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# ON DEGENERATIONS OF MODULES WITH NONDIRECTING INDECOMPOSABLE SUMMANDS

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ABSTRACT. Let $A$ be a finite dimensional associative $K$-algebra with an identity over an algebraically closed field $K$, $d$ a natural number, and $\mathrm{mod}_A(d)$ the affine variety of $d$-dimensional $A$-modules. The general linear group $\mathrm{Gl}_d(K)$ acts on $\mathrm{mod}_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of $d$-dimensional modules. For $M$ and $N$ in $\mathrm{mod}_A(d)$, $N$ is called a degeneration of $M$, if $N$ belongs to the closure of the orbit of $M$. This defines a partial order $\leq_{\mathrm{deg}}$ on $\mathrm{mod}_A(d)$. There has been a work [1], [10], [11], [21] connecting $\leq_{\mathrm{deg}}$ with other partial orders $\leq_{\mathrm{ext}}$ and $\leq$ on $\mathrm{mod}_A(d)$ defined in terms of extensions and homomorphisms. In particular, it is known that these partial orders coincide in the case $A$ is representation-finite and its Auslander-Reiten quiver is directed. We study degenerations of modules from the additive categories given by connected components of the Auslander-Reiten quiver of $A$ having oriented cycles. We show that the partial orders $\leq_{\mathrm{ext}}$, $\leq_{\mathrm{deg}}$ and $\leq$ coincide on modules from the additive categories of quasi-tubes [24], and describe minimal degenerations of such modules. Moreover, we show that $M \leq_{\mathrm{deg}} N$ does not imply $M \leq_{\mathrm{ext}} N$ for some indecomposable modules $M$ and $N$ lying in coils in the sense of [4].

1. **Introduction and main results.** Throughout the paper $K$ denotes a fixed algebraically closed field. By an algebra we mean an associative finite dimensional $K$-algebra with an identity, and by an $A$-module a finite dimensional (unital) right $A$-module. We shall denote by mod $A$ the category of $A$-modules, by $\Gamma_A$ the Auslander-Reiten quiver of $A$, and by $\tau_A$ the Auslander-Reiten translation in $\Gamma_A$.

In this article we are interested in geometric properties of modules with indecomposable summands in connected Auslander-Reiten components of a prescribed form. Let $A$ be an algebra with a basis $a_1 = 1, a_2, \ldots, a_n$ and the associated structure constants $a_{ijk}$. For any natural number $d$ we have the affine variety $\mathrm{mod}_A(d)$ of $d$-dimensional $A$-modules consisting in $n$-tuples $m = (m_1, \ldots, m_n)$ of $d \times d$ matrices with coefficients in $K$ such that $m_1$ is the identity matrix and $m_i m_j = \sum m_k a_{kij}$ for all indices $i$ and $j$. The general linear group $\mathrm{Gl}_d(K)$ acts on $\mathrm{mod}_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of $d$-dimensional $A$-modules (see [16]). We shall agree to identify a $d$-dimensional $A$-module $M$ with its isomorphism class, and with the point of $\mathrm{mod}_A(d)$ corresponding to it. Then one says that a module $M$ in $\mathrm{mod}_A(d)$ degenerates to a module $N$ in $\mathrm{mod}_A(d)$, and writes $M \leq_{\mathrm{deg}} N$, if the $\mathrm{Gl}_d(K)$-orbit $O(N)$ of $N$ is contained in the closure $\overline{O(M)}$ of the $\mathrm{Gl}_d(K)$-orbit $O(M)$ of $M$ in $\mathrm{mod}_A(d)$. Thus $\leq_{\mathrm{deg}}$ is a partial order on the set of isomorphism classes of $d$-dimensional $A$-modules. There has been an important

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work by S. Abeasis and A. del Fra [1], K. Bongartz [10], [11] and Ch. Riedtmann [21] connecting $\leq_{\text{deg}}$ with other partial orders $\leq_{\text{ext}}$, $\leq_{\text{virt}}$ and $\leq$ on the isomorphism classes in $\text{mod}_A(d)$ which are defined in terms of representation theory as follows:

- $M \leq_{\text{ext}} N$: $\Longleftrightarrow$ there are modules $M_i$, $U_i$, $V_i$ and short exact sequences $0 \longrightarrow U_i \longrightarrow M_i \longrightarrow V_i \longrightarrow 0$ in $\text{mod}\, A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number $s$.
- $M \leq_{\text{virt}} N$: $\Longleftrightarrow M \oplus X \leq_{\text{deg}} N \oplus X$ for some $A$-module $X$.
- $M \leq N$: $\Longleftrightarrow [X, M] \leq [X, N]$ holds for all modules $X$.

