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A LOCAL ERGODIC THEOREM FOR MULTIPARAMETER SUPERADDITIVE PROCESSES

BY

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Dedicated to Professor Shigeru Tsurumi on his 60th birthday

ABSTRACT. In this paper a local ergodic theorem is proved for positive (multiparameter) superadditive processes with respect to (multiparameter) semiflows of nonsingular point transformations on a σ-finite measure space. The theorem obtained here generalizes Akcoglu-Krengel's [2] local ergodic theorem for superadditive processes with respect to semiflows of measure preserving transformations. The proof is a refinement of Akcoglu-Krengel's argument in [2]. Also, ideas of Feyel [3] and the author [4], [5] are used.

1. **Preliminaries and the Theorem**. Let k be a fixed positive integer and R^k_+ denote the additive semigroup of all vectors $t = (t_1, \ldots, t_k)$ with $t_i > 0$. If $a = (a_i)$ and $b = (b_i)$ are two vectors with $0 \leq a_i < b_i$, $(a, b]$ denotes the set $\{t \in R^k_+ : a_i < t_i \leq b_i\}$, and $\mathcal{J}$ denotes the class of sets of this form. If r is a positive real number, we write $J_r = (0, re]$, where 0 and e are the vectors with all coordinates equal to zero or one, respectively.

Let $\theta = \{\theta_t : t \in R^k_+\}$ be a k-parameter measurable semiflow of nonsingular point transformations on a σ-finite measure space $(X, \mathcal{F}, \mu)$. Thus each θ_t is a measurable point transformation from X into itself such that $\mu(\theta_t^{-1}E) = 0$ whenever $\mu(E) = 0$; and the transformation $(t, x) \mapsto \theta_t x$ from $R^k_+ \times X$ into X is measurable. In this paper we shall assume that θ satisfies:

(1) $$\mu(E) > 0 \text{ implies } \mu(\theta_t^{-1}E) > 0 \text{ for some } t \in R^k_+,$$

(2) $$f \in L^1_+(\mu) \text{ implies } \int_X f(\theta_t x)\, d\mu(x) < \infty \text{ for all } t \in R^k_+,$$

(3) $$f \in L^1_+(\mu) \text{ implies } \int_X \int_{J_1} f(\theta_t x)\, dt\, d\mu(x) < \infty.$$

By a *process in* $L^p(\mu)$, where $1 \leq p \leq \infty$, we mean a family $F = \{F_I\}_{I \in \mathcal{J}}$ of functions in $L^p(\mu)$. A process F is called *positive* if $F_I \in L^p_+(\mu)$ for all $I \in \mathcal{J}$, *linearly bounded* if there exists a positive real number r such that

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$$\sup_{I \subset J_r} \frac{1}{|I|} \|F_I\|_p < \infty$$

where $|I|$ denotes the Lebesgue measure of $I \in \mathcal{I}$, and *superadditive* (with respect to θ) if

$$F_I \cdot \theta_t \leq F_{t+I} \text{ for all } I \in \mathcal{I} \text{ and } t \in R_+^k, \tag{4}$$

$$\sum_{i=1}^{n} F_{I_i} \leq F_I \text{ whenever } I_1, \ldots, I_n \tag{5}$$

are disjoint sets in $\mathcal{I}$ and

$$I = \bigcup_{i=1}^{n} I_i$$

is also in $\mathcal{I}$.

If both F and $-F$ are superadditive, F is called *additive*.

DEFINITION. ([2]). *Let $\{I_r\}$ be a family of sets in $\mathcal{I}$, where r ranges over the positive rational numbers. $\{I_r\}$ is called* regular *(with constant C) if there exists another family $\{I_r'\}$ of sets in $\mathcal{I}$ such that $I_r \subset I_r'$ for all r, $I_r' \subset I_s'$ whenever $r < s$*, and $|I_r'| \leq C|I_r|$ *for all r. We shall write* $\lim_{r\to 0} I_r = 0$ *if for each $\alpha > 0$ there is an $r_0 > 0$ such that $I_r' \subset J_\alpha$ for all $r < r_0$.*

We are in a position to state the theorem.

