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On the relation of a distributive lattice to its lattice of ideals

Herbert S. Gaskill

In this note we examine the relationship of a distributive lattice to its lattice of ideals. Our main result is that a distributive lattice and its lattice of ideals share exactly the same collection of finite sublattices. In addition we give a related result characterizing those finite distributive lattices L which can be embedded in a lattice L' whenever they can be embedded in its lattice of ideals $T(L')$.

In this note our main result is the following: if L is a distributive lattice and $T(L)$ its lattice of ideals, then L and $T(L)$ have the same collection of finite substructures. In addition we give a characterization of those finite distributive lattices L for which if L' is any lattice and L is embeddable in $T(L')$ then L can be embedded in L' .

Preliminaries

In general we follow the notation of Grätzer [3]. By a *lattice* we mean a structure $\langle L; +, \cdot \rangle$ where $+$ and $\cdot$ are binary, associative, commutative, idempotent, and related by,

$$x + (y \cdot x) = x \text{ and } x(x+y) = x .$$

We often omit the $\cdot$. If $x + y = x$, then $xy = y$, and we then write $y \leq x$. By an *ideal* of a lattice L we mean a non-empty subset I of L

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such that if a and $b \in I$ and $c \leq a + b$ then $c \in I$. We will denote the collection of all ideals of L by $I(L)$. It is well known that the structure $T(L) = \langle I(L); +, \cap \rangle$ is a lattice where $\cap$ is set theoretic intersection and $+$ is defined by

$$I_1 + I_2 = \{a : a \in L \text{ and } \exists b_1 \in I_1, \exists b_2 \in I_2 \text{ and } a \leq b_1 + b_2\} .$$

If $I_0 \in I(L)$ and for some $a \in L$,

$$I_0 = \{b : b \in L \text{ and } b \leq a\} ,$$

then we say I_0 is the *principal* ideal generated by a. We will denote the principal ideal generated by a by $\bar{a}$. Lastly a lattice is distributive if it satisfies

$$x(y+z) = xy + xz .$$

1.

In this section we give a complete characterization of those finite distributive lattices L which satisfy the condition that whenever L is embeddable in $T(L')$ then L can be embedded in L'. We have termed such lattices *weakly transferable*. The problem of characterizing weakly transferable lattices was first raised by Grätzer in [4, p. 207], and at that time he pointed out the following:

LEMMA 1. *If L is any finite lattice and L has a point which is both join and meet reducible, then L is weakly transferable.*

We shall show that if L is a finite distributive lattice and no point of L is both join and meet reducible, then L is weakly transferable. In fact we show an even stronger result. We say that a finite lattice is *transferable* if whenever ϕ embeds L in $T(L')$ there is a ψ embedding L in L' such that $x\psi \in y\phi$ if and only if $x \leq y$. Thus the embedding of L in L' relates in a substantial way to the embedding of L in $T(L')$. For the remainder of this paper $L = \langle L; +, \cdot \rangle$ denotes a *fixed finite distributive lattice.*

THEOREM 1. *If no point of L is both join and meet reducible, then L is transferable.*

Before proceeding with the proof of this result we shall need some information about the structure of finite distributive lattices.

LEMMA 2 [1, p. 58]. *If* L *is a finite distributive lattice, then every element has a unique representation as a join of a join-irredundant set of join irreducibles.*

Using Lemma 2, for each $x \in L$, let J_x denote that unique join irredundant set of join irreducibles satisfying $\sum J_x = x$. As an immediate consequence of the lemma we obtain:

LEMMA 3. *If* ϕ *is any map of* L *into a lattice* L' *such that* ϕ *is order preserving on the set of join irreducibles and such that* $x\phi = \sum(J_x\phi)$ *for each* $x \in L$, *then* ϕ *is a join homomorphism.*

Proof. This is immediate from the fact that Lemma 2 tells us that every join irredundant set of join irreducibles sums to a unique element. For greater detail see [1].