Here and later on we abbreviate $\dim_K \text{Hom}_A(X, Y)$ by $[X, Y]$. Then for modules $M$ and $N$ in $\text{mod}_A(d)$ the following implications hold:

$$M \leq_{\text{ext}} N \Rightarrow M \leq_{\text{deg}} N \Rightarrow M \leq_{\text{virt}} N \Rightarrow M \leq N$$

(see [10], [21]). Unfortunately, the reverse implications are not true in general, and it is interesting to find out when they are. This is the case for all modules over representation-finite algebras $A$ with $\Gamma_A$ directed, and hence for representations of Dynkin quivers [10], [11]. Finally, for a module $M$ in $\text{mod}\, A$, we shall denote by $[M]$ the image of $M$ in the Grothendieck group $K_0(A)$ of $A$. Thus $[M] = [N]$ if and only if $M$ and $N$ have the same simple composition factors including the multiplicities. Observe that, if $M$ and $N$ have the same dimension and $M \leq N$, then $[M] = [N]$.

We are interested in the following problem. Let $\mathcal{C}$ be a family of connected components of an Auslander-Reiten quiver $\Gamma_A$ and $\text{add}(\mathcal{C})$ the additive category of $\mathcal{C}$. We may ask when $M \leq_{\text{deg}} N$ for $M$ and $N$ in $\text{add}(\mathcal{C})$ with $[M] = [N]$? For preprojective components this problem has been investigated in [10]. In particular, it was shown in [10] that, for $\mathcal{C}$ preprojective, the partial orders $\leq_{\text{ext}}$ and $\leq$ coincides on $\text{add}(\mathcal{C})$. An important feature of preprojective components is that they consists of modules not lying on oriented cycles of nonzero nonisomorphisms between indecomposable modules (directing modules [22]), and hence such modules are uniquely determined (up to isomorphism) by their composition factors. Here, we are interested in degenerations of modules from $\text{add}(\mathcal{C})$ for connected components $\mathcal{C}$ of $\Gamma_A$ containing oriented cycles. Our interest in such components is motivated by a result due to L. Peng – J. Xie [19] and the first named author [25] saying that the Auslander-Reiten quiver $\Gamma_A$ of any algebra $A$ has at most finitely many $\tau_A$-orbits containing directing modules. A distinguished role in the representation theory is played by components consisting of $\tau_A$-periodic modules, called stable tubes (see [13], [14], [15], [22], [26]), that is, components of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$. In [14] d'Este and Ringel investigated components, called (coherent) tubes, which can be obtained from stable tubes by ray and coray insertions. In recent investigations of tame simply connected algebras appeared a natural generalization of the notion of tube called coil, introduced by I. Assem and the first named author in [3], [4]. Roughly speaking a coil is a translation quiver whose underlying topological space, modulo projective-injective points, is homeomorphic to a crowned cylinder. Special types of coils are quasi-tubes [24] whose underlying topological space, modulo projective-injective vertices, is homeomorphic to a tube. It is shown in [4] that coils can be obtained from stable tubes by a sequence of admissible operations. Moreover, it was shown in [29] (see also [28]) that a strongly simply connected algebra $A$ is (tame) of polynomial growth if and only if every nondirecting indecomposable $A$-module lies in a standard coil of a multicoil of $\Gamma_A$. We note also that quasi-tubes frequently appear in the Auslander-Reiten quivers of selfinjective algebras (see [24]). Recall that a component $\mathcal{C}$ of $\Gamma_A$ is called standard if the full subcategory of mod $A$ formed by modules from $\mathcal{C}$ is equivalent to the mesh-category $K(\mathcal{C})$ of $\mathcal{C}$ [12], [22].

Our first main result shows that the partial orders $\leq_{\text{ext}}$, $\leq_{\text{deg}}$, $\leq_{\text{virt}}$ and $\leq$ coincide on the additive categories of quasi-tubes.

THEOREM 1. *Let $A$ be an algebra, $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ be a family of pairwise orthogonal standard quasi-tubes in $\Gamma_A$, and $M$, $N$ modules in* add$(\mathcal{C})$ *with $[M] = [N]$. Then the following conditions are equivalent:*

*(i) $M \leq_{\text{ext}} N$,*

*(ii) $M \leq N$,*

*(iii) $[X, M] \leq [X, N]$ for all modules $X$ in $\mathcal{C}$.*

Note that the condition (iii) is rather easy to check, so the above theorem gives a handy criterion to decide when $N$ is a degeneration of $M$.

Our second theorem shows the convexity of the degenerations between modules from the additive categories of pairwise orthogonal standard quasi-tubes of $\Gamma_A$ in the lattices of all degenerations between $A$-modules of a given dimension.

