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FRACTAL INTERPOLATION SURFACES ON RECTANGULAR GRIDS

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Abstract

In this paper, we present a general framework to construct fractal interpolation surfaces (FISs) on rectangular grids. Then we introduce bilinear FISs, which can be defined without any restriction on interpolation points and vertical scaling factors.

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1. Introduction

By using the method of iterated function systems, Barnsley [1] introduced (one-dimensional) *fractal interpolation functions* (FIFs). Basically, an FIF is an interpolation function whose graph is the invariant set of an iterated function system. Since then, much work has been done on FIFs, leading to theoretical progress and practical applications (see, for example, [3, 4, 16, 18–21]).

It is natural to ask whether we can define FIFs in higher-dimensional cases, in particular, the two-dimensional case. While it is straightforward to define a similar iterated function system to that in the one-dimensional case, it is hard to guarantee that the invariant set of such an iterated function system is the graph of a continuous function.

In [15], Massopust defined *fractal interpolation surfaces* (FISs) on triangles, where the interpolation points on the boundary are required to be coplanar. This work was generalised by Geronimo and Hardin [12] and Zhao [23], where the interpolation points have more freedom.

Dalla [9] constructed FISs on rectangular grids, where the interpolation points on the boundary are collinear. Feng [11] presented a more general construction

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of FISs on rectangular grids, but the restrictive condition for continuity is hard to check.

By introducing a 'fold-out' technique, Małysz [14] constructed FISs on rectangles for arbitrary interpolation points. The main deficiency in [14] is that vertical scaling factors are required to be equal. This method was generalised by Metzler and Yun [17], with a function as vertical scaling factor. However, examples in [17] still assumed that vertical scaling factors are equal.

In this paper we present a general framework to generate FISs. Then we define a special class of FISs which are called bilinear FISs. We remark that bilinear FISs can be defined on rectangular grids without any restriction on interpolation points and vertical scaling factors.

While we were preparing our manuscript, we learned of the work on one-dimensional bilinear fractal interpolation functions by Barnsley and Massopust [5]. We remark that some ideas of the two papers are similar.

The paper is organised as follows. In Section 2 we recall some ideas needed in constructing FIFs. In Section 3 we present a general framework to construct FISs. Bilinear FISs are introduced in Section 4. In Section 5 we give some remarks on future work.

2. Preliminaries

2.1. One-dimensional fractal interpolation functions. Let $x_0 < x_1 < \cdots < x_N$ be real numbers. Let $L_i : [x_0, x_N] \to [x_{i-1}, x_i]$ be a contractive homeomorphism satisfying

$$L_i(x_0) = x_{i-1}, \quad L_i(x_N) = x_i, \quad i = 1, 2, \ldots, N. \tag{2.1}$$

Denote $K = [x_0, x_N] \times \mathbb{R}$. Let $y_0, y_1, \ldots, y_N$ be real numbers. For $i = 1, 2, \ldots, N$, define a continuous map $F_i : K \to \mathbb{R}$ such that, for a constant $0 < \alpha_i < 1$,

$$\begin{aligned} F_i(x_0, y_0) = y_{i-1}, \quad F_i(x_N, y_N) = y_i, \\ |F_i(x, y') - F_i(x, y'')| \le \alpha_i \cdot |y' - y''| \end{aligned} \tag{2.2}$$

for all $x \in [x_0, x_N]$ and $y', y'' \in \mathbb{R}$. Now, define functions $W_i : K \to K$ for $i = 1, 2, \ldots, N$ by

$$W_i(x, y) = (L_i(x), F_i(x, y)).$$

Then W_i is a continuous function from K to K for each i so that $\{K, W_i : i = 1, 2, \ldots, N\}$ is an *iterated function system* (IFS).

Barnsley [1] proved that there exists a unique nonempty compact subset G of K satisfying $G = \bigcup_{i=1}^{N} W_i(G)$, that is, G is the invariant set of the IFS. Furthermore, G is the graph of a continuous function $f : [x_0, x_N] \to \mathbb{R}$ which obeys $f(x_i) = y_i$, $i = 0, 1, \ldots, N$. We call such a function f a (one-dimensional) *fractal interpolation function*.

The fractal interpolation function f defined above is called a linear FIF if for all $1 \le i \le N$,

$$L_i(x) = a_i x + e_i, \quad F_i(x, y) = c_i x + s_i y + g_i,$$

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where a_i, e_i, c_i, s_i, g_i are constants and $|s_i| < 1$. It is clear that a_i and e_i can be found from (2.1). By (2.2), only one of c_i, s_i and g_i can be arbitrary chosen. We always choose s_i to be the free parameter since we require that $|s_i| < 1$. We call s_i, $1 \le i \le N$, *the vertical scaling factors* of the FIF f.

Linear FIFs have been widely used in applications. For example, one efficient algorithm was presented by Mazel and Hayes [16] to model discrete data.