Proof of Theorem 1. Let ϕ be the embedding of L in $T(L')$ where L' is any lattice such that L is embeddable in $T(L')$. Now to each $x \in L$ choose an $x_0 \in L'$ such that $x_0 \in y\phi$ if and only if $x \leq y$. Such choices are possible since ϕ is an embedding. Note that if $x\phi$ is principal then $x\phi = \overline{a}$ for some $a \in L$ and we may take $x_0 = a$. It is now clear that if ψ is a homomorphism of L into L' such that $x_0 \leq x\psi \in x\phi$ then ψ is one-to-one. Further it is easily seen that for each $x \in L$ we can choose a ψ_x such that ψ_x is defined exactly on the set of join irreducibles of L, ψ_x is order preserving, for each join irreducible y, $y_0 \leq y\psi_x \in y\phi$, and such that $x_0 \leq \sum(J_x\psi_x)$. We then define ψ^* by

$$y\psi^* = \begin{cases} \sum\{y\psi_x : x \in L\} & \text{if } y \text{ is join irreducible,} \\ \sum(J_y\psi^*) & \text{if } y \text{ is join reducible.} \end{cases}$$

It is clear that ψ^* is a join embedding and that $x\psi^* \in y\phi$ if and only if $x \leq y$. To complete the proof we define ψ by

$$x\psi = \prod (M_x\psi^*) ,$$

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where M_x is the unique meet irredundant meet representation of x as a meet of meet irreducibles given by the dual of Lemma 2. Since $x\psi^* \le x\psi \in x\phi$, we have that ψ is a meet isomorphism satisfying $x\psi \le y\phi$ if and only if $x \le y$. Now we assert that ψ is a lattice isomorphism. It is well known that in any distributive lattice, if $z \le x + y$ then there exists $x_1 \le x$ and $y_1 \le y$ such that $x_1 + y_1 = z$. Similarly the dual of this result is valid. Consider a, b, c and $d \in L$ such that $cd \le a + b$ where $a + b$ is a proper join and cd a proper meet. Since cd is not join reducible we must have $cd \le a$ or $cd \le b$. Since $a + b$ is not meet reducible we must have either $c \le a + b$ or $d \le a + b$. Now ψ is a meet isomorphism whence ψ preserves the valid inequality of $cd \le a$ and $cd \le b$. Without loss of generality assume $c \le a + b$. There are two cases.

Case 1. c is meet irreducible. Since $c\psi = c\psi^*$ for this case, we obtain

$$c\psi = c\psi^* \le a\psi^* + b\psi^* \le a\psi + b\psi ,$$

since ψ^* is a join isomorphism, which completes Case 1.

Case 2. c is meet reducible. For this case, c is join irreducible whence for some $x_0 \in J_{a+b}$, $c \le x_0$ by Lemma 2. Since ψ is an order isomorphism and $J_{a+b} \subseteq J_a \cup J_b$, we have

$$c\psi \le x_0\psi \le a\psi + b\psi ,$$

which completes Case 2.

It follows that for arbitrary a, b, c and $d \in L$, whenever $cd \le a + b$ then $(c\psi)(d\psi) \le a\psi + b\psi$. It is immediate that ψ is an isomorphism, and this completes the proof of Theorem 1.

Distributive lattices in which no point is both join and meet reducible have a particularly nice structure. This description was first obtained by Galvin and Jónsson [2] and for the sake of completeness we give this description. Given two non-empty subsets L_1 and L_2 of L, we will write $L_1 \le L_2$ if and only if $L_1 = L_2$ or for each $x \in L_1$ and $y \in L_2$, $x < y$. L is said to be linearly indecomposable if there do not

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exist L_1 and L_2 such that $L_1 < L_2$ and $L_1 \cup L_2 = L$. It is straightforward that L is the union of a unique finite linearly ordered family C_L of linearly indecomposable lattices.