THEOREM 2. *Let $A$ be an algebra and $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ a family of pairwise orthogonal standard quasi-tubes in $\Gamma_A$. Assume that $M$, $N$, $V$ are $A$-modules such that $[M] = [V] = [N]$, $M \leq_{\text{deg}} V \leq_{\text{deg}} N$, and $M$ and $N$ belong to* add$(\mathcal{C})$. *Then $V$ belongs to* add$(\mathcal{C})$.

It is well known that if $\mathcal{O}(M)$ is a $\mathrm{Gl}_d(K)$-orbit in $\mathrm{mod}_A(d)$ then the set $\overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)$ is a union of orbits of smaller dimension than $\dim \mathcal{O}(M)$, and $\dim \mathcal{O}(M) = \dim \mathrm{Gl}_d(K) - \dim \mathrm{Stab}_{\mathrm{Gl}_d(K)}(M) = d^2 - [M, M]$ (see [16]). Hence any chain of neighbours

$$M = M_0 <_{\text{deg}} M_1 <_{\text{deg}} \cdots <_{\text{deg}} M_r = N$$

in $\mathrm{mod}_A(d)$ has at most $[N, N] - [M, M]$ members (see also [10]). We shall now describe the minimal degenerations in the additive categories of quasi-tubes. With each coil $\Gamma$ one associates in [5] two numerical invariants $\big(p(\Gamma), q(\Gamma)\big)$ which measure respectively the number of rays and corays in $\Gamma$. For $\Gamma$ a quasi-tube, we define in Section 4 canonical short exact sequences

$$\Sigma(U, s, t) \colon 0 \longrightarrow U \longrightarrow E(U, s, t) \longrightarrow \varphi^{-s}\psi^{t} U \longrightarrow 0$$

with $U$ and $\varphi^{-s}\psi^{t}U$ indecomposable modules in $\Gamma$, and $s$ and $t$ measuring the size of the rectangle

$$\mathcal{R}(U, s, t) = \{\varphi^{-i}\psi^{j}U;\ 0 \leq i < s, 0 \leq j < t\}$$

determined by $U$ and $\tau_A V = \varphi^{-s+1}\psi^{t-1}U$. Then our next main result is as follows.

THEOREM 3. *Let $A$ be an algebra, $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ a family of pairwise orthogonal standard quasi-tubes in $\Gamma_A$, and $M$, $N$ modules in* add$(\mathcal{C})$ *with $[M] = [N]$. Then $N$ is a minimal degeneration of $M$ if and only if $M = E \oplus U^{m-1} \oplus V^{r-1} \oplus X$, $N = U^m \oplus V^r \oplus X$, $m, r \geq 1$, and the following conditions are satisfied:*

*(i) $U \oplus V$ and $E \oplus X$ have no common nonzero direct summands.*

*(ii) $U$ and $V$ are indecomposable modules lying in one quasi-tube $\Gamma = \mathcal{C}_{i_0}$ of $\mathcal{C}$.*

*(iii) There exists a canonical exact sequence*

$$0 \longrightarrow U \longrightarrow E(U,s,t) \longrightarrow \varphi^{-s}\psi^t U \longrightarrow 0$$

*with $E \simeq E(U,s,t)$, $V \simeq \varphi^{-s}\psi^t U$, and $s$, $t$ satisfying one of the following conditions:*

*(a) $s < p(\Gamma)$.*

*(b) $t < q(\Gamma)$.*

*(c) $s = p(\Gamma)$ and $t = kq(\Gamma)$, for some $k \geq 1$.*

*(d) $s = kp(\Gamma)$ and $t = q(\Gamma)$, for some $k \geq 1$.*

*(iv) Any common indecomposable direct summand $W \not\simeq \varphi^{-s}\psi^t U$ of $M$ and $N$ does not belong to the rectangle $\mathcal{R}(\tau_A^- U, s, t)$.*

*(v) Any common indecomposable direct summand $W \not\simeq U$ of $M$ and $N$ does not belong to the rectangle $\mathcal{R}(U,s,t)$.*

From the description of the exact sequences $\Sigma(U,s,t)$ given in Section 4 we then get the following fact (*cf.* [11, Lemma 5]).

COROLLARY 1. *Let $A$ be an algebra, $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ a family of pairwise orthogonal standard quasi-tubes in $\Gamma_A$, and $M$, $N$ modules in* add$(\Gamma)$ *with $[M] = [N]$ and without common nonzero direct summands. If there is a minimal degeneration $M <_{\deg} N$, then no indecomposable direct summand $X$ occurs twice in $M$.*

For coils which are not quasi-tubes we shall prove the following fact.

