{\xi_\alpha(c_\alpha) : \alpha \in \Lambda\}$, $\xi_\alpha : G_\alpha \to H_\alpha$ for each $\alpha \in \Lambda$. The mapping ξ is bijective, since for $\xi(c) = e_H \in H$, where e_H is the neutral element in H, one gets $p_{c,\gamma} I_{\mathcal{F}} = T_\beta \hat{U}_{c_\gamma} T_\beta^{-1}$ and hence $\hat{U}_{c_\gamma} = p_{c,\gamma} I_{\mathcal{E}}$, where $I_{\mathcal{E}}$ denotes the unit operator on $\mathcal{E}$. Therefore, $c = e_G$ and hence $p_{c,\gamma} = 1$ for each γ.

Then the mapping $G \ni c \mapsto \hat{U}_c$ is continuous from G in the box topology τ_G^b and relative to the strong operator topology according to Proposition 10 [**19**], consequently, the mapping $H \ni t \mapsto \hat{V}_t$ is also continuous, since T and T^{-1} are bounded linear operators.

Then for each $\epsilon = (\epsilon_\alpha > 0 : \alpha \in \Lambda)$, there exists a neighbourhood $Y = \prod_{\alpha \in \Lambda} Y_\alpha$ of e_H in (H, τ_H^b) such that each Y_α is an (open) neighbourhood of the neutral element e_α in H_α for which $\epsilon_\alpha/2 < \lambda_\alpha(Y_\alpha) < \epsilon_\alpha$ for each $\alpha \in \Lambda$, since λ_α is the quasi-invariant borelian measure on H_α relative to the dense subgroup H_β and hence non-atomic. Moreover, if Z is an arbitrary neighbourhood of e_H in (H, τ_H^b), then there exists Y such that $YY^{-1} \subseteq Z$. Then the function $g = (g_\alpha = \chi_{Y_\alpha} : \alpha \in \Lambda)$ belongs to $\mathcal{F}$, where χ_{A_α} denotes the characteristic function of a subset A_α in H_α. Suppose that p is a marked element in B^Λ. Let $t \in H$ be such that

$$\|p_\beta g_\beta \tilde{\star}(\hat{V}_{t_\beta}^* g_\alpha) - g_\beta \tilde{\star} g_\alpha\| < [\lambda_\beta|_{Y_\beta} \tilde{\star} \lambda_\alpha](Y_\alpha), \text{ where}$$
$$[\lambda_\beta|_{Y_\beta} \tilde{\star} \lambda_\alpha](Y_\alpha) := \int_{Y_\beta} \int_{Y_\alpha} \lambda_\beta(dx_\beta) \lambda_\alpha(\theta_\alpha^\beta(x_\beta) dx_\alpha), \tag{1}$$

where $\theta_\alpha^\beta : H_\beta \hookrightarrow H_\alpha$ are embeddings (see Section 1). If $t_\beta \notin Z_\beta$, then $s_\beta Y_\beta$ and $s_\beta t_\beta Y_\beta$ are the disjoint subsets in the group H_β for each element s_β in H_β, consequently,

$$\|p_\beta g_\beta \tilde{\star}[\hat{V}_{t_\beta}^* g_\alpha] - g_\beta \tilde{\star} g_\alpha\| = \sup_{s_\beta \in H_\beta} \int_{H_\alpha} |p_\beta [\hat{V}_{s_\beta t_\beta} g_\beta] \tilde{\star} g_\alpha(x_\alpha) - [\hat{V}_{s_\beta} g_\beta] \tilde{\star} g_\alpha(x_\alpha)| \lambda_\alpha(dx_\alpha)$$
$$= \sup_{s_\beta \in H_\beta} \int_{H_\beta} \int_{H_\alpha} |p_\beta g_\beta(s_\beta t_\beta x_\beta) g_\alpha(\theta_\alpha^\beta(x_\beta) x_\alpha)| \lambda_\beta(dx_\beta) \lambda_\alpha(dx_\alpha)$$
$$+ \sup_{s_\beta \in H_\beta} \int_{H_\beta} \int_{H_\alpha} |g_\beta(s_\beta x_\beta) g_\alpha(\theta_\alpha^\beta(x_\beta) x_\alpha)| \lambda_\beta(dx_\beta) \lambda_\alpha(dx_\alpha) \geq [\lambda_\beta|_{Y_\beta} \tilde{\star} \lambda_\alpha](Y_\alpha).$$

