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MAXIMUM AVERAGE DISTANCE IN COMPLEX FINITE DIMENSIONAL NORMED SPACES

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A number $r > 0$ is called a *rendezvous number* for a metric space (M, d) if for any $n \in \mathbb{N}$ and any $x_1, \ldots x_n \in M$, there exists $x \in M$ such that $(1/n) \sum_{i=1}^{n} d(x_i, x) = r$. A rendezvous number for a normed space X is a rendezvous number for its unit sphere. A surprising theorem due to O. Gross states that every finite dimensional normed space has one and only one average number, denoted by $r(X)$. In a recent paper, A. Hinrichs solves a conjecture raised by R. Wolf. He proves that $r(X) \leqslant r(\ell_1^n) = 2 - 1/n$ for any n-dimensional real normed space. In this paper, we prove the analogous inequality in the complex case for $n \geqslant 3$.

1. INTRODUCTION

A number $r > 0$ is called a *rendezvous number* for a metric space (M, d) if for any $n \in \mathbb{N}$ and any $x_1, \ldots x_n \in M$, there exists $x \in M$ such that

$$\frac{1}{n} \sum_{i=1}^{n} d(x_i, x) = r.$$

In 1964, Gross [4] proved that every compact connected metric space has one and only one rendezvous number. In this case, the unique rendezvous number is denoted by $r(M, d)$, and it is said that (M, d) has the *average distance property.* The general inequalities $D/2 \leqslant r \leqslant D$ can be easily checked for any rendezvous number r of a metric space with diameter D. Moreover, for a compact metric space, the second inequality is $r < D$ (see [10]).

Consider a normed space X. It is known (see [2] or [13]) that the unit ball of X has the average distance property, with 1 as the unique rendezvous number. A much more interesting case is the unit sphere $S(X)$ of X. If X is a finite dimensional normed space, a direct application of Gross's theorem implies that $S(X)$ has the average distance property, and its rendezvous number, called the rendezvous number of X, is

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denoted simply by $r(X)$. In this case we also know that $1 < r(X) < 2$ (see [15]). Calculations for these numbers in classical spaces have been carried out:

(1) $r(\ell_1^n) = 2 - 1/n$; $r(\ell_\infty^n) = 3/2$ (see [13])
(2) $r(\ell_2^n) = \bigl(2^{n-2}\Gamma(n/2)^2\bigr)/\bigl(\sqrt{\pi}\Gamma(2n-1/2)\bigr)$ $(\to \sqrt{2}$ as $n \to \infty)$ (see [8])
(3) $r(\ell_p^n) \to 2^{1/p}$ as $n \to \infty$ (see [7])

Since $X \mapsto r(X)$ is continuous on the Minkowski compactum of normed spaces of fixed dimension n (see [1]), it follows that there is an n-dimensional normed space X_0 such that $r(X) \leqslant r(X_0) < 2$ for any n-dimensional normed space X. It was conjectured by Wolf in [13] that the maximum, for $n \geqslant 2$, is attained for ℓ_1^n. Thus the conjecture can be written

$$r(X) \leqslant 2 - \frac{1}{n}$$

for any n-dimensional normed space X. The same author proved the inequality for $n = 2$ [13], for any X with a 1-unconditional basis ([14]) and for any X isometrically isomorphic to a subspace of $L^1[0,1]$ ([17]). Moreover, he proved that equality holds in these three cases if and only if X is isometrically isomorphic to ℓ_1^n. A general upper bound

$$r(X) \leqslant 2 - \frac{1}{2 + (n-1)2^{n-1}}$$

was proved in [1]. The conjecture was finally solved positively by Hinrichs in [6], using properties of the John ellipsoid.

All the previous results are related to real spaces. In [3], the values for some complex spaces are computed. In particular, in that paper it is shown that

$$r\bigl(\ell_1^n(\mathbb{C})\bigr) = 1 - \frac{1}{n} + \frac{1}{2\pi}\int_0^{2\pi} |e^{i\theta} - 1/n| \, d\theta$$

(where $|\cdot|$ denotes the complex modulus). The goal of this article is to show the inequality

$$r(X) \leqslant r\bigl(\ell_1^n(\mathbb{C})\bigr)$$

for any n-dimensional complex normed space X. This will be proved for $n \geqslant 3$, using the same techniques developed in [6], but with more elaborate computations. The inequality should hold for $n = 2$, and probably the same computations should work, sharped in a smart way.