2.2. Fractal interpolation surfaces on rectangular grids. Let $I = [a, b]$ and $J = [c, d]$. Given interpolation data $\{(x_i, y_j, z_{ij}) \in \mathbb{R}^3 \mid i = 0, 1, \ldots, N; j = 0, 1, \ldots, M\}$ such that $a = x_0 < x_1 < \cdots < x_N = b$ and $c = y_0 < y_1 < \cdots < y_M = d$, it is natural to ask the following questions:

Question 2.1. Can we present a general framework as in [1] to define a fractal function f on $I \times J$ such that $f(x_i, y_j) = z_{ij}$ for all $(i, j) \in \{0, 1, \ldots, N\} \times \{0, 1, \ldots, M\}$?

Question 2.2. Can we define a fractal function f on $I \times J$ which is similar to a linear FIF such that $f(x_i, y_j) = z_{ij}$ for all $(i, j) \in \{0, 1, \ldots, N\} \times \{0, 1, \ldots, M\}$?

Denote $K = I \times J \times \mathbb{R}$. One may define $W_{ij} : K \to K$ as follows:

$$\begin{aligned} W_{ij}(x, y, z) &= (u_i(x), v_j(y), F_{ij}(x, y, z)) \\ &= (a_i x + b_i, c_j y + d_j, e_{ij} x + f_{ij} y + g_{ij} xy + s_{ij} z + k_{ij}), \end{aligned}$$

where the constant s_{ij}, called *the vertical scaling factor*, can be arbitrary chosen in $(-1, 1)$, while other constants $a_i, b_i, c_j, d_j, e_{ij}, f_{ij}, g_{ij}, k_{ij}$ are determined by s_{ij} and the equations

$$\begin{aligned} &W_{ij}(x_0, y_0, z_{00}) = (x_{i-1}, y_{j-1}, z_{i-1,j-1}), && W_{ij}(x_N, y_0, z_{N0}) = (x_i, y_{j-1}, z_{i,j-1}), \\ &W_{ij}(x_0, y_M, z_{0M}) = (x_{i-1}, y_j, z_{i-1,j}), && W_{ij}(x_N, y_M, z_{NM}) = (x_i, y_j, z_{ij}). \end{aligned}$$

Using the technique introduced in [1], we can prove that there exists a unique nonempty compact subset G of K satisfying $G = \bigcup_{i=1}^{N} \bigcup_{j=1}^{M} W_{ij}(G)$. However, in general, G is not the graph of a continuous function on $I \times J$. Dalla [9] showed that if each of the sets

$$\begin{aligned} &\{(x_0, y_j, z_{0j}) : j = 0, 1, \ldots, M\}, && \{(x_N, y_j, z_{Nj}) : j = 0, 1, \ldots, M\}, \\ &\{(x_i, y_0, z_{i0}) : i = 0, 1, \ldots, N\}, && \{(x_i, y_M, z_{iM}) : i = 0, 1, \ldots, N\} \end{aligned}$$

is collinear, then G is the graph of a continuous function on $I \times J$. This result corrected a construction by Xie and Sun [22].

Another attempt is to introduce a new 'IFS' $\{K, \widetilde{W}_{ij} : 1 \le i \le N; 1 \le j \le M\}$, where $\widetilde{W}_{ij}(x, y, z) = (u_i(x), v_j(y), \widetilde{F}_{ij}(x, y, z))$ and

$$\widetilde{F}_{ij}(x, y, z) = \begin{cases} F_{i+1,j}(x_0, y, z) & x = x_N, i = 1, 2, \ldots, N-1, j = 1, 2, \ldots, M, \\ F_{i,j+1}(x, y_0, z) & y = y_M, i = 1, 2, \ldots, N, j = 1, 2, \ldots, M-1, \\ F_{ij}(x, y, z) & \text{otherwise.} \end{cases}$$

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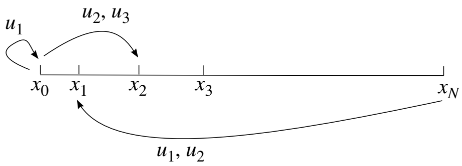

FIGURE 1. Functions u_i on I.

However, it is easy to see that, in general, $\widetilde{F}_{ij}$ is not continuous on K. For example, for fixed $(y', z') \in J \times \mathbb{R}$ and $(i, j) \in \{1, 2, \dots, N-1\} \times \{1, 2, \dots, M\}$, we generally do not have

$$\lim_{x \to x_N} \widetilde{F}_{ij}(x, y', z') = \widetilde{F}_{ij}(x_N, y', z').$$

It follows that $\{K, \widetilde{W}_{ij} : 1 \le i \le N, 1 \le j \le M\}$ is not an IFS. As a result, we remark that the method introduced by Chand and Kapoor [8] is not feasible.

In this paper, we will try to answer Questions 2.1 and 2.2 in Sections 3 and 4, respectively. In Section 3 we extend the fold-out technique used in [14, 17]. In Section 4 we define bilinear FISs.