Thus Inequality (1) implies that $t_\beta \in Z_\beta$. Hence, the mapping $p\hat{V}_{\xi_\beta(c_\beta)} \mapsto \xi_\beta(c_\beta) = t_\beta \in H_\beta$, with H_β in the topology τ_β, is continuous for each β, when linear operators $p\hat{V}$ are considered relative to the strong operator topology, since the set of all (μ_α-measurable) simple functions is dense in $L^1_{G_\beta}(G_\alpha)$. The mapping $c_\beta \mapsto \xi_\beta(c_\beta)$ is the

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composition of three mappings $c_\beta \mapsto \hat{U}_{c_\beta} \mapsto T_\alpha \hat{U}_{c_\beta} T_\alpha^{-1} = p_{c,\beta} \hat{V}_{\xi_\beta(c_\beta)} \mapsto \xi_\beta(c_\beta) = t_\beta$ which are continuous for each $\beta \in \Lambda$ as it was proved above, consequently, the mapping $\xi : (G, \tau_G^b) \to (H, \tau_H^b)$ is also continuous.

The mapping $c \mapsto p_c$ is continuous, since $c \mapsto p_c I$ is continuous as the composition of two uniformly bounded and continuous mappings $T\hat{U}_c T^{-1}$ and $\hat{K}_{\xi(c)}$.

LEMMA 13. *The mapping $\xi : G \to H$ is the homeomorphism of (G, τ_G^b) onto (H, τ_H^b).*

Proof. If $\{\xi_\beta(x_{\beta,b}) : b\}$ is a net converging to $y_\beta \in H_\beta$, where $x_{\beta,b} \in G_\beta$, then $\{\hat{V}_{\xi_\beta(x_{\beta,b})} : b\}$ converges to $\hat{V}_{y_\beta}$ in the strong operator topology. Therefore, $\{T_\alpha^{-1} \hat{V}_{\xi_\beta(x_{\beta,b})} T_\alpha : b\}$ converges to $T_\alpha^{-1} \hat{V}_{y_{\beta,b}} T_\alpha$. From Lemma 11 we have the equality

$$T_\alpha^{-1} \hat{V}_{\xi_\beta(x_{\beta,b})} T_\alpha = p_{x_b,\beta}^{-1} \hat{U}_{x_{\beta,b}},$$

hence, the net of operators $\{p_{x_b,\beta}^{-1} \hat{U}_{x_{\beta,b}} : b\}$ strongly converges to $p_\beta \hat{U}_{x_\beta}$ for some $p_\beta \in B$ and $x_\beta \in G_\beta$. Thus the equality

$$p_\beta T_\alpha \hat{U}_{x_\beta} T_\alpha^{-1} = \hat{V}_{y_\beta},$$

is fulfilled with $y_\beta = \xi_\beta(x_\beta)$ and $p_\beta = p_{x,\beta}^{-1}$ for each $\beta \in \Lambda$. This implies that $\xi_\beta(G_\beta)$ is closed in H_β for each β and hence $\xi(G)$ is closed in (H, τ_H^b).

The inverse operator T^{-1} is bounded (see Section 11). Then $T_\alpha^{-1} \hat{V}_{y_\beta} T_\alpha = (sT_\alpha)^{-1} \hat{V}_{y_\beta} (sT_\alpha)$ for each $s \in \mathbf{F} \setminus \{0\}$. Hence, without loss of generality we can consider that $0 < \|T_\alpha^{-1}\| \leq 1$ for each $\alpha \in \Lambda$. On the other hand, from the equality $T_\alpha^{-1} \hat{V}_{y_\beta} T_\alpha = p_{x,\beta}^{-1} \hat{U}_{x_\beta}$ with $x_\beta = \xi_\beta^{-1}(y_\beta)$ analogously to ξ in Section 12 the continuity of $\xi_\beta^{-1} : \xi_\beta(G_\beta) \to G_\beta$ follows.