THEOREM 2 (Galvin and Jónsson). *If no point of* L *is both join and meet reducible and* $L_1 \in C_L$ *then either* L_1 *is the* 1 *element lattice or* L_1 *is the* 8 *element boolean lattice or* L_1 *is a direct product of the* 2 *element chain with a finite chain having* 2 *or more elements.*

2.

In this section we show that if L' is a distributive lattice then L' and $T(L')$ have exactly the same collection of finite sublattices. For the remainder, let L' be a fixed infinite distributive lattice. Let L be a fixed finite distributive lattice with ϕ embedding L in $T(L')$. Further let ψ^* and ψ be obtained as in the proof of Theorem 1. With J_x and M_x as before, it is easily seen that if for each $x \in L$, $\sum(J_x\psi) = \prod(M_x\psi)$, then ψ is a lattice isomorphism. Further from the definition of ψ we have $\sum(J_x\psi) \le \prod(M_x\psi)$. Thus if ψ is not a lattice isomorphism then there is an x such that

$$\sum(J_x\psi) < \prod(M_x\psi) .$$

Such an x is clearly meet and join reducible.

LEMMA 4. *Let* $[L\psi]$ *be the lattice closure of* $L\psi$ *in* L'. *Then if* $y \in [L\psi] \sim L\psi$, *there is an* $x_y \in L$ *such that* $\sum\left(J_{x_y}\psi\right) \le y < \prod\left(M_{x_y}\psi\right)$.

Proof. We define sets $K_0, \ldots, K_n, \ldots$ as follows: $K_0 = L\psi$, K_{2i+1} is the join closure of K_{2i} and K_{2i+2} is the meet closure of K_{2i+1}. Now for some $n_0 \in \omega$, $[L\psi] = K_{n_0}$ since L' is distributive. Suppose that for each j such that $0 \le j < n$, if $j \in K_j \sim L\psi$ then the lemma is satisfied. Consider $y \in K_n \sim L\psi$.

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Case 1. $n = 2m + 1 \geq 1$. Now if $y \in K_n \sim K_{n-1}$, then there is a set of elements $J_y \subseteq K_{n-1}$ which is join irredundant and such that $\sum J_y = y$. Now for each $z \in J_y$ there is an $x_z \in L$ such that $\sum\left(J_{x_z}\psi\right) \leq z \leq \prod\left(M_{x_z}\psi\right)$. Let

$$x_y = \sum\left(\cup\left\{J_{x_z} : z \in J_y\right\}\right) .$$

Now $\sum J_{x_y} = \prod M_{x_y}$, whence for each $z \in J_y$ we have $z < \prod\left(M_{x_z}\psi\right) \leq \prod\left(M_{x_y}\psi\right)$, whence we conclude that

$$\sum\left(J_{x_y}\psi\right) \leq y < \prod\left(M_{x_y}\psi\right)$$

as desired. This completes Case 1.

Case 2. $n = 2m + 2 \geq 2$. The treatment of this case is similar to that of Case 1, and we omit the details.

Observe that Lemma 4 allows us to draw the conclusion that, if $x \in L$ is join irreducible, then $x\psi$ is join irreducible in $\langle [L\psi]; +, \cdot \rangle$.

LEMMA 5. *Let* a *be a fixed maximal member of* L *such that* $\sum(J_a\psi) < \prod(M_a\psi)$. *Then for all* $x \in L$ *either* $x \leq a$ *or* $a \leq x$.