THEOREM. *Assume that θ satisfies conditions* (1), (2) *and* (3). *If F is a positive superadditive process in $L^p(\mu)$, where $1 \leq p \leq \infty$, and if $\{I_r\}$ is a regular family of sets in $\mathcal{I}$ with* $\lim_{r\to 0} I_r = 0$, *then* q-$\lim_{r\to 0} (1/|I_r|) F_{I_r}$ *exists and is finite a.e. on X, where* q-$\lim_{r\to 0}$ *means that the limit is taken along the positive rational numbers.*

2. **Proof of the Theorem**. Since the proof is rather long, we shall divide it into several steps.

(I) First, by conditions (2) and (3), if we define, for a measurable function f on X,

$$K_t f(x) = f(\theta_t x) \qquad (t \in R_+^k, \quad x \in X),$$

then $\{K_t\}$ may be regarded as a k-parameter semigroup of bounded linear operators on $L^1(\mu)$, strongly integrable with respect to the Lebesgue measure over ever finite interval $I \in \mathcal{I}$. This together with condition (1), implies (cf. the proof of Theorem 2 in [5]) that there exists a vector $a = (a_i) \in R_+^k$ and a function $u \in L^\infty(\mu)$, with $u > 0$ a.e. on X, such that $\|K_t\|_{L^1(u\,d\mu)} \leq \exp(\Sigma\, a_i t_i)$ for all $t = (t_i) \in R_+^k$, and strong-$\lim_{t\to 0} K_t = I$ (the identity operator) on $L^1(u\, d\mu)$. Therefore, considering the measure $u\, d\mu$ instead of μ, there is no loss of generality in assuming that $\|K_t\|_1 \leq \exp(\Sigma\, a_i t_i)$ for all $t \in R_+^k$ and strong-$\lim_{t\to 0} K_t = I$ on $L^1(\mu)$.

(II) By step (I), if we let

$$T_t = \exp(-\Sigma\, a_i t_i)\, K_t \qquad (t = (t_i) \in R_+^k),$$

then $T = \{T_t\}$ is a k-parameter semigroup of positive contractions on $L^1(\mu)$.

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Let us now fix an $h \in L^1(\mu)$, with $h > 0$ a.e. on X, such that

$$q\text{-}\lim_{r\to 0} \frac{1}{|I_r|} \int_{I_r} T_t h \, dt = h \text{ a.e. on } X.$$

Write

$$F_I^h = \min \{F_I, \int_I T_t h \, dt\} \qquad (I \in \mathcal{I}).$$

Clearly, $F^h = \{F_I^h\}$ is a linearly bounded superadditive process in $L^1(\mu)$ with respect to T, i.e., $T_t F_I^h \leq F_{t+I}^h$ for all $I \in \mathcal{I}$ and $t \in R_+^k$. Thus by a standard argument (cf. e.g. the proof of Lemma 4.7 in [2]), there is a positive additive process G in $L^1(\mu)$, with $G \leq F^h$, such that if we set $H = F^h - G$ then

$$\lim_n \int \sum_{i_1,\ldots,i_k=1}^{2^n} T_{(i_1 2^{-n},\ldots,i_k 2^{-n})} H_{J_{2^{-n}}} \, d\mu = 0. \tag{6}$$

(III) In this step we shall prove a *local maximal lemma* which is essential in the proof of the theorem. Although the proof is similar to that of Akcoglu-Krengel's maximal inequality [2], we prefer to give the complete proof.