Applying Lemmas 11 and 12 and the proof in this section above to $T^{-1} : \mathcal{F} \to \mathcal{E}$, we get that there exists a continuous bijective homomorphism $\eta : (H, \tau_H^b) \to (G, \tau_G^b)$ such that $\eta(H)$ is closed in (G, τ_G^b) and

(1) $\hat{Q}_y = r_y \hat{U}_t$ for $t = \eta(y)$ and

(2) $r_y = \{r_{y,\alpha} : |r_{y,\alpha}| = 1 \ \forall \alpha \in \Lambda\} \in \mathbf{F}^\Lambda$, where $\hat{Q}_y = T^{-1} \hat{V}_y T$ for each $y \in H$, $r : (G, \tau_G^b) \to B^\Lambda$ is a continuous homomorphism. The operators $\hat{K}_c$ and $\hat{Q}_y$ are the left meta-centralizers on $\mathcal{F}$ and $\mathcal{E}$ respectively for each $c \in G$ and $y \in H$. But from 11(1, 2) it follows that $\eta = \xi^{-1}$ and $p_{\eta(y)} = r_y^{-1}$ for each $y \in H$, since η and ξ are bijective homomorphisms. Therefore, Formulas (1, 2) and 11(1, 2) imply that $\eta(\xi(G)) = G$ and hence $\xi(G) = H$.

THEOREM 14. *Let $T : \mathcal{E} \to \mathcal{F}$ be an isomorphism of normed algebras such that $Tf = (T_\alpha f_\alpha : \alpha \in \Lambda)$, $T_\alpha : L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) \to L^1_{H_\beta}(H_\alpha, \lambda_\alpha, \mathbf{F})$ and $\|T_\alpha\| \leq 1$ for each α, where $\mathcal{F} = L^\infty(L^1_{H_\beta}(H_\alpha, \lambda_\alpha, \mathbf{F}) : \alpha < \beta \in \Lambda)$ (see Sections 11 and 12). Then a homeomorphism ξ of topological groups exists from (G, τ_G^b) onto (H, τ_H^b) and a continuous homomorphism $\psi : G \to B^\Lambda$ such that*

(1) $T\hat{U}_x T^{-1} = \psi(x^{-1}) \hat{V}_{\xi(x)}$ and

(2) $(Tf)_\alpha(\xi(x)) = \psi_\beta(x_\beta) f_\alpha(x_\alpha)$ for each $x \in G$, $f \in \mathcal{E}$ and $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$, where $\psi(x) = (\psi_\alpha(x_\alpha) : \alpha \in \Lambda)$, $\psi_\alpha : G_\alpha \to B$,

$$T_\alpha \hat{U}_{x_\beta} T_\alpha^{-1} = \psi_\beta\big(x_\beta^{-1}\big) \hat{V}_{\xi_\beta(x_\beta)}.$$

Moreover, T is an isometry.

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Proof. We define a homomorphism $\psi(x) = p_x^{-1}$, hence $\psi(x) = (\psi_\alpha(x_\alpha) = p_{x,\alpha}^{-1} : \alpha \in \Lambda\} \in B^\Lambda$, hence $\psi_\alpha : G_\alpha \to B$ is a character for each $\alpha \in \Lambda$. From Lemmas 11–13, Statement (1) of this theorem follows such that $\xi : (G, \tau_G^b) \to (H, \tau_H^b)$ and $\xi^{-1} : (H, \tau_H^b) \to (G, \tau_G^b)$ and $\psi : G \to B^\Lambda$ are continuous homomorphisms with $\xi(G) = H$.

If $S : \mathcal{E} \to \mathcal{F}$ is an isomorphism of normed algebras such that $Sf = (S_\alpha f_\alpha : \alpha \in \Lambda)$, $S_\alpha : L^1_{G_\beta}(G_\alpha, \mu_\alpha, \mathbf{F}) \to L^1_{H_\beta}(H_\alpha, \lambda_\alpha, \mathbf{F})$ and $\|S_\alpha\| \leq 1$ for each α such that S satisfies Equality (2).

$(Sf)_\alpha(\xi(x)) = \psi_\beta(x_\beta) f_\alpha(x_\alpha)$ for each $x \in G$ and $f \in \mathcal{E}$, then $(S^{-1}g)_\alpha(x) = \psi_\beta(x_\beta^{-1}) g_\alpha(\xi_\alpha(x_\alpha))$ for each $g \in \mathcal{F}$ and $x \in G$. Therefore, one infers that