3. General construction of fractal interpolation surfaces

Let I, J and $\{(x_i, y_j, z_{ij}) \in \mathbb{R}^3 \mid i = 0, 1, \dots, N; j = 0, 1, \dots, M\}$ be the same as in Section 2.2. For convenience, we write $\Sigma_N = \{1, 2, \dots, N\}$, $\Sigma_{N,0} = \{0, 1, \dots, N\}$, $\partial\Sigma_{N,0} = \{0, N\}$ and $\mathrm{int}\Sigma_{N,0} = \{1, 2, \dots, N-1\}$. Similarly, we can define $\Sigma_M, \Sigma_{M,0}, \partial\Sigma_{M,0}$ and $\mathrm{int}\Sigma_{M,0}$.

Denote $I_i = [x_{i-1}, x_i]$ and $J_j = [y_{j-1}, y_j]$ for $i \in \Sigma_N$ and $j \in \Sigma_M$. For any $i \in \Sigma_N$, let $u_i : I \to I_i$ be a contractive homeomorphism satisfying

$$u_i(x_0) = x_{i-1}, \quad u_i(x_N) = x_i \quad \text{if } i \text{ is odd,} \tag{3.1}$$
$$u_i(x_0) = x_i, \quad u_i(x_N) = x_{i-1} \quad \text{if } i \text{ is even, and} \tag{3.2}$$
$$|u_i(x') - u_i(x'')| \le \alpha_i |x' - x''| \quad \forall x', x'' \in I, \tag{3.3}$$

where $0 < \alpha_i < 1$ is a given constant. Clearly, this implies that $u_1(x_0) = x_0$, $u_1(x_N) = u_2(x_N) = x_1$, $u_2(x_0) = u_3(x_0) = x_2$ and so on (see Figure 1). Similarly, for any $j \in \Sigma_M$, let $v_j : J \to J_j$ be a contractive homeomorphism satisfying

$$v_j(y_0) = y_{j-1}, \quad v_j(y_M) = y_j \quad \text{if } j \text{ is odd,} \tag{3.4}$$
$$v_j(y_0) = y_j, \quad v_j(y_M) = y_{j-1} \quad \text{if } j \text{ is even, and} \tag{3.5}$$
$$|v_j(y') - v_j(y'')| \le \beta_j |y' - y''| \quad \forall y', y'' \in J, \tag{3.6}$$

where $0 < \beta_j < 1$ is a given constant. By the definitions of u_i and v_j, it is easy to check that

$$u_i^{-1}(x_i) = u_{i+1}^{-1}(x_i), \quad \forall i \in \mathrm{int}\Sigma_{N,0}, \quad \text{and}$$
$$v_j^{-1}(y_j) = v_{j+1}^{-1}(y_j), \quad \forall j \in \mathrm{int}\Sigma_{M,0}.$$

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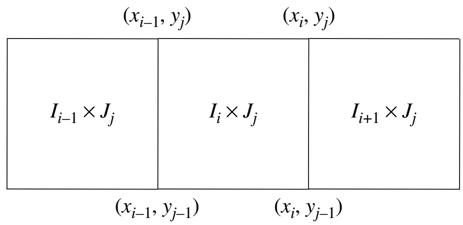

FIGURE 2. $I_i \times J_j$.

Now we will rewrite (3.1)–(3.6) for simplicity. Let $\tau : \mathbb{Z} \times \{0, N, M\} \to \mathbb{Z}$ be defined by

$$\tau(i,0) = \begin{cases} i-1, & \tau(i,N) = \tau(i,M) = i \quad \text{if } i \text{ is odd,} \\ i, & \tau(i,N) = \tau(i,M) = i-1 \quad \text{if } i \text{ is even.} \end{cases}$$

Then $u_i(x_k) = x_{\tau(i,k)}$ for all $i \in \Sigma_N$ and $k \in \partial\Sigma_{N,0}$. Similarly, $v_j(y_k) = y_{\tau(j,k)}$ for all $j \in \Sigma_M$ and $k \in \partial\Sigma_{M,0}$.

For example, if both i and j are odd,

$$(x_{i-1}, y_{j-1}, z_{i-1,j-1}) = (x_{\tau(i,0)}, y_{\tau(j,0)}, z_{\tau(i,0),\tau(j,0)}) = (x_{\tau(i-1,0)}, y_{\tau(j,0)}, z_{\tau(i-1,0),\tau(j,0)}),$$
$$(x_i, y_j, z_{ij}) = (x_{\tau(i,N)}, y_{\tau(j,M)}, z_{\tau(i,N),\tau(j,M)}) = (x_{\tau(i+1,N)}, y_{\tau(j,M)}, z_{\tau(i+1,N),\tau(j,M)})$$

(see Figure 2).

Denote $K = I \times J \times \mathbb{R}$. For each $(i, j) \in \Sigma_N \times \Sigma_M$, let $F_{ij} : K \to \mathbb{R}$ be a continuous function satisfying

$$F_{ij}(x_k, y_\ell, z_{k\ell}) = z_{\tau(i,k),\tau(j,\ell)}, \quad \forall (k,\ell) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}, \quad \text{and} \tag{3.7}$$
$$|F_{ij}(x,y,z') - F_{ij}(x,y,z'')| \le \gamma_{ij}|z' - z''|, \quad \forall (x,y) \in I \times J \text{ and } z', z'' \in \mathbb{R}, \tag{3.8}$$

where $0 < \gamma_{ij} < 1$ is a given constant.