LEMMA. *Let $W = \{W_I\}$ be a positive superadditive process in $L^1(\mu)$ with respect to T and $\{I_r\}$ a regular family of sets in $\mathcal{I}$ with $\lim_{r\to 0} I_r = 0$. Let $\alpha > 0$ and $E \in \mathcal{F}$ with $\mu(E) < \infty$. Suppose*

$$q\text{-}\lim_{r\to 0}\sup \frac{1}{|I_r|} W_{I_r} > \alpha \text{ on } E.$$

Given a set $D \in \mathcal{F}$ define

$$\delta(D) = q\text{-}\lim_{r\to 0}\inf \frac{1}{|J_r|} \int_D W_{J_r} \, d\mu.$$

Then we have

$$\delta(D) \geq \frac{\alpha}{6^k} C^{-1} \exp\left(-\sum_{i=1}^k a_i\right) \mu(E \cap D).$$

PROOF. For an $\epsilon > 0$, choose $0 < r < 1$ so that

$$\frac{1}{|J_r|} \int_D W_{J_r} \, d\mu < \delta(D) + \epsilon, \tag{7}$$

and also so that

$$\mu((E \cap D)\Delta\theta_t^{-1}(E \cap D)) < \epsilon \text{ whenever } t \in J_r,$$

where the sign Δ stands for the symmetric difference. Further, choose $0 < r_1 < \ldots < r_M$, with $I_{r_n} \subset J_{r/2}$ for all $1 \leq n \leq M$, so that if we put

$$E' = E \cap D \cap \left\{ \max_{1 \le n \le M} \frac{1}{|I_{r_n}|} W_{I_{r_n}} > \alpha \right\}$$

then $\mu((E \cap D) \backslash E') < \epsilon$ and $\mu(E' \Delta \theta_t^{-1} E') < 3\epsilon$ for all $t \in J_r$. Since $E' \subset E \cap D$, it follows that

$$\mu(D \cap \theta_t^{-1} aE') > \mu(E') - 3\epsilon > \mu(E \cap D) - 4\epsilon$$

for all $t \in J_r$. Define $A(x) = \{t \in J_{r/2} : \theta_t x \in E'\}$. By Fubini's theorem we then have

$$\int_D |A(x)| \, d\mu(x) = \int_{J_{r/2}} \mu(D \cap \theta_t^{-1} E') \, dt > |J_{r/2}| \, (\mu(E \cap D) - 4\epsilon).$$

On the other hand, to each $t \in A(x)$ there corresponds an $n(t) \in \{r_1, \ldots, r_M\}$ such that

$$\alpha |I_{n(t)}| < W_{I_{n(t)}}(\theta_t x) \le \exp\,(\Sigma\, a_i t_i)\, W_{t + I_{n(t)}}(x),$$

and since $\{I_r\}$ is regular, there exists a nested family $I'_{r_1} \subset \ldots I'_{r_M} \subset J_{r/2}$ in $\mathcal{I}$ such that $I_{r_i} \subset I'_{r_i}$ and $|I'_{r_i}| \le C|I_{r_i}|$. Fix an $a \in I'_{r_1}$, and put $U_t = (t - a) + I'_{n(t)}$ for $t \in A(x)$. Since $t \in U_t$ for all $t \in A(x)$, Lemma 4.1 in [2] may be applied to infer that there are finitely many vectors $t^1, \ldots, t^m$ in $A(x)$ such that the sets $t^j + I'_{n(t^j)}$ are disjoint and

$$|A(x)| \le 3^k \sum_{j=1}^{m} |I'_{n(t^j)}| \le 3^k C \sum_{j=1}^{m} |I_{n(t^j)}|.$$

Therefore

$$\begin{aligned} \frac{\alpha}{3^k} C^{-1} |A(x)| &\le \alpha \sum_{j=1}^{m} |I_{n(t^j)}| \\ &< \sum_{j=1}^{m} \exp\left(\sum_{i=1}^{k} a_i t_i^j\right) W_{t^j + I_{n(t^j)}}(x) \\ &< \exp\left(\sum_{i=1}^{k} a_i\right) \sum_{j=1}^{m} W_{t^j + I_{n(t^j)}}(x) \\ &\le \exp\left(\sum_{i=1}^{k} a_i\right) W_{J_r}(x), \end{aligned}$$

and thus

$$\begin{aligned} \int_D W_{J_r}\, d\mu &\ge \frac{\alpha}{3^k} C^{-1} \exp\,(-\Sigma\, a_i) \int_D |A(x)|\, d\mu(x) \\ &> \frac{\alpha}{3^k} C^{-1} \exp\,(-\Sigma\, a_i)\, |J_{r/2}|\, (\mu(E \cap D) - 4\epsilon). \end{aligned}$$

By (7) we obtain

$$\delta(D) + \epsilon > \frac{\alpha}{6^k} C^{-1} \exp\,(-\Sigma\, a_i)(\mu(E \cap D) - 4\epsilon),$$

and letting $\epsilon \to 0$, the desired inequality follows.