$$\begin{aligned}(S_\alpha \hat{U}_{c_\beta} S_\alpha^{-1} g_\alpha)(\xi_\alpha(x_\alpha)) &= \psi_\beta(x_\beta)(\hat{U}_{c_\beta} S_\alpha^{-1} g_\alpha)(x_\alpha)\\ &= \psi_\beta(x_\beta)(S_\alpha^{-1} g_\alpha)(\theta_\alpha^\beta(c_\beta) x_\alpha) = \psi_\beta(x_\beta)\psi_\beta(x_\beta^{-1} c_\beta^{-1}) g_\alpha(\theta_\alpha^\beta(\xi_\beta(c_\beta))\xi_\alpha(x_\alpha))\\ &= \psi_\beta(c_\beta^{-1}) g_\alpha(\theta_\alpha^\beta(\xi_\beta(c_\beta))\xi_\alpha(x_\alpha)) = \psi_\beta(c_\beta^{-1})(\hat{U}_{\xi_\beta(c_\beta)} g_\alpha)(\xi_\alpha(x_\alpha)),\end{aligned}$$

consequently, $S_\alpha \hat{U}_{c_\beta} S_\alpha^{-1} = \psi_\beta(c_\beta^{-1}) \hat{U}_{\xi_\beta(c_\beta)}$ for each $c \in G$, $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$, where embeddings $H_\beta \hookrightarrow H_\alpha$ also are denoted by θ_α^β for the notation simplicity (see Section 1). This means that $S \hat{U}_c S^{-1} = T \hat{U}_c T^{-1}$ and hence

$$T_\alpha^{-1} S_\alpha \hat{U}_{c_\beta} = \hat{U}_{c_\beta} T_\alpha^{-1} S_\alpha, \tag{3}$$

for each $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$. In view of Lemmas 11–13 and the conditions of this theorem the linear operators T, T^{-1}, S and S^{-1} are continuous. Thus, the operator

$$T^{-1} S =: Y, \tag{4}$$

is the isomorphism of the algebra $\mathcal{E}$ onto itself commuting with all operators $\hat{U}_c$ such that Y and Y^{-1} are continuous. As in Section 13, it is sufficient to consider the case $0 < \|Y_\alpha\| \leq 1$ for each $\alpha \in \Lambda$, since $\hat{U}_{c_\beta} = Y_\alpha^{-1} \hat{U}_{c_\beta} Y_\alpha = (kY_\alpha)^{-1} \hat{U}_{c_\beta}(kY_\alpha)$ for every $k \in \mathbf{F} \setminus \{0\}$, $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$ and $c \in G$. Take $f, q \in \mathcal{E}$ and consider the left meta-centralizer A defined by a radonian measure $\nu_\alpha \in M_t(G_\alpha, \mathbf{F})$ such that

$$\nu_\alpha(dx_\alpha) = q_\alpha(x_\alpha)\mu_\alpha(dx_\alpha), \tag{5}$$

for each $\alpha \in \Lambda$, that is $Af = \nu \tilde{\star} f$. On the other hand,

$$(Af)_\alpha(x_\alpha) = \int_{G_\beta} q_\beta(y_\beta)[\hat{U}_{y_\beta} f_\alpha(x_\alpha)]\mu_\beta(dy_\beta), \tag{6}$$

that is relative to the strong operator topology

$$A_\alpha = \int_{G_\beta} q_\beta(y_\beta)\hat{U}_{y_\beta}\mu_\beta(dy_\beta), \tag{7}$$

for each $\alpha \in \Lambda$ with $\beta = \phi(\alpha)$, where $Af = (A_\alpha f_\alpha : \alpha \in \Lambda)$. In each Banach space $L^1_{G_\gamma}(G_\beta, \mu_\beta, \mathbf{F})$ the space of (μ_β-measurable) simple functions $\sum_{j=1}^n v_j \chi_{Z_j}$ is dense, where $v_j \in \mathbf{F}$ is a constant and Z_j is a μ_β-measurable subset in G_β for each $j = 1, \ldots, n$, $n \in \mathbf{N}$. Therefore, from Formulas (3–7) it follows that

$$YAf = Y(q \tilde{\star} f) = (Yq)\tilde{\star}(Yf) = AYf = q\tilde{\star}(Yf),$$

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consequently, $Yq = q$ for each $q \in \mathcal{E}$, since $f \in \mathcal{E}$ is arbitrary. Thus $Y = I_{\mathcal{E}}$ and hence $T = S$, where $I_{\mathcal{E}}$ denotes the unit operator on $\mathcal{E}$. From this Formula (2) follows. The last statement follows from Formulas (2) and 3(1).

15. Remark. The results of this paper can be used for further studies of non-locally compact group algebras, representations of groups, completions and extensions of groups, etc.

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