Now, for each $(i, j) \in \Sigma_N \times \Sigma_M$, we define $W_{ij} : K \to I_i \times J_j \times \mathbb{R}$ by

$$W_{ij}(x,y,z) = (u_i(x), v_j(y), F_{ij}(x,y,z)). \tag{3.9}$$

Then $\{K, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ is an IFS. By definition, for any $(i, j) \in \Sigma_N \times \Sigma_M$, we have

$$W_{ij}(x_k, y_\ell, z_{k\ell}) = (x_{\tau(i,k)}, y_{\tau(j,\ell)}, z_{\tau(i,k),\tau(j,\ell)}), \quad \forall (k,\ell) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}.$$

THEOREM 3.1. *Let $\{K, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ be the IFS defined as in (3.9). Assume that $\{F_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ satisfies the following matching conditions:*

(1) *for all $i \in \mathrm{int}\Sigma_{N,0}$, $j \in \Sigma_M$ and $x^* = u_i^{-1}(x_i) = u_{i+1}^{-1}(x_i)$,*

$$F_{ij}(x^*, y, z) = F_{i+1,j}(x^*, y, z), \quad \forall y \in J, z \in \mathbb{R}, \quad \textit{and} \tag{3.10}$$

(2) *for all $i \in \Sigma_N$, $j \in \mathrm{int}\Sigma_{M,0}$ and $y^* = v_j^{-1}(y_j) = v_{j+1}^{-1}(y_j)$,*

$$F_{ij}(x, y^*, z) = F_{i,j+1}(x, y^*, z), \quad \forall x \in I, z \in \mathbb{R}. \tag{3.11}$$

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Then there exists a unique continuous function $f : I \times J \to \mathbb{R}$ *such that* $f(x_i, y_j) = z_{ij}$ *for all* $(i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}$ *and* $G = \bigcup_{(i,j)\in\Sigma_N\times\Sigma_M} W_{ij}(G)$, *where* $G = \mathrm{Graph}(f) = \{(x, y, f(x, y)) : (x, y) \in I \times J\}$ *is the graph of* f. *We call* G *the FIS and* f *the FIF with respect to the IFS* $\{K, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$.

Proof. Let $C(I \times J)$ be the set of all continuous functions on $I \times J$. Define $T : C(I \times J) \to C(I \times J)$ as follows: given $p \in C(I \times J)$,

$$Tp(x, y) = F_{ij}(u_i^{-1}(x), v_j^{-1}(y), p(u_i^{-1}(x), v_j^{-1}(y))), \quad (x, y) \in I_i \times J_j, \tag{3.12}$$

for all $(i, j) \in \Sigma_N \times \Sigma_M$. From (3.10) and (3.11), we know that Tp is well defined on the boundary of $I_i \times J_j$ for all $(i, j) \in \Sigma_N \times \Sigma_M$. It follows that $T : C(I \times J) \to C(I \times J)$ is well defined.

Let $C^*(I \times J) = \{p \in C(I \times J) : p(x_i, y_j) = z_{ij}, \text{for all } (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}\}$ and let $p \in C^*(I \times J)$. For any $(i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}$, choose $k \in \partial\Sigma_{N,0}$ and $\ell \in \partial\Sigma_{M,0}$ such that $i = \tau(i, k)$ and $j = \tau(j, \ell)$. By the definition of τ, $x_k = u_i^{-1}(x_i)$ and $y_\ell = v_j^{-1}(y_j)$. Using (3.7) and (3.12),

$$Tp(x_i, y_j) = F_{ij}(x_k, y_\ell, p(x_k, y_\ell)) = F_{ij}(x_k, y_\ell, z_{k\ell}) = z_{\tau(i,k),\tau(j,\ell)} = z_{ij}.$$

It follows that T is a map from $C^*(I \times J)$ to $C^*(I \times J)$.

For any $p \in C^*(I \times J)$, we define $|p|_\infty = \max\{p(x, y) : (x, y) \in I \times J\}$. From (3.8), we can easily see that T is contractive on the complete metric space $(C^*(I \times J), |\cdot|_\infty)$. Thus there exists a unique function $f \in C^*(I \times J)$ such that $Tf = f$, that is,

$$f(x, y) = F_{ij}(u_i^{-1}(x), v_j^{-1}(y), f(u_i^{-1}(x), v_j^{-1}(y))), \quad (x, y) \in I_i \times J_j. \tag{3.13}$$