For more information about generalisations of Gross's theorem, and some properties of rendezvous numbers of finite dimensional normed spaces, we refer the reader to [5, 9, 12, 16, 18]. A survey of contributions to this topic is given in [2].

Given a normed space X, we denote by B and S its unit ball and its unit sphere respectively By B_2^n we denote the Euclidean unit ball in $\mathbb{C}^n$.

2. Previous results

We shall prove the following result.

Theorem 1. *Let X be a complex n-dimensional normed space, with $n \geqslant 3$. Then $r(X) \leqslant r\big(\ell_1^n(\mathbb{C})\big)$.*

In the proof, we use properties of the John ellipsoid. We recall briefly the properties we shall use (see [**11**] for proofs).

Given a complex n-dimensional normed space $X = \big(\mathbb{C}^n, \|\cdot\|\big)$, there is a unique ellipsoid of maximal volume contained in B. By an affine transformation, we may assume that this ellipsoid is the Euclidean ball $\{x \in \mathbb{C}^n : |x| \leqslant 1\}$. For $x = (x_1, \dots x_n) \in \mathbb{C}^n$, $|x|$ will denote the standard Euclidean norm $|x| = \sqrt{(x,x)}$, where (x,y) denotes the complex scalar product. Then there exist m contact points $v_1, \dots v_m \in \mathbb{C}^n$ and real scalars $c_1, \dots c_m > 0$ satisfying

$$\|v_k\| = |v_k| = 1 \text{ for } k = 1, \dots n; \qquad \sum_{k=1}^{m} c_k = n,$$

$$x = \sum_{k=1}^{m} c_k (x, v_k) v_k, \text{ or equivalently } |x|^2 = \sum_{k=1}^{m} c_k \big|(x, v_k)\big|^2 \text{ for all } x \in X$$

$$\big|(x, v_k)\big| \leqslant 1 \text{ for all } x \in B,$$

$$\|x\| \leqslant |x| \leqslant \sqrt{n}\|x\| \text{ for all } x \in X.$$

In order to prove

$$r(X) \leqslant r\big(\ell_1^n(\mathbb{C})\big) = 1 - \frac{1}{n} + \frac{1}{2\pi}\int_0^{2\pi} \left|e^{i\theta} - \frac{1}{n}\right| d\theta,$$

we shall need some properties of the function $f(t) = 1/(2\pi)\int_0^{2\pi} \left|e^{i\theta} - t\right| d\theta$ which we state in the following two lemmas. The first one is just a verification.

Lemma 1. *The function $f : \mathbb{R} \to \mathbb{R}$ is convex, even, 1-Lipschitz and increasing in $[0, +\infty)$. Moreover*

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = 1/2.$$

Lemma 2. *The function $g(t) = 1 - t + f(t)$ is decreasing in $[0, 1]$.*

Proof: Let $t_1 < t_2$. The inequality $g(t_1) \geqslant g(t_2)$ is equivalent to $f(t_2) - f(t_1) \leqslant t_2 - t_1$, which is true since f is 1-Lipschitz. ∎

Remarks. (1) Lemma 1 implies the inequality $r(\ell_1^n) < r\big(\ell_1^n(\mathbb{C})\big)$, since $f(1/n) > f(0) = 1$. Moreover, asymptotically $r\big(\ell_1^n(\mathbb{C})\big) = r(\ell_1^n) + 1/(4n^2) + o(n^{-3})$.

(2) A consequence of Lemma 2 is that the sequence $r\big(\ell_1^n(\mathbb{C})\big) = g(1/n)$ is increasing (its limit equals 2).

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3. Proof of Theorem 1

Theorem 1 will be deduced from the following inequality:

$$\frac{1}{n}\sum_k c_k \frac{1}{2\pi}\int_0^{2\pi} \|v_i - e^{i\theta}x\|\,d\theta \leqslant r\big(\ell_1^n(\mathbb{C})\big) \tag{1}$$

for any $x \in S$. Fix $x \in S$, and let $r = |x|$. We know that $1 \leqslant r \leqslant \sqrt{n}$. Let P be the ortogonal projection onto $\operatorname{span}_{\mathbb{C}}\{x\}$, namely $Pz = r^{-2}(z,x)x$. Let Q be the complementary projection, $Qz = z - Pz$. We may assume, changing v_k by $e^{-i\alpha_k}v_k$ if neccesary, that $(x, Pv_k) = |x|\,|Pv_k|$. In this case, set

$$t_k = (x, Pv_k) = (x, v_k) = |x|\,|Pv_k| \geqslant 0.$$

We shall use the following properties of these numbers, which can be easily checked (see [**6**, Lemma 2 and Lemma 6] for proofs).

