BULL. AUSTRAL. MATH. SOC. MOS 0640, 0635
VOL. I (1969), 279-280.

The completion by cuts of an orthocomplemented modular lattice

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In this note we give an example of an orthocomplemented modular lattice whose completion by cuts is not orthomodular. This solves negatively Problem 36, p. 131, in G. Birkhoff: Lattice theory (3rd edition).

Let V be an infinite dimensional prehilbert space over the complex field C. Assume that V is incomplete with respect to the inner product $(\cdot,\cdot)$. Let $L(V)$ be the lattice of subsets of V closed under the closure operation $S \to S^{\perp\perp}$ where

$$S^{\perp} = \{f : f \in V , (f,g) = 0 , \forall g \in S\} .$$

It is easy to see that the elements of $L(V)$ are linear subspaces of V, and it follows from the lemma on p. 425 of [1] that all finite dimensional subspaces of V belong to $L(V)$. It is straightforward to show that the map $^{\perp}: L(V) \to L(V)$ is an orthocomplementation in $L(V)$ ([2], p. 123).

Now let $L_1(V)$ be the sublattice of $L(V)$ consisting of all finite dimensional subspaces of V and their orthocomplements. Then $L_1(V)$ is an orthocomplemented modular lattice, and it is also join-dense in $L(V)$. It follows from a theorem of M. Donald MacLaren that the completion by cuts of $L_1(V)$ is isomorphic to the complete lattice $L(V)$ ([3], Th. 2.5). As V was assumed to be incomplete, $L(V)$ is not orthomodular by the main theorem in [1].

Received 23 March 1969.

https://doi.org/10.1017/S0004972700041526 Published online by Cambridge University Press

We must now show the existence of such an incomplete prehilbert space V. The space of all continuous, absolutely square integrable functions on the closed interval $[0,1]$ is such a space with the usual L_2 inner product. So the lattice $L_1(V)$ of finite and cofinite dimensional subspaces of V is the promised example.

References

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