Now, let $G = \mathrm{Graph}(f)$. By (3.9) and (3.13),

$$\begin{aligned}
&\bigcup_{(i,j)\in\Sigma_N\times\Sigma_M} W_{ij}(G) \\
&\quad= \bigcup_{(i,j)\in\Sigma_N\times\Sigma_M} \{(u_i(x), v_j(y), F_{ij}(x, y, f(x, y))) : (x, y) \in I \times J\} \\
&\quad= \bigcup_{(i,j)\in\Sigma_N\times\Sigma_M} \{(x, y, F_{ij}(u_i^{-1}(x), v_j^{-1}(y), f(u_i^{-1}(x), v_j^{-1}(y)))) : (x, y) \in I_i \times J_j\} \\
&\quad= \bigcup_{(i,j)\in\Sigma_N\times\Sigma_M} \{(x, y, f(x, y)) : (x, y) \in I_i \times J_j\} = G.
\end{aligned}$$

Assume that $\widetilde{f} \in C^*(I \times J)$ satisfies $\widetilde{G} = \bigcup_{(i,j)\in\Sigma_N\times\Sigma_M} W_{ij}(\widetilde{G})$, where $\widetilde{G} = \mathrm{Graph}(\widetilde{f})$. Then we must have

$$F_{ij}(x, y, \widetilde{f}(x, y)) = \widetilde{f}(u_i(x), v_j(y)), \quad \forall (x, y) \in I \times J,$$

so that $\widetilde{f}$ satisfies $T\widetilde{f} = \widetilde{f}$. Since T is contractive on $(C^*(I \times J), |\cdot|_\infty)$, we know that $\widetilde{f} = f$. This completes the proof of the theorem. □

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REMARK 3.1. The main difficulty of constructing FISs is the continuity. All other aspects are similar to the classical arguments in [1, 3].

Generally, the IFS $\{K, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ is not hyperbolic. However, we can still show that $G = \mathrm{Graph}(f)$ is the attractor of the IFS. The spirit of the proof follows from [1, 3]. In the rest of this section, we will use $d(\cdot, \cdot)$ to denote the Euclidean metric. For any two nonempty compact subsets B, C of $\mathbb{R}^k$, we define their Hausdorff metric by

$$d_H(B, C) = \max\Big\{\max_{x\in B} \min_{y\in C} d(x, y), \max_{y\in C} \min_{x\in B} d(x, y)\Big\}.$$

For any closed subset X of $\mathbb{R}^k$, let $\mathcal{H}(X)$ be the family of all nonempty compact subsets of X. It is well known that $(\mathcal{H}(X), d_H)$ is a complete metric space. For details about the Hausdorff metric, see [2, 10].

Define $L : \mathcal{H}(I \times J) \to \mathcal{H}(I \times J)$ by

$$L(B) = \bigcup_{(i,j)\in\Sigma_N\times\Sigma_M} \{(u_i(x), v_j(y)) : (x, y) \in B\}, \quad B \in \mathcal{H}(I \times J).$$

It is clear that L is contractive on $\mathcal{H}(I \times J)$, and $I \times J$ is the attractor of L.

THEOREM 3.2. *Let f be the fractal interpolation function with respect to the IFS $\{K, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ and $G = \mathrm{Graph}(f)$. Then for any $A \in \mathcal{H}(K)$,*

$$\lim_{n\to\infty} d_H(W^n(A), G) = 0,$$

where $W^0(A) = A$ and $W^{n+1}(A) = W(W^n(A))$ for any $n \geq 0$.

PROOF. For any $A \in \mathcal{H}(K)$, we define

$$A_{XY} = \{(x, y) \in I \times J : \text{there exists } z \in \mathbb{R} \text{ such that } (x, y, z) \in A\}.$$

It is clear that $A_{XY} \in \mathcal{H}(I \times J)$. Let $E_n = \{(x, y, f(x, y)) : (x, y) \in L^n(A_{XY})\}$. Since $I \times J$ is the attractor of L,

$$\lim_{n\to\infty} d_H(L^n(A_{XY}), I \times J) = 0.$$

Thus, noticing that f is uniformly continuous on $I \times J$,

$$\lim_{n\to\infty} d_H(E_n, G) = 0. \tag{3.14}$$

Let

$$t_n = \sup\{|f(x, y) - z| : (x, y, z) \in W^n(A)\}, \quad \forall n \geq 1.$$

By the definition of Hausdorff metric, we have $d_H(E_n, W^n(A)) \leq t_n$. Given $n \geq 1$, for any $(x, y, z) \in W^n(A)$, there exists $(x^*, y^*, z^*) \in W^{n-1}(A)$ and $(i, j) \in \Sigma_N \times \Sigma_M$ such that

$$(x, y, z) = (u_i(x^*), v_j(y^*), F_{ij}(x^*, y^*, z^*)).$$

Let $\gamma = \max\{\gamma_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$. By (3.8) and (3.13),

$$\begin{aligned}|f(x, y) - z| &= |F_{ij}(x^*, y^*, f(x^*, y^*)) - F_{ij}(x^*, y^*, z^*)|\\ &\leq \gamma|f(x^*, y^*) - z^*| \leq \gamma t_{n-1}\end{aligned}$$