Lemma 3. *For any $k = 1, \dots m$, $0 \leqslant t_k \leqslant 1$. We also have $\sum_{k=1}^{m} c_k t_k^2 = r^2$.*

Consider K be the convex hull of $B_2^n \cup \{e^{i\theta}x : \theta \in [0, 2\pi]\}$. It is clearly a ball in $\mathbb{C}^n$, whose norm is given by the expression:

$$|||z||| = \begin{cases} |z|, & \text{if } |Qz|^2 \geqslant (r^2-1)|Pz|^2 \\ \frac{1}{r}\big(|Pz| + \sqrt{r^2-1}\,|Qz|\big), & \text{if } |Qz|^2 \leqslant (r^2-1)|Pz|^2 \end{cases}$$

This is shown by reducing it to the real case proved in [**6**]. Since $B_2^n \subset K \subset B$, we clearly have $\|z\| \leqslant |||z||| \leqslant |z|$. Therefore, in order to prove (1), it is enough to prove

$$\frac{1}{n}\sum_k c_k \frac{1}{2\pi}\int_0^{2\pi} |||v_k - e^{i\theta}x|||\,d\theta \leqslant r\big(\ell_1^n(\mathbb{C})\big). \tag{2}$$

To estimate the above integrals, we have to evaluate the Euclidean norms of $P(v_k - e^{i\theta}x) = Pv_k - e^{i\theta}x$ and $Q(v_k - e^{i\theta}x) = Qv_k$:

$$\big|P(v_k - e^{i\theta}x)\big|^2 = |Pv_k - e^{i\theta}x|^2 = \frac{t_k^2}{r^2} + r^2 - 2\Re(Pv_k, e^{i\theta}x) = \frac{t_k^2}{r^2} + r^2 - 2t_k\cos\theta$$

$$\big|Q(v_k - e^{i\theta}x)\big|^2 = |Qv_k|^2 = 1 - |Pv_k|^2 = 1 - \frac{t_k^2}{r^2}.$$

We shall use the following two lemmas. Let γ be the number $(1+\sqrt{5})/2$.

Lemma 4. *If $r^2 \geqslant 2$ then*

$$\left|Q\left(v_k - e^{i\theta}x\right)\right|^2 \leqslant \left(r^2 - 1\right)\left|P\left(v_k - e^{i\theta}x\right)\right|^2$$

for any $1 \leqslant k \leqslant m$ and any $\theta \in [0, 2\pi]$.

Lemma 5. *If $r^2 \geqslant \gamma$ then*

$$\left|Q\left(v_k - e^{i\theta}x\right)\right|^2 \leqslant \left(r^2 - 1\right)\left|P\left(v_k - e^{i\theta}x\right)\right|^2$$

for any $1 \leqslant k \leqslant m$ and any $\theta \in [\pi/2, 3\pi/2]$.

Accordingly, the proof will be divided into three cases. In the most important one, the first case, computations also work for $n = 2$. In the other two cases they are valid only for $n \geqslant 3$, but probably inequality (2) is also true for $n = 2$.

Case 1. $r^2 \geqslant 2$. Lemma 4 implies

$$\frac{1}{2\pi}\int_0^{2\pi} |||v_k - e^{i\theta}x|||\, d\theta = \frac{1}{2\pi r}\int_0^{2\pi} |Pv_k - e^{i\theta}x|\, d\theta + \sqrt{1 - \frac{1}{r^2}}\sqrt{1 - \frac{t_k^2}{r^2}},$$

for any $1 \leqslant k \leqslant m$. The vectors Pv_k and $e^{i\theta}x$ are in the same 1-dimensional vector space, $\text{span}_{\mathbb{C}}\{x\}$. Hence

$$\frac{1}{2\pi r}\int_0^{2\pi} |Pv_k - e^{i\theta}x|\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{Pv_k}{r} - e^{i\theta}\frac{x}{r}\right| d\theta = f\left(|Pv_k/r|\right) = f\left(t_k/r^2\right).$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} |||v_k - e^{i\theta}x|||\, d\theta = h_r\left(t_k^2/r^4\right) \tag{3}$$

where

$$h_r(t) = f\left(\sqrt{t}\right) + \sqrt{1 - \frac{1}{r^2}}\sqrt{1 - tr^2}.$$

The following lemma does the job.