Proof. As noted earlier, a must be both join and meet reducible. Suppose for the sake of contradiction that there is a $d \in L$ with $a \nleq d$ and $d \nleq a$. Now $\sum(J_a\psi) < \prod(M_a\psi)$ and $\sum(J_a\psi) \in [L\psi]$. Let

$$H = \{x : x \in L,\ x \text{ is meet irreducible and } x \nleq a\} .$$

Then for each $y \in L$ either $y \leq \prod H$ or $\prod H \leq y$. To see this we note that for any meet irreducible c , if $c \leq a$ then $c \leq \prod H$ by the dual of Lemma 2. Further by assumption $\prod H < a$. Now either $\prod H$ is join irreducible or there is a join irreducible c such that $\prod H < c < a$. Hence fix c such that c is join irreducible and

$\prod H \le c < a$. Then $c\psi = \sum(J_c\phi) = \prod M_c\psi$, whence

$$\prod (M_c\psi) < \sum(J_a\psi) < \prod (M_a\psi) .$$

For convenience let $\sum(J_a\psi) = b$ and $M_c = \{b_0, \ldots, b_{m-1}\}$. Since $a\psi^* \le b < \prod (M_a\psi) \in a\phi$, if for some b_i , $b_i\psi \le b$ then we can conclude that $b_i\psi \in a\phi$ contrary to hypothesis. Thus for each b_i , $b_i \nleq b$. Now by distributivity we obtain

$$b = b + \prod_{i\in m} (b_i\psi) = \prod_{c\in m} (b+b_i\psi) .$$

Fix an i and let $c_i \in L$ be such that

$$\sum\left(J_{c_i}\psi\right) \le (b+b_i\psi) \le \prod \left(M_{c_i}\psi\right) .$$

We immediately conclude that

$$a\psi^* \le b \le \prod \left(M_{c_i}\psi\right) \in c_i\phi ,$$

whence $a \le c_i$ by definition of ψ^* . By the maximality of a we obtain $c_i\psi = b + b_i\psi$ whence

$$b < a\psi \le \prod_{i\in m} (c_i\psi) = b$$

which is absurd. Thus for all $x \in L$ either $a \le x$ or $x \le a$.

LEMMA 6. *Let* L *and* L' *be as before with* ϕ *embedding* L *in* $T(L)$. *Let* 1 *be the greatest element of* L . *If* $a \in 1\phi$ *such that for all* $y \in L \sim \{1\}$, $a \nleq y\phi$ *then the map* ϕ' *defined by* $x\phi' = \bar{a} \cap x\phi$ *is an isomorphism.*

Proof. That ϕ' is a meet isomorphism is obvious. Let $b \in L$. We must show that if b is join reducible with $J_b = \{b_0, \ldots, b_{m-1}\}$ and $c \in b\phi'$ then we can choose $c_i \in b_i\phi'$ such that $\sum c_i \ge c$. But this is obvious, whence ϕ' is an isomorphism.

THEOREM 3. *Let* L *and* L' *be as before. If* L *can be embedded*

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in $T(L')$ *then* L *can be embedded in* L

Proof. Let $b_0, \ldots, b_{m-1}$ be a list in descending order of those elements of L excluding 0 and 1 which satisfy for all $x \in L$, $x \le y$ or $y \le x$. Let ψ be obtained as in Theorem 1. Then if ψ is not the desired isomorphism it fails for a maximal b_{j_0} where for each $x \in L$ if $b_{j_0} < x$ then $\sum(J_x\psi) = \prod(M_x\psi)$. We define a new isomorphism $\phi_1 : L \to I(L')$ by:

$$x\phi = \begin{cases} \overline{x\psi} & \text{if } b_{j_0} \le x , \\ \overline{x_{j_0}\psi} \cap x\phi & \text{if } x < b_{j_0} . \end{cases}$$

By procedures outlined in the proof of Theorem 1 we obtain a ψ_1 such that for all $x \in L$, $x\psi_1 \in y\phi_1$ if and only if $x \le y$. Further we may also require that $x\psi_1 = x\psi$ for each x such that $b_{j_0} \le x$, and in addition that $x\psi_1 \le y\phi$ if and only if $x \le y$. Since if ψ_1 is not the desired isomorphism, its failure occurs at $b_{j_1} < b_{j_0}$, we are done since the set of b_j's is finite. This concludes the proof and the note.

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Department of Mathematics,
Simon Fraser University,
Burnaby,
British Columbia,
Canada.