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so that $t_n \le \gamma t_{n-1}$. Since $\gamma < 1$, we have $\lim_{n\to\infty} t_n = 0$ so that

$$\lim_{n\to\infty} d_H(E_n, W^n(A)) = 0.$$

Combining this with (3.14), we see that the theorem holds. □

4. Bilinear fractal interpolation surfaces

Let g, h, s be continuous functions on $I \times J$ satisfying

$$g(x_i, y_j) = z_{ij}, \quad \forall\, (i, j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0},$$
$$h(x_i, y_j) = z_{ij}, \quad \forall\, (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}, \quad \text{and}$$
$$s^* = \max\{|s(x, y)| : (x, y) \in I \times J\} < 1.$$

Recall that $K = I \times J \times \mathbb{R}$. For each $(i, j) \in \Sigma_N \times \Sigma_M$, we define $F_{ij} : K \to \mathbb{R}$ by

$$F_{ij}(x, y, z) = s(u_i(x), v_j(y))(z - g(x, y)) + h(u_i(x), v_j(y)), \tag{4.1}$$

where $u_i \in C(I)$ and $v_j \in C(J)$ satisfy (3.1)–(3.6). Then for all $(i, j) \in \Sigma_N \times \Sigma_M$ and $(k, \ell) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}$,

$$F_{ij}(x_k, y_\ell, z_{k\ell}) = h(u_i(x_k), v_j(y_\ell)) = h(x_{\tau(i,k)}, y_{\tau(j,\ell)}) = z_{\tau(i,k),\tau(j,\ell)}$$

so that (3.7) holds. Since $s^* < 1$, we can easily see that (3.8) holds.

Given $i \in \mathrm{int}\Sigma_{N,0}$ and $j \in \Sigma_M$, let $x^* = u_i^{-1}(x_i) = u_{i+1}^{-1}(x_i)$. For any $y \in J$ and $z \in \mathbb{R}$,

$$F_{ij}(x^*, y, z) = F_{i+1,j}(x^*, y, z) = s(x_i, v_j(y))(z - g(x^*, y)) + h(x_i, v_j(y))$$

so that (3.10) holds. Similarly, (3.11) holds for all $i \in \Sigma_N$, $j \in \mathrm{int}\Sigma_{M,0}$ and $y^* = v_j^{-1}(y_j) = v_{j+1}^{-1}(y_j)$.

By Theorem 3.1, we have the following result.

Theorem 4.1 [17]. *Let $\{K, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ be the IFS defined by (3.9), where F_{ij} is defined by (4.1) for $(i, j) \in \Sigma_N \times \Sigma_M$. Then there exist a unique continuous function f such that $f(x_i, y_j) = z_{ij}$ for all $(i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}$ and $G = \bigcup_{(i,j)\in\Sigma_N\times\Sigma_M} W_{ij}(G)$, where $G = \mathrm{Graph}(f) = \{(x, y, f(x, y)) : (x, y) \in I \times J\}$ is the graph of f.*

The function s in the above theorem is called the vertical scaling factor function of the FIF f. We remark that all s are constant functions in examples in [17]. In order to solve Question 2.2, we will present a class of FISs which are easily constructed and in which s is a piecewise bilinear function.

Firstly, we define u_i and v_j to be linear functions satisfying (3.1), (3.2), (3.4) and (3.5) for all $i \in \Sigma_N$ and $j \in \Sigma_M$. We define g to be the bilinear function on $I \times J$ satisfying

$$g(x_i, y_j) = z_{ij}, \quad \forall (i, j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}.$$

Equivalently,

$$g(x, y) = \frac{1}{(b-a)(d-c)}((b-x)(d-y)z_{0,0} + (x-a)(d-y)z_{N,0}$$
$$+ (b-x)(y-c)z_{0,M} + (x-a)(y-c)z_{M,N}).$$

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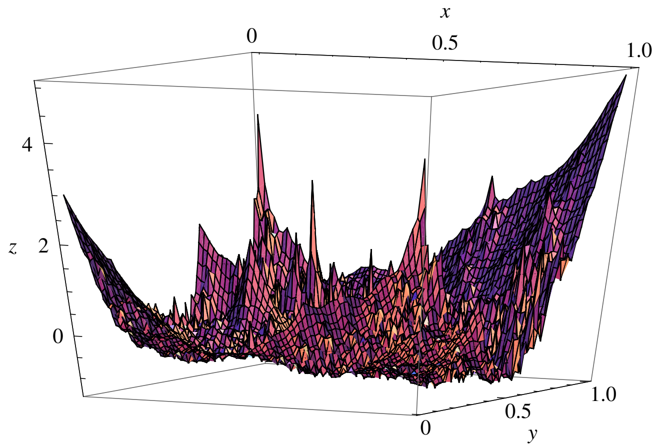

Figure 3. Bilinear FIS in Example 4.1.