Lemma 6. *For $r^2 \geqslant 2$, the function $h_r : [0, 1/r^2] \to \mathbb{R}$ is concave.*

Proof: We have to show that $h_r'(t) = 1\left(2\sqrt{t}\right)f'\left(\sqrt{t}\right) - \left(r\sqrt{r^2 - 1}/2\sqrt{1 - tr^2}\right)$ is a decreasing function in $\left(0, 1/r^2\right)$. Letting $t = s^2$, we have to prove that

$$s \in (0, 1/r) \mapsto \frac{1}{s}f'(s) - \frac{r\sqrt{r^2 - 1}}{\sqrt{1 - r^2 s^2}}$$

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is decreasing. Differentiation with respect to s and symplification give the following inequality to prove:

$$sf''(s) - f'(s) \leqslant r^3\sqrt{r^2-1}\frac{s^3}{(1-r^2s^2)^{3/2}} \quad 0 < s < 1/r < 1/\sqrt{2}.$$

In order to get this inequality, we shall use repeatedly the following elementary property: given two derivable functions $F, G : [a,b] \to \mathbb{R}$ such that $F(a) = G(a)$ and $F'(x) \leqslant G'(x)$ for all $x \in [a,b]$, then $F \leqslant G$ in $[a,b]$.

Both sides of the inequality above are null for $s = 0$, so what is left to show is that

$$f'''(s) \leqslant 3r^3\sqrt{r^2-1}\frac{s}{(1-r^2s^2)^{5/2}}.$$

Differentiating under the integral sign gives

$$f'''(s) = \frac{-3}{2\pi}\int_0^{2\pi}\frac{(s+\cos\theta)\sin^2\theta}{(1+s^2+2s\cos\theta)^{5/2}}\,d\theta$$

and hence $f'''(0) = 0$. Therefore, both sides of the inequality are equal (null) for $s = 0$, and using the property again, we are reduced to proving

$$f^{iv}(s) \leqslant 3r^3\sqrt{r^2-1}\frac{1+4r^2s^2}{(1-r^2s^2)^{7/2}}.$$

The function $r \in (\sqrt{2}, 1/s) \mapsto 3r^3\sqrt{r^2-1}\left(1 + 4r^2s^2/(1-r^2s^2)^{7/2}\right)$ is increasing, and so the second term in the inequality is greater than or equal to

$$6\sqrt{2}\frac{1+8s^2}{(1-2s^2)^{7/2}};$$

and thus it remains to prove that

$$f^{iv}(s) \leqslant 6\sqrt{2}\frac{1+8s^2}{(1-2s^2)^{7/2}}, \quad s \in \left(0, 1/\sqrt{2}\right).$$

Differentiation again under the integral sign shows that

$$f^{iv}(s) = \frac{6}{\pi}\int_0^{2\pi}\frac{\sin^2\theta}{(1+s^2+2s\cos\theta)^{5/2}}\,d\theta - \frac{15}{2\pi}\int_0^{2\pi}\frac{\sin^4\theta}{(1+s^2+2s\cos\theta)^{7/2}}\,d\theta.$$

The second term in the right side of the inequality is negative, so

$$f^{iv}(s) \leqslant \frac{6}{\pi}\int_0^{2\pi}\frac{\sin^2\theta}{(1+s^2+2s\cos\theta)^{5/2}}\,d\theta$$

$$= \frac{12}{\pi}\int_0^{\pi} \frac{\sin^2\theta}{(1+s^2+2s\cos\theta)^{5/2}}\,d\theta$$
$$\leqslant \frac{12}{\pi}\int_0^{\pi} \frac{\sin\theta}{(1+s^2+2s\cos\theta)^{5/2}}\,d\theta$$
$$= \frac{8}{\pi}\frac{3+s^2}{(1-s^2)^3}.$$

Finally, the proof is completed by showing that the latter is less than or equal to

$$6\sqrt{2}\frac{1+8s^2}{(1-2s^2)^{7/2}}.$$

□

To deduce inequality (2), average (3) and use Lemma 6 to obtain

$$\frac{1}{n}\sum_k c_k \frac{1}{2\pi}\int_0^{2\pi} |||v_k - e^{i\theta}x|||\,d\theta = \frac{1}{n}\sum_k c_k h_r\left(t_k^2/r^4\right) \leqslant h_r\left(\frac{1}{n}\sum_k c_k t_k^2/r^4\right).$$

