A NOTE ON AN INEQUALITY WITH NON-CONJUGATE PARAMETERS

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The inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dydx < \pi \operatorname{cosec}(\pi/p)\,\| f \|_p \| g \|_q,$$

which is valid for positive, non-null f, g in the spaces $L^p(0, \infty)$, $L^q(0, \infty)$, where $p>1$, $(1/p)+(1/q) = 1$, is a well-known generalisation of the classical inequality of Hilbert (see for instance Chapter 9 of Hardy, Littlewood, and Polya (**1**)).

It is also shown in (**1**) that the constant $\pi \operatorname{cosec}(\pi/p)$ is best possible.

The case when p, q are not conjugate parameters, but are restricted only by $p>1$, $q>1$, $(1/p)+(1/q) \geqq 1$ is also considered in (**1**). If $p' = p/(p-1)$, $q' = q/(q-1)$ are the conjugate indices to p, q, and we write

$$\lambda = (1/p')+(1/q') = 2-(1/p)-(1/q), \text{ then } \lambda \in (0, 1],$$

and the inequality takes the form

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dydx < K(p, q)\,\| f \|_p \| g \|_q. \tag{i}$$

The upper estimate

$$K(p, q) \leqq \{\pi \operatorname{cosec}(\pi/\lambda p')\}^\lambda \tag{ii}$$

was first obtained by Levin (**2**) and later, more elegantly by Bonsall (**3**). Evidently this value agrees with that in the conjugate case when $\lambda = 1$.

The object of the present note is to show that although the problem of whether (ii) gives the best possible of $K(p, q)$ in all cases is still apparently open, the estimate is asymptotically best possible, in the sense that if say q is fixed then $K(p, q)\{\pi \operatorname{cosec}(\pi/\lambda p')\}^{-\lambda} \to 1$ as $p \to 1$ (and so $p' \to \infty$).

In order to obtain lower estimates for $K(p, q)$, we take the following functions for f, g:

$$\{f(x)\}^p = \begin{cases} x^{-1-a} & (x \geqq 1), \\ 0 & (0 \leqq x < 1), \end{cases} \qquad \{g(y)\}^q = \begin{cases} y^{-1-b} & (y \geqq 1), \\ 0 & (0 \leqq y < 1). \end{cases}$$

In this case $\| f \|_p = a^{-r}$, $\| g \| = b^{-s}$, where we have written r, s for $(1/p)$, $(1/q)$ respectively. Then

$$\int_0^\infty \int_0^\infty (x+y)^{-\lambda} f(x)g(y)\,dxdy = \int_1^\infty x^{-r(1+a)} \int_1^\infty (x+y)^{-\lambda} y^{-s(1+b)}\,dydx$$

$$= \int_0^\infty x^{-1-ar-bs} \int_{1/x}^\infty (1+t)^{-\lambda} t^{-s(1+b)}\,dtdx,$$

where we have written $y = xt$.

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We now interchange the order of integration to obtain

$$(ar+bs)^{-1}\left[\int_0^1 (1+t)^{-\lambda}t^{ar-s}dt+\int_1^\infty (1+t)^{-\lambda}t^{-s(1+b)}dt\right]$$

$$=(ar+bs)^{-1}\int_0^{\frac{1}{2}}\{u^{ar-s}(1-u)^{-r(1+a)}+u^{bs-r}(1-u)^{-s(1+b)}\}du,$$

where the substitutions $t = u(1+t)$, and $1 = u(1+t)$ have been used.

It follows that

$$K(p, q)>a^r b^s(ar+bs)^{-1}\int_0^{\frac{1}{2}}\{u^{ar-s}+u^{bs-r}\}du$$

where the negative powers of $(1-u)$ have been suppressed.

Hence

$$K(p, q)>a^r b^s(ar+bs)^{-1}[(ar+s')^{-1}2^{-(ar+s')}+(bs+r')^{-1}2^{-(bs+r')}],$$

where $s' = 1-s = 1/q'$, $r' = 1-r = 1/p'$.

In order to obtain an asymptotic estimate as $p'\to\infty$, we allow b to tend to zero as c/p' for constant c. Elementary calculus shows that the most favourable value of c is given by $c = q'$; hence we take $c = q'$, $b = q'/p'$, and symmetrically, $a = p'/q'$. In this case $bs+r' = q'/p'$ also, and we obtain from the second term of the above estimate for $K(p, q)$, that

$$K(p, q)>(q'/p')^{s-r-1}(ar+bs)^{-1}2^{-q'r'}.$$

It follows that

$$K(p, q)\{\pi \operatorname{cosec}(\pi/\lambda p')\}^{-\lambda}>$$

$$\{(p'/q')^r(p'/q'+q'/p'-\lambda)^{-1}\}\,.\,\{p'\sin(\pi/\lambda p')/q'\pi\}^{s'}\,.\,\{\sin(\pi/\lambda p')2^{-q'}\pi^{-1}\}^{r'}.$$

As $p\to 1$, we have $p'\to\infty$, $r'\to 0$, $\lambda\to 1/q'$, and it is easily seen that each bracketed term tends to unity.

REFERENCES

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