Then we define $h : I \times J \to \mathbb{R}$ to be the function such that $h|_{I_i \times J_j}$ is bilinear for all $(i, j) \in \Sigma_N \times \Sigma_M$ and

$$h(x_i, y_j) = z_{ij}, \quad \forall (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}.$$

Let $\{s_{ij} \mid (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}\}$ be a given subset of $\mathbb{R}$ with $|s_{ij}| < 1$ for all i, j. We define $s : I \times J \to \mathbb{R}$ to be the function such that $s|_{I_i \times J_j}$ is bilinear for all $(i, j) \in \Sigma_N \times \Sigma_M$ and

$$s(x_i, y_j) = s_{ij}, \quad \forall (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}.$$

For all $(i, j) \in \Sigma_N \times \Sigma_M$, we define $F_{ij} : I \times J \times \mathbb{R} \to \mathbb{R}$ by (4.1). Then the fractal interpolation function f determined by $\{K, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ is called a *bilinear FIF*. We call $G = \text{Graph}(f)$ a *bilinear FIS*. Also $s_{ij}, (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}$, are called *vertical scaling factors* of f.

Clearly, a bilinear FIS is determined by interpolation points $\{(x_i, y_j, z_{ij}) : (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}\}$ and vertical scaling factors $\{s_{ij} : (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}\}$. This property is similar to the linear FIF in the one-dimensional case.

Example 4.1. Let $N = M = 3$. Let $x_i = i/N$ and $y_j = j/M$, for all $i \in \Sigma_{N,0}$ and $j \in \Sigma_{M,0}$. Let $Z = (z_{ij})_{(i,j) \in \Sigma_{N,0} \times \Sigma_{M,0}}$ and $S = (s_{ij})_{(i,j) \in \Sigma_{N,0} \times \Sigma_{M,0}}$ be chosen as

$$Z = \begin{pmatrix} 3 & 0 & 2 & 4 \\ 0 & 2 & 3 & 1 \\ 1 & 0 & 1 & 3 \\ 4 & 1 & 2 & 5 \end{pmatrix}, \quad S = \begin{pmatrix} 0.2 & 0.3 & 0.4 & 0.8 \\ 0.1 & 0.5 & 0.9 & 0.2 \\ 0.3 & 0.2 & 0.4 & 0.3 \\ 0.5 & 0.4 & 0.8 & 0.2 \end{pmatrix}.$$

The corresponding bilinear FIS is shown in Figure 3.

Example 4.2. Let $p(x, y) = \sin(\pi(x^2 + y^2))$, $x, y \in [0, 1]$. See Figure 4 for the graph of p. Let $N = 3$ and $M = 2$. Define $x_i = i/N$ and $y_j = j/M$, for all $i \in \Sigma_{N,0}$ and $j \in \Sigma_{M,0}$.

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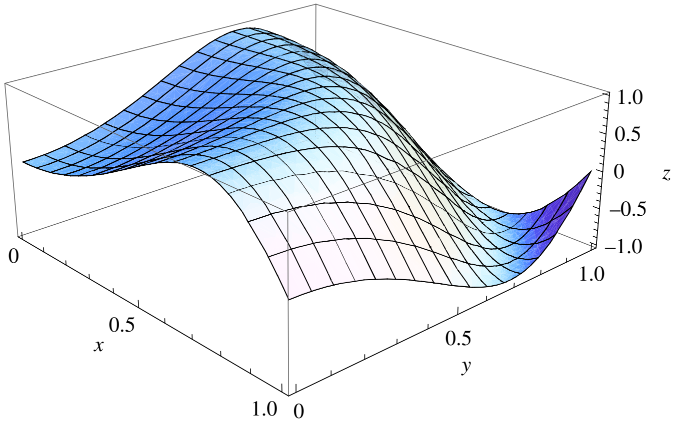

FIGURE 4. The graph of p in Example 4.2.

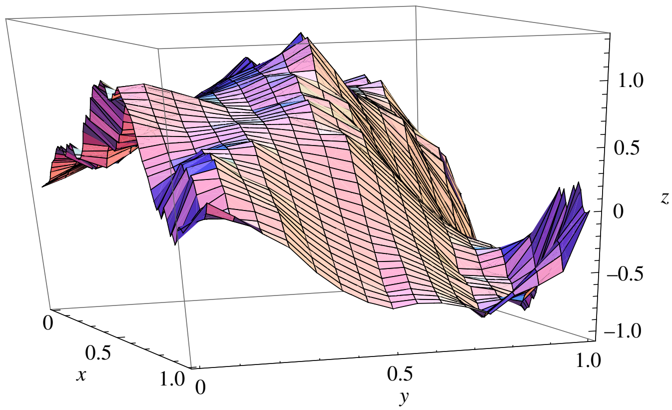

FIGURE 5. The bilinear FIS in Example 4.2.

Various algorithms can be devised to obtain $S = (s_{ij})_{(i,j)\in\Sigma_{N,0}\times\Sigma_{M,0}}$ so that the corresponding FIS fits the graph of p as well as possible. Here, by using a genetic algorithm (see [13] for details), we obtain

$$S = \begin{pmatrix} 0.035 & -0.458 & -0.130 & 0.837 \\ 0.018 & 0.472 & 0.402 & -0.289 \\ -0.442 & 0.99 & -0.231 & -0.99 \end{pmatrix}.$$

The corresponding bilinear FIS is shown in Figure 5.