Using Lemma 3, we have

$$= h_r\left(1/nr^2\right) = f\left(1/r\sqrt{n}\right) + \sqrt{1-\frac{1}{r^2}}\sqrt{1-\frac{1}{n}} \leqslant f\left(1/r\sqrt{n}\right) + 1 - \frac{1}{r\sqrt{n}} = g\left(1/r\sqrt{n}\right).$$

Finally, Lemma 2 and the inequality $r \leqslant \sqrt{n}$ yield

$$\leqslant g(1/n) = (1-1/n) + f(1/n) = r\left(\ell_1^n(\mathbb{C})\right)$$

as desired.

CASE 2. $\gamma \leqslant r^2 < 2$.

By Lemma 5, for $\theta \in [\pi/2, 3\pi/2]$,

$$|||v_k - e^{i\theta}x||| = \frac{1}{r}\left[|Pv_k - e^{i\theta}x| + \sqrt{r^2-1}|Qv_k|\right]$$
$$= \frac{1}{r}\sqrt{r^2 + \frac{t_k^2}{r^2} - 2t_k\cos\theta} + \sqrt{1-\frac{1}{r^2}}\sqrt{1-\frac{t_k^2}{r^2}}$$

In an other case,

$$|||v_k - e^{i\theta}x||| \leqslant |v_k - e^{i\theta}x| = \sqrt{1+r^2-2t_k\cos\theta}.$$

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Set $I = [\pi/2, 3\pi/2]$ and $I^c = [0, 2\pi] \setminus I$. Then

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} |||v_k - e^{i\theta}x|||\,d\theta \\
&\quad\leqslant \frac{1}{2\pi}\int_{I^c}\sqrt{1+r^2-2t_k\cos\theta}\,d\theta + \frac{1}{2\pi}\int_I \sqrt{1+\frac{t_k^2}{r^4}-2\frac{t_k}{r^2}\cos\theta}\,d\theta \\
&\qquad\qquad + \frac{1}{2}\sqrt{1-\frac{1}{r^2}}\sqrt{1-\frac{t_k^2}{r^2}} \\
&\quad= \frac{1}{2\pi}\int_{I^c}\left(\sqrt{1+r^2-2t_k\cos\theta}+\sqrt{1+\frac{t_k^2}{r^4}+2\frac{t_k}{r^2}\cos\theta}\right)d\theta + \frac{1}{2}\sqrt{1-\frac{1}{r^2}}\sqrt{1-\frac{t_k^2}{r^2}}.
\end{aligned}$$

Now use $\|\cdot\|_{L^1} \leqslant \|\cdot\|_{L^2}$ in both integrals for the probability measure $(d\theta)/\pi$ to get

$$\begin{aligned}
&\frac{1}{2}\left(\sqrt{\frac{1}{\pi}\int_{I^c}(1+r^2-2t_k\cos\theta)\,d\theta}+\sqrt{\frac{1}{\pi}\int_{I^c}\left(1+\frac{t_k^2}{r^4}+\frac{2t_k}{r^2}\cos\theta\right)d\theta}\right) \\
&\qquad\qquad = \frac{1}{2}\left(\sqrt{1+r^2-\frac{4t_k}{\pi}}+\sqrt{1+\frac{t_k^2}{r^4}+\frac{4t_k}{\pi r^2}}\right).
\end{aligned}$$

Applying the inequality $\sqrt{a+x} \leqslant \sqrt{a} + (x/2\sqrt{a})$ in both square roots we obtain that the latter expression is at most

$$\frac{\sqrt{1+r^2}+1}{2} - \frac{t}{\pi\sqrt{1+r^2}} + \frac{t^2}{4r^4} + \frac{t}{\pi r^2}$$

The expression above is a convex parabola as a function of $t \in [0,1]$, and so its maximum is attained at one of the extreme points of the interval. For $t = 0$, it equals

$$\frac{\sqrt{1+r^2}+1}{2} \leqslant \frac{\sqrt{3}+1}{2},$$

and for $t = 1$,

$$\frac{\sqrt{1+r^2}+1}{2} + \frac{1}{4r^4} - \frac{1}{\pi}\left(\frac{1}{\sqrt{1+r^2}} - \frac{1}{r^2}\right) \tag{4}$$