(IV) In this step, using the lemma, we shall prove that $q\text{-}\lim_{r\to 0}(1/|I_r|)H_{I_r} = 0$ a.e. on X. To do this, let $\alpha > 0$ and $E \in \mathcal{F}$ with $\mu(E) < \infty$. Suppose

$$q\text{-}\limsup_{r\to 0} \frac{1}{|I_r|} H_{I_r} > \alpha \text{ on } E.$$

Since $\|T_t\|_1 \le 1$ and strong-$\lim_{t\to 0} T_t = I$, it follows that $q\text{-}\lim_{r\to 0} T^*_{re}1 = 1$ a.e. on X, where T^*_{re} denotes the adjoint operator of T_{re}. Therefore the limit function

$$v = \lim_n |J_{2^{-n}}| \sum_{i_1,\ldots,i_k=1}^{2^n} T^*_{(i_1 2^{-n},\ldots,i_k 2^{-n})} 1$$

(the a.e. existence of the limit is easily checked) satisfies $0 < v \le 1$ a.e. on X. This together with (6) proves that given an $\epsilon > 0$ there exists a set $D \in \mathcal{F}$, with $\mu(E \backslash D) < \epsilon$, such that

$$\lim_n \frac{1}{|J_{2^{-n}}|} \int_D H_{J_{2^{-n}}}\, d\mu = 0.$$

Since H is a positive superadditive process in $L^1(\mu)$ with respect to T, the lemma implies that

$$0 \ge \frac{\alpha}{6^k} C^{-1} \exp(-\Sigma\, a_i)(\mu(E) - \epsilon).$$

Letting $\epsilon \to 0$, we have that $\mu(E) = 0$, and the desired result follows.

(V) We shall next consider the process G. By [1], G can be written as $G = G' + G''$, where G' and G'' are positive additive processes in $L^1(\mu)$ (with respect to T) such that (i) G' is singular and (ii) G'' is absolutely continuous. Since G' has the localization property ([1]), i.e., given an $\epsilon > 0$ and a set $E \in \mathcal{F}$, wth $\mu(E) < \infty$, there is an $r > 0$ and a set $D \in \mathcal{F}$, with $\mu(E\backslash D) < \epsilon$, such that $\int_D G'_I\, d\mu \le \epsilon |I|$ for all $I \in \mathcal{I}$ satisfying $I \subset J_r$, it may be seen, as in step (IV), that $q\text{-}\lim_{r\to 0}(1/|I_r|)\,G'_{I_r} = 0$ a.e. on X. The details are omitted.

For the process G'', there is an $f \in L^1_+(\mu)$ such that $G''_I = \int_I T_t f\, dt$ for all $I \in \mathcal{I}$. On the other hand, it is well known and easily checked that the class

$$\left\{ g \in L^1(\mu) : q\text{-}\lim_{r\to 0} \frac{1}{|I_r|} \int_{I_r} T_t g\, dt = g \text{ a.e. on } X \right\}$$

is dense in $L^1(\mu)$. Further, using the lemma, it follows easily that

$$q\text{-}\sup_{r<1} \frac{1}{|I_r|} \int_{I_r} T_t g\, dt < \infty \text{ a.e. on } X$$

for all $g \in L^1_+(\mu)$. Thus Banach's convergence theorem proves that

$$q\text{-}\lim_{r\to 0} \frac{1}{|I_r|} G''_{I_r} = q\text{-}\lim_{r\to 0} \frac{1}{|I_r|} \int_{I_r} T_t f \,\mathrm{d}t$$

exists a.e. on X.