5. Further remarks

In this section, we give some remarks on further work.

QUESTION 5.1. How can we obtain the box dimension of a bilinear FIS?

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It seems that the method in [5] is helpful, where Barnsley and Massopust obtained the box dimension of a one-dimensional bilinear FIF with $x_i = i/N$ for all $i = 0, 1, \ldots, N$. However, dealing with FISs is more involved.

QUESTION 5.2. How can we construct recurrent FISs on rectangular grids which can be easily generated? In particular, we hope that vertical scaling factors are easily determined (for example, by a genetic algorithm) when we want to use recurrent FISs to fit given data.

From Example 4.1, we expect that bilinear FISs will be used to generate some natural scenes. However, in order to fit given data more effectively, we need recurrent FISs. In [6, 7], Bouboulis, Dalla and Drakopoulos presented nice methods to generate recurrent FISs, although the restrictive conditions for continuity are hard to check.

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References

[1] M. F. Barnsley, 'Fractal functions and interpolation', *Constr. Approx.* **2** (1986), 303–329.
[2] M. F. Barnsley, *Fractals Everywhere*, 2nd edn (Academic Press Professional, Boston, 1993), 27–41.
[3] M. F. Barnsley, J. Elton, D. Hardin and P. Massopust, 'Hidden variable fractal interpolation functions', *SIAM J. Math. Anal.* **20** (1989), 1218–1242.
[4] M. F. Barnsley and A. N. Harrington, 'The calculus of fractal interpolation functions', *J. Approx. Theory* **57** (1989), 14–34.
[5] M. F. Barnsley and P. R. Massopust, Bilinear fractal interpolation and box dimension, Preprint, arXiv:1209.3139v1.
[6] P. Bouboulis and L. Dalla, 'A general construction of fractal interpolation functions on grids of $\mathbb{R}^n$', *Eur. J. Appl. Math.* **18** (2007), 449–476.
[7] P. Bouboulis, L. Dalla and V. Drakopoulos, 'Construction of recurrent bivariate fractal interpolation surfaces and computation of their box-counting dimension', *J. Approx. Theory* **141** (2006), 99–117.
[8] A. K. B. Chand and G. P. Kapoor, 'Hidden variable bivariate fractal interpolation surfaces', *Fractals* **11** (2003), 277–288.
[9] L. Dalla, 'Bivariate fractal interpolation functions on grids', *Fractals* **10** (2002), 53–58.
[10] K. J. Falconer, *The Geometry of Fractal Sets* (Cambridge University Press, Cambridge, 1985), 36–37.
[11] Z. Feng, 'Variation and Minkowski dimension of fractal interpolation surfaces', *J. Math. Anal. Appl.* **345** (2008), 322–334.
[12] J. S. Geronimo and D. Hardin, 'Fractal interpolation surfaces and a related 2D multiresolution analysis', *J. Math. Anal. Appl.* **176** (1993), 561–586.
[13] R. L. Haupt and S. E. Haupt, *Practical Genetic Algorithms*, 2nd edn (John Wiley & Sons, Hoboken, NJ, 2004), 27–50.
[14] R. Małysz, 'The Minkowski dimension of the bivariate fractal interpolation surfaces', *Chaos Solitons Fractals* **27** (2006), 1147–1156.
[15] P. R. Massopust, 'Fractal surfaces', *J. Math. Anal. Appl.* **151** (1990), 275–290.
[16] D. S. Mazel and M. H. Hayes, 'Using iterated function systems to model discrete sequences', *IEEE Trans. Signal Process.* **40** (1992), 1724–1734.

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[17] W. Metzler and C. H. Yun, 'Constuction of fractal interpolation surfaces on rectangular grids', *Internat. J. Bifur. Chaos* **20** (2010), 4079–4086.
[18] H.-J. Ruan, Z. Sha and W.-Y. Su, 'Counterexamples in parameter identification problem of the fractal interpolation functions', *J. Approx. Theory* **122** (2003), 121–128.
[19] H.-J. Ruan, W.-Y. Su and K. Yao, 'Box dimension and fractional integral of linear fractal interpolation functions', *J. Approx. Theory* **161** (2009), 187–197.
[20] J. L. Véhel, K. Daoudi and E. Lutton, 'Fractal modeling of speech signals', *Fractals* **2** (1994), 379–382.
[21] H.-Y. Wang and J.-S. Yu, 'Fractal interpolation functions with variable parameters and their analytical properties', *J. Approx. Theory* **175** (2013), 1–18.
[22] H. Xie and H. Sun, 'The study on bivariate fractal interpolation functions and creation of fractal interpolated surfaces', *Fractals* **5** (1997), 625–634.
[23] N. Zhao, 'Construction and application of fractal interpolation surfaces', *Visual Computer* **12** (1996), 132–146.

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