Let $s = r^2$. The function

$$s \in [\gamma, 2] \mapsto \frac{\sqrt{1+s}+1}{2} + \frac{1}{4s^2} - \frac{1}{\pi}\left(\frac{1}{\sqrt{1+s}} - \frac{1}{s}\right) \tag{5}$$

is convex. Computing at the extreme points of the interval yields

$$\begin{aligned}
s = \gamma &\mapsto 1.40451\ldots \\
s = 2 &\mapsto 1.4039\ldots
\end{aligned}$$

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and therefore the maximum is attained at $s = \gamma$, and (4) is (recall that $\gamma^2 = \gamma + 1$) $\leqslant \gamma + 1/2 + 1/(4\gamma^2)$. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} |||v_k - e^{i\theta}x|||\,d\theta \leqslant \frac{\gamma+1}{2} + \frac{1}{4\gamma^2} + \frac{1}{2}\sqrt{1-\frac{1}{r^2}}\sqrt{1-\frac{t_k^2}{r^2}}.$$

Multiplication by c_k/n, summation over $k = 1, \dots m$, and Cauchy-Schwarz inequality give

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{m}\frac{c_k}{2\pi}\int_0^{2\pi} |||v_k - e^{i\theta}x|||\,d\theta &\leqslant \frac{\gamma+1}{2} + \frac{1}{4\gamma^2} + \frac{1}{2}\sqrt{1-\frac{1}{r^2}}\sqrt{1-\frac{1}{n}} \\ &\leqslant \frac{\gamma+1}{2} + \frac{1}{4\gamma^2} + \frac{1}{2\sqrt{2}}\sqrt{1-\frac{1}{n}}.\end{aligned}$$

For $n = 3$, the latter is $(\gamma+1)/2 + 1/(4\gamma^2) + 1/(2\sqrt{3}) \simeq 1.69318\ldots < r(\ell_1^3(\mathbb{C})) \simeq 1.69464\ldots.$ For $n \geqslant 4$, the bound above is $< (\gamma+1)/2 + 1(4\gamma^2) + 1/(2\sqrt{2}) \simeq 1.75806\ldots < r(\ell_1^4(\mathbb{C})) \simeq 1.76569\ldots \leqslant r(\ell_1^n(\mathbb{C}))$. For $n = 2$, the estimate equals to $(\gamma+1)/2 + 1/(4\gamma^2) + 1/4 \simeq 1.65451\ldots$, but $r(\ell_1^2(\mathbb{C})) \simeq 1.56354\ldots$, so this calculation does not prove (2) in the case $n = 2$.

The only point remaining is the convexity of the function given by (5). Differentiating twice, we get

$$\begin{aligned}&-(1+s)^{-3/2}/8 + \frac{3}{2}s^{-4} - \frac{3}{4\pi}(1+s)^{-5/2} + \frac{2}{\pi}s^{-3} \\ &\qquad > -\frac{1}{8}(1+\gamma)^{-3/2} + \frac{3}{2}2^{-4} - \frac{3}{4\pi}(1+\gamma)^{-5/2} + \frac{2}{\pi}2^{-3} \simeq 0.20187\ldots > 0.\end{aligned}$$

Case 3. $1 \leqslant r^2 \leqslant \gamma$.

The inequality $|||\cdot||| \leqslant |\cdot|$ gives

$$\frac{1}{2\pi}\int_0^{2\pi} |||v_k - e^{i\theta}x|||\,d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} |v_k - e^{i\theta}x|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{1+r^2-2t_k\cos\theta}\,d\theta.$$

Finally, apply again the inequality $\|\cdot\|_{L^1} \leqslant \|\cdot\|_{L^2}$ for the probability measure $(1/2\pi)d\theta$ to obtain

$$\leqslant \left(\frac{1}{2\pi}\int_0^{2\pi}\left(1+r^2-2t_k\cos\theta\right)d\theta\right)^{1/2} = \sqrt{1+r^2} \leqslant \sqrt{1+\gamma} = \gamma.$$

Since $\gamma \simeq 1.61803\ldots$ and $r(\ell_1^3(\mathbb{C})) \simeq 1.69464\ldots$, we have the inequality $\gamma < r(\ell_1^n(\mathbb{C}))$ for $n \geqslant 3$. Again the argument does not work for $n = 2$.

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