THEOREM 4. *Let $A$ be an algebra and $\mathcal{C}$ a standard coil of $\Gamma_A$ which is not a quasi-tube. Then there exist indecomposable modules $M$ and $N$ in $\mathcal{C}$ such that $[M] = [N]$ and $M <_{\deg} N$.*

As a direct consequence of Theorems 1 and 4 we get the following corollary.

COROLLARY 2. *Let $A$ be an algebra and $\mathcal{C}$ a standard coil in $\Gamma_A$. Then $\mathcal{C}$ is a quasi-tube if and only if, for any $M$ and $N$ in* add$(\mathcal{C})$ *with $[M] = [N]$, $M \leq_{\deg} N$ implies $M \leq_{\mathrm{ext}} N$.*

The paper is organized as follows. In Section 2 we fix the notation, recall the relevant definitions and facts, and prove some preliminary results on modules which we apply in our investigations. Section 3 is devoted to coils and their construction from stable tubes by admissible operations. We prove also there that the additive category add$(\Gamma)$ of a standard coil $\Gamma$ of an Auslander-Reiten quiver $\Gamma_A$ is closed under extensions. In Section 4 we prove several facts on additive functions determined by short exact sequences in the additive categories of standard quasi-tubes. Sections 5, 6 and 7 are devoted to the proofs of Theorems 1 and 2, 3, and 4, respectively.

For a basic background on the topics considered here we refer to [11], [16], [22] and [26].

## 2. **Preliminaries on modules.**

2.1. Throughout the paper $A$ denotes a fixed finite dimensional associative $K$-algebra with an identity over an algebraically closed field $K$. We denote by $\operatorname{mod} A$ the category of finite dimensional right $A$-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by indecomposable modules, by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$, and by $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^{i}(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. By an $A$-module is meant an object from $\operatorname{mod} A$. Further, we denote by $\Gamma_A$ the Auslander-Reiten quiver of $A$ and by $\tau = \tau_A$ and $\tau^- = \tau_A^-$ the Auslander-Reiten translations $D\mathrm{Tr}$ and $\mathrm{TrD}$, respectively. We shall agree to identify the vertices of $\Gamma_A$ with the corresponding indecomposable modules. For $M$ in $\operatorname{mod} A$ we denote by $[M]$ the image of $M$ in the Grothendieck group $K_0(A)$ of $A$. Further, for $X$, $Y$ from $\operatorname{mod} A$ we abbreviate $\dim_K \operatorname{Hom}_A(X, Y)$ by $[X, Y]$. Finally, for a family $\Gamma$ of $A$-modules, we denote by $\operatorname{add}(\Gamma)$ the additive category given by $\Gamma$, that is, the full subcategory of $\operatorname{mod} A$ formed by all modules isomorphic to the direct sums of modules from $\Gamma$.

2.2 Following [21], for $M, N$ from $\operatorname{mod} A$, we set $M \leq N$ if and only if $[X, M] \leq [X, N]$ for all $A$-modules $X$. The fact that $\leq$ is a partial order on the isomorphism classes of $A$-modules follows from a result by M. Auslander (see [6], [9]). M. Auslander and I. Reiten have shown in [7] that, if $[M] = [N]$ for $A$-modules $M$ and $N$, then for all nonprojective indecomposable $A$-modules $X$ and all noninjective indecomposable modules $Y$ the following formulas hold:

$$[X,M] - [M,\tau X] = [X,N] - [N,\tau X],$$
$$[M,Y] - [\tau^- Y,M] = [N,Y] - [\tau^- Y,N].$$

Hence, if $[M] = [N]$, then $M \leq N$ if and only if $[M,X] \leq [N,X]$ for all $A$-modules $X$.

2.3. Let $M$ and $N$ be $A$-modules with $[M] = [N]$ and

$$\Sigma\colon 0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0$$

an exact sequence in $\operatorname{mod} A$. Following [21] we define the additive functions $\delta_{M,N}$, $\delta'_{M,N}$, $\delta_\Sigma$ and $\delta'_\Sigma$ for an $A$-module $X$ as follows:

$$\delta_{M,N}(X) = [N,X] - [M,X]$$
$$\delta'_{M,N}(X) = [X,N] - [X,M]$$
$$\delta_\Sigma(X) = \delta_{E,D\oplus F}(X) = [D \oplus F, X] - [E,X]$$
$$\delta'_\Sigma(X) = \delta'_{E,D\oplus F}(X) = [X, D \oplus F] - [X,E]$$

From the Auslander-Reiten formulas (2.2) we get the following very useful equalities

$$\delta_{M,N}(X) = \delta'_{M,N}(\tau^- X), \quad \delta_{M,N}(\tau X) = \delta'_{M,N}(X)$$

and

$$\delta_\Sigma(X) = \delta'_\Sigma(\tau^- X), \quad \delta_\Sigma(\tau X) = \delta'_\Sigma(X)$$

for all $A$-modules $X$. Observe also that $\delta_{M,N}(I) = 0$ for any injective $A$-module $I$, and $\delta'_{M,N}(P) = 0$ for any projective $A$-module $P$. In particular, we get that the following conditions are equivalent:

(1) $M \leq N$.
(2) $\delta_{M,N}(X) \geq 0$ for all $X \in \operatorname{ind} A$.
(3) $\delta'_{M,N}(X) \geq 0$ for all $X \in \operatorname{ind} A$.