(VI) We shall conclude the proof as follows. We have proved that $q\text{-}\lim_{r\to 0} (1/|I_r|) F^h_{I_r}$ exists a.e. on X. Here, replacing h by Nh, with $N \geq 1$, and noticing that $\lim_{N\to\infty} Nh = \infty$ a.e. on X, we observe immediately that $q\text{-}\lim_{r\to 0} 1/(|I_r|) F_{I_r} = f$ exists a.e. on X. To see that $f < \infty$ a.e. on X, we first notice that $f = q\text{-}\lim_{r\to 0} (1/|J_r|) F_{J_r}$ a.e. on X. Next, fix a function $g \in L^1(\mu) \cap L^\infty(\mu)$ with $g > 0$ a.e. on X Since F is superadditive with respect to T, it follows that for $n \geq 1$,

$$\begin{aligned}\int F_{J_2} g \,\mathrm{d}\mu &\geq \int \Big(\sum_{i_1,\dots,i_k=1}^{2^n} T_{(i_1 2^{-n},\dots,i_k 2^{-n})} F_{J_{2^{-n}}} \Big) g \,\mathrm{d}\mu \\ &= \int F_{J_{2^{-n}}} \Big(\sum_{i_1,\dots,i_k=1}^{2^n} T^*_{(i_1 2^{-n},\dots,i_k 2^{-n})} g \Big) \mathrm{d}\mu \\ &= \int \frac{1}{|J_{2^{-n}}|} F_{J_{2^{-n}}} \Big(\frac{1}{2^{kn}} \sum_{i_1,\dots,i_k=1}^{2^n} T^*_{(i_1 2^{-n},\dots,i_k 2^{-n})} g \Big) \mathrm{d}\mu,\end{aligned}$$

where $\lim_n 1/(|J_{2^{-n}}|) F_{J_{2^{-n}}} = f$ a.e. on X and where

$$L^1(\mu)\text{-}\lim_n \frac{1}{2^{kn}} \sum_{i_1,\dots,i_k=1}^{2^n} T^*_{(i_1 2^{-n},\dots,i_k 2^{-})} g = g'$$

exists and g' satisfies $g' > 0$ a.e. on X, because T^*_t, as an operator of $L^1(\mu)$, converges strongly to the identity operator when $t = (t_1, \dots, t_k)$ tends to 0 (cf. Theorem 1 in [5]). Using Fatou's lemma, $\int fg' \,\mathrm{d}\mu \leq \int F_{J_2} g \,\mathrm{d}\mu < \infty$, and this in turn implies that $f < \infty$ a.e. on X. The proof is completed.

3. **A concluding remark**. We assumed in the Theorem that the process F in $L^p(\mu)$ is positive; but this assumption may be replaced by the following *weak* boundedness assumption:

$$\sup_{I \subset J_r} \frac{1}{|I|} \|F_I^-\|_p = K < \infty$$

for some positive number r, where F_I^- denotes the negative part of the function F_I.

To see this, let $F^- = \{F_I^-\}_{I\in\mathcal{I}}$. Clearly, $-F^-$ is a linearly bounded superadditive process. Since $\|K_t\|_\infty = 1$ and $\|K_t\|_{L^1(u\,d\mu)} \leq \exp(\Sigma\, a_i t_i)$ for all $t = (t_i) \in R^k_+$ by step (I), it follows from the Riesz convexity theorem that $\sup_{t\in I} \|K_t\|_{L^p(u\,\mathrm{d}\mu)} < \infty$ for all $I \in \mathcal{I}$. Thus, modifying an argument in the proof of Lemma 4.7 in [2], we can construct a positive additive process G in $L^p(u \,\mathrm{d}\mu)$ such that $F^- \leq G$ and such that G is linearly bounded. (In case $p = \infty$, we put $G_I(x) = K|I|$ for $x \in X$ and $I \in \mathcal{I}$. Then $G = \{G_I\}$ is a positive additive process in $L^\infty(u\,\mathrm{d}\mu)$ satisfying $F^- \leq G$.) Since $F = (F + G) - G$, where $F + G$ is a positive superadditive process in $L^p(u\,\mathrm{d}\mu)$, the Theorem ends the proof of our assertion.

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