2.4 For an $A$-module $M$ and an indecomposable $A$-module $Z$, we denote by $\mu(M, Z)$ the multiplicity of $Z$ as a direct summand of $M$. For a noninjective indecomposable $A$-module $U$ we denote by $\Sigma(U)$ an Auslander-Reiten sequence

$$\Sigma(U)\colon 0 \longrightarrow U \longrightarrow E(U) \longrightarrow \tau^- U \longrightarrow 0,$$

and define $\pi(U)$ to be the unique indecomposable projective-injective direct summand of $E(U)$, if such a summand exists, or 0 otherwise.

We shall need the following lemmas.

LEMMA 2.5. *Let $G$ be an $A$-module and $U$ an indecomposable $A$-module. Then*
*(i) If $U$ is noninjective, then $\delta_{\Sigma(U)}(G) = \mu(G, U)$.*
*(ii) If $U$ is nonprojective, then $\delta'_{\Sigma(\tau U)}(G) = \mu(G, U)$.*

PROOF. (i) The Auslander-Reiten sequence $\Sigma(U)$ induces an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(\tau^- U, G) \longrightarrow \operatorname{Hom}_A(E(U), G) \longrightarrow \operatorname{rad}(U, G) \longrightarrow 0,$$

and hence we get that

$$\delta_{\Sigma(U)}(G) = [U \oplus \tau^- U, G] - [E(U), G] = [U, G] - \dim_K \operatorname{rad}(U, G) = \mu(G, U)$$

(ii) The Auslander-Reiten sequence $\Sigma(\tau U)$ induces an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(G, \tau U) \longrightarrow \operatorname{Hom}_A\big(G, E(\tau U)\big) \longrightarrow \operatorname{rad}(G, U) \longrightarrow 0$$

and hence we get the equalities

$$\delta'_{\Sigma(\tau U)}(G) = [G, \tau U \oplus U] - [G, E(\tau U)] = [G, U] - \dim_K \operatorname{rad}(G, U) = \mu(G, U)$$

LEMMA 2.6. *Let $\Gamma$ be a standard component of $\Gamma_A$ and assume that there exists in $\Gamma$ a mesh-complete subquiver of the form*

$$\begin{array}{ccccccccc}
U_1 & \longrightarrow & U_2 & \longrightarrow \cdots \longrightarrow & U_i & \longrightarrow & U_{i+1} & \longrightarrow \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & \\
V_1 & \longrightarrow & V_2 & \longrightarrow \cdots \longrightarrow & V_i & \longrightarrow & V_{i+1} & \longrightarrow \cdots
\end{array}$$

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*with all $U_i$, $V_i$, $i \geq 1$, pairwise nonisomorphic. Then for any $Z \in \mathrm{add}(\Gamma)$ the following equality holds*

$$[V_1, Z] - [U_1, Z] = \sum_{i \geq 1} \mu(Z, V_i)$$

PROOF. Since $\Gamma$ is standard there exist irreducible maps $f_i \colon V_i \longrightarrow V_{i+1}$, $g_i \colon U_i \longrightarrow U_{i+1}$, $h_i \colon V_i \longrightarrow U_i$, $i \geq 1$, such that $g_i h_i = h_{i+1} f_i$ for all $i \geq 1$. Moreover, by [18], for any indecomposable modules $X$ and $Y$ in $\Gamma$, $\mathrm{rad}^\infty(X, Y) = 0$ ($\Gamma$ is generalized standard in the sense of [27]), and hence any nonzero morphism in $\mathrm{rad}(X, Y)$ is a linear combination of the composites of irreducible morphisms between indecomposable modules in $\Gamma$. Clearly, in order to prove the lemma, we may consider an indecomposable module $Z$ in $\Gamma$. First observe that the induced map $\mathrm{Hom}_A(h_1, Z) \colon \mathrm{Hom}_A(U_1, Z) \longrightarrow \mathrm{Hom}_A(V_1, Z)$ is a monomorphism. Indeed, take a nonzero map $w$ in $\mathrm{Hom}_A(U_1, Z)$. Then by the above remarks there exists $r \geq 0$ such that $w \in \mathrm{rad}^r(U_1, Z) \setminus \mathrm{rad}^{r+1}(U_1, Z)$. Applying now the dual of Corollary 1.6 in [17], we get that $h_1 \colon V_1 \longrightarrow U_1$ is of infinite right degree, and consequently $wh_1 \in \mathrm{rad}^{r+1}(V_1, Z) \setminus \mathrm{rad}^{r+2}(V_1, Z)$. In particular, $wh_1 \neq 0$ and we are done. Further, we know that any irreducible map $V_i \longrightarrow W$ with $W$ indecomposable is of the form $\alpha f_i + \varphi$, $\varphi \in \mathrm{rad}^2(V_i, V_{i+1})$, or $\alpha h_i + \psi$, $\psi \in \mathrm{rad}^2(V_i, U_i)$, for some $\alpha \in K$. Hence, if $Z \not\simeq V_i$, for any $i \geq 1$, then using the equalities $g_i h_i = h_{i+1} f_i$ we get that the map $\mathrm{Hom}_A(h_1, Z)$ is an isomorphism. Then

$$[V_1, Z] - [U_1, Z] = 0 = \sum_{i \geq 1} \mu(Z, V_i).$$

Assume $Z = V_j$ for some $j \geq 1$. Then we get

$$\mathrm{Hom}_A(V_1, Z) = \mathrm{im}\, \mathrm{Hom}_A(h_1, Z) + K f_{j-1} \cdots f_1$$

where, in case $j = 1$, $f_0$ is the identity map $V_1 \longrightarrow V_1$. Moreover, by [8], $f_{j-1} \cdots f_1$ does not belong to $\mathrm{im}\, \mathrm{Hom}_A(h_1, Z)$, because $\tau^- V_i = U_{i+1} \not\simeq V_{i+2}$ for any $i \geq 1$. Therefore, we get

$$[V_1, Z] - [U_1, Z] = 1 = \mu(Z, V_j) = \sum_{i \geq 1} \mu(Z, V_i)$$

because the modules $V_1, V_2, \ldots$ are pairwise nonisomorphic.

LEMMA 2.7. *Let $\Gamma_A = \Gamma' \cup \Gamma''$ be a decomposition of $\Gamma_A$ into a disjoint sum of connected components. Assume that $M$ and $N$ are $A$-modules such that $[M] = [N]$ and $\delta_{M,N}(X) = 0$ for all $X \in \mathrm{add}(\Gamma')$. Then the following statements hold:*

*(i) If $M, N \in \mathrm{add}(\Gamma')$ then $M \simeq N$.*

*(ii) $M \in \mathrm{add}(\Gamma'')$ if and only if $N \in \mathrm{add}(\Gamma'')$.*

PROOF. Since each $X \in \mathrm{mod}\, A$ has a decomposition $X = X' \oplus X''$ with $X' \in \mathrm{add}(\Gamma')$ and $X'' \in \mathrm{add}(\Gamma'')$ it is sufficient to prove that $\mu(M, U) = \mu(N, U)$ for any indecomposable module $U$ in $\Gamma'$. Take an indecomposable module $U$ in $\Gamma'$. Assume first that $U$ is not

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projective. Then by our assumption and Lemma 2.5(ii) we get the equalities

$$\begin{aligned}\mu(N,U)-\mu(M,U) &= \delta'_{\Sigma(\tau U)}(N)-\delta'_{\Sigma(\tau U)}(M)\\ &= [N,\tau U\oplus U]-[N,E(\tau U)]-[M,\tau U\oplus U]+[M,E(\tau U)]\\ &= \delta_{M,N}(\tau U)+\delta_{M,N}(U)-\delta_{M,N}\big(E(\tau U)\big)=0\end{aligned}$$

because $U,\tau U$, and $E(\tau U)$ belong to add$(\Gamma')$. Assume now that $U$ is projective. Then we get the equalities

$$\mu(M,U)=[M,U]-[M,\operatorname{rad}U]=[N,U]-[N,\operatorname{rad}U]=\mu(N,U)$$

because $\operatorname{rad}U\in\operatorname{add}(\Gamma')$ and $\delta_{M,N}(U)=0$, $\delta_{M,N}(\operatorname{rad}U)=0$. This finishes the proof.

2.8. Let $\Gamma$ be a connected component of $\Gamma_A$. For modules $M$ and $N$ in add$(\Gamma)$ we set

$$M\le_\Gamma N\iff [X,M]\le[X,N]\text{ for all modules }X\text{ in }\operatorname{add}(\Gamma).$$

Clearly, $M\le N$ implies $M\le_\Gamma N$. The following direct consequence of the above lemma shows that $\le_\Gamma$ is a partial order on the isomorphism classes of modules in add$(\Gamma)$ having the same composition factors.

COROLLARY. *Let $M$ and $N$ be two modules in* add$(\Gamma)$ *such that* $[M]=[N]$. *Then $M\simeq N$ if and only if $M\le_\Gamma N$ and $N\le_\Gamma M$.*

Moreover, if $M$ and $N$ belongs to add$(\Gamma)$ and $[M]=[N]$ then the following conditions are equivalent (see (2.3)):

(1) $M\le_\Gamma N$.
(2) $\delta_{M,N}(X)\ge 0$ for all modules $X$ in $\Gamma$.
(3) $\delta'_{M,N}(X)\ge 0$ for all modules $X$ in $\Gamma$.

3. **Coils.** We shall recall some basic facts on coils introduced by I. Assem and the first named author in [3] (see also [4]) and prove that the additive categories of standard coils are closed under extensions.

3.1. A translation quiver $\Gamma$ is called a *tube* [14], [22] if it contains a cyclical path and its underlying topological space is homeomorphic to $S^1\times\mathbb{R}^+$ (where $S^1$ is the unit circle, and $\mathbb{R}^+$ the non-negative real half-line). Tubes containing neither projective vertices nor injective vertices are called stable. The *rank* of a stable tube $\Gamma$ is the least positive integer such that $\tau^r X=X$ for all $X\in\Gamma$.

3.2. The one-point extension of an algebra $B$ by a $B$-module $X$ is the matrix algebra

$$B[X]=\begin{bmatrix}K & X\\ 0 & B\end{bmatrix}$$

with the usual addition and multiplication of matrices. The $B[X]$-modules are usually identified with the triples $(V,M,\varphi)$, where $V$ is a $K$-vector space, $M$ is a $B$-module and $\varphi\colon V\longrightarrow \operatorname{Hom}_A(X,M)$ is a $K$-linear map. A $B[X]$-linear map $(V,M,\varphi)\longrightarrow(V',M',\varphi')$ is then identified with a pair $(f, g)$, where $f\colon V \longrightarrow V'$ is $K$-linear, $g\colon M \longrightarrow M'$ is $B$-linear and $\varphi' f = \mathrm{Hom}_B(X, g)\varphi$. One defines dually the one-point coextension $[X]B$ of $B$ by $X$ (see [22]).

3.3. A coil is a translation quiver constructed inductively from a stable tube by a sequence of operations called admissible. Our first task is to define the latter. Let $B$ be an algebra and $\Gamma$ be a standard component of $\Gamma_B$. Recall that $\Gamma$ is called *standard* if the full subcategory of mod $B$ formed by modules from $\Gamma$ is equivalent to the mesh-category $K(\Gamma)$ of $\Gamma$ (see [22]). For an indecomposable module $X$ in $\Gamma$, the support $S(X)$ of the functor $\mathrm{Hom}_B(X, -)|_\Gamma$ is the factor category of $K(\Gamma)$ by the ideal $I_X$ of $K(\Gamma)$ generated by all morphisms $f\colon M \longrightarrow N$ such that $\mathrm{Hom}_B(X, f) = 0$. For an indecomposable module $X$ in $\Gamma$, called the pivot, one defines admissible operations (ad 1), (ad 2), (ad 3) and their duals (ad 1*), (ad 2*), (ad 3*), modifying $(\Gamma, \tau)$ to a new translation quiver $(\Gamma', \tau')$, depending on the shape of the support $S(X)$.

(ad 1) Assume that $S(X)$ is the $K$-linear category of an infinite sectional path starting at $X$:

$$X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

In this case, we let $t \geq 1$ be a positive integer, $D$ denote the full $t \times t$-lower triangular matrix algebra and $Y_1, \ldots, Y_t$ denote the indecomposable injective $D$-modules with $Y = Y_1$ the unique indecomposable projective-injective module. We define the modified algebra $B'$ of $B$ to be the one-point extension

$$B' = [B \times D][X \oplus Y]$$

and the modified component $\Gamma'$ of $\Gamma$ to be obtained by inserting in $\Gamma$ a rectangle consisting of the modules $Z_{ij} = (K, X_i \oplus Y_j, \binom{1}{1})$ for $i \geq 0$, $1 \leq j \leq t$, and $X_i' = (K, X_i, 1)$ for $i \geq 0$. The translation $\tau'$ of $\Gamma'$ is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1, j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geq 2$, $Z_{01} = P$ is projective, $\tau' X_0' = Y_t$, $\tau' X_i' = Z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau^- X_i) = X_i'$ provided $X_i$ is not an injective $B$-module, otherwise $X_i$ is injective in $\Gamma'$. For the remaining vertices of $\Gamma$ (or $\Gamma_D$), the translation $\tau'$ coincides with $\tau$ (or $\tau_D$, respectively).

If now $t = 0$, we define the modified algebra $B'$ to be the one-point extension $B' = B[X]$ and the modified component $\Gamma'$ to be the component obtained from $\Gamma$ by inserting only the sectional path consisting of the $X_i'$, $i \geq 0$.

(ad 2) Assume $S(X)$ is the $K$-linear category given by two sectional paths starting at $X$, the first infinite and the second finite with at least one arrow

$$Y_t \longleftarrow \cdots \longleftarrow Y_2 \longleftarrow Y_1 \longleftarrow X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

where $t \geq 1$. In particular, $X$ is necessarily injective. We define the modified algebra $B'$ of $B$ to be the one-point extension $B' = B[X]$ and the modified component $\Gamma'$ of $\Gamma$ to be obtained by inserting in $\Gamma$ a rectangle consisting of the modules $Z_{ij} = (K, X_i \oplus Y_j, \binom{1}{1})$ for $i \geq 1$, $1 \leq j \leq t$, and $X_i' = (K, X_i, 1)$ for $i \geq 1$. The translation $\tau'$ of $\Gamma'$ is defined as follows: $P = X'_0$ is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{1j} = Y_{j-1}$ if $j \geq 2$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq 2$, $\tau' X'_1 = Y_t$, $\tau'(\tau^- X_i) = X'_i$ if $i \geq 1$, provided $X_i$ is not injective $B$-module, otherwise $X'_i$ is injective in $\Gamma'$. For the remaining vertices of $\Gamma'$, the translation $\tau'$ coincides with the translation $\tau$.

(ad 3) Assume $S(X)$ is the mesh-category of two parallel sectional paths

$$\begin{array}{ccccccccccc}
Y_1 & \longrightarrow & Y_2 & \longrightarrow \cdots \longrightarrow & Y_t & & & & & & \\
\uparrow & & \uparrow & & \uparrow & & & & & & \\
X = X_0 & \longrightarrow & X_1 & \longrightarrow \cdots \longrightarrow & X_{t-1} & \longrightarrow & X_t & \longrightarrow & X_{t+1} & \longrightarrow & \cdots
\end{array}$$

where $t \geq 2$. In particular, $X_{t-1}$ is necessarily injective. We define the modified algebra $B'$ of $B$ to be the one-point extension $B' = B[X]$ and the modified component $\Gamma'$ to be obtained by inserting in $\Gamma$ a rectangle consisting of the modules $Z_{ij} = (K, X_i \oplus Y_j, \binom{1}{1})$ for $i \geq 1$, $1 \leq j \leq i$, and $X'_i = (K, X_i, 1)$ for $i \geq 1$. The translation $\tau'$ of $\Gamma'$ is defined as follows: $P = X'_0$ is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $2 \leq j \leq i$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' X'_i = Y_i$ if $1 \leq i \leq t$, $\tau' X'_i = Z_{i-1,t}$ if $i > t$, $\tau' Y_j = X'_{j-2}$ if $2 \leq j \leq t$, $\tau'(\tau^- X_i) = X'_i$ if $i \geq t$ provided $X_i$ is not an injective $B$-module, otherwise $X'_i$ is injective in $\Gamma'$. For the remaining vertices of $\Gamma'$, the translation $\tau'$ coincides with $\tau$. We note that $X'_{t-1}$ is injective.

Finally, together with each of the admissible operations (ad 1), (ad 2) and (ad 3), we must consider its dual, denoted by (ad 1*), (ad 2*) and (ad 3*), respectively.

3.4. A translation quiver $\Gamma$ is called a *coil* if there exists a sequence of algebras $B_0, B_1, \ldots, B_m = \Lambda$ and components $\Gamma_i$ of $\Gamma_{B_i}$; $0 \leq i \leq m$, such that $\Gamma = \Gamma_m$, $\Gamma_0$ is a standard stable tube, and for each $i$ $(0 \leq i < m)$, $B_{i+1}$ is the modified algebra $B_i$ of $B_i$ and $\Gamma_{i+1}$ is the modified component of $\Gamma_i$, by one of the admissible operations (ad 1), (ad 2), (ad 3), (ad 1*), (ad 2*), or (ad 3*). It is shown in [3] that such a coil $\Gamma$ is a standard component of $\Gamma_\Lambda$. We refer to [4] for an axiomatic definition of a coil and examples. Hence any stable tube is trivially a coil. A (*coherent*) *tube* in the sense of [14] is a coil having the property that each admissible operation in the sequence defining it is of the form (ad 1) or (ad 1*). If we apply only operations of the type (ad 1) (respectively, of the type (ad 1*)) then such a coil is called a *ray tube* (respectively, *coray tube*). Observe that a coil without injective (respectively, projective) vertices is a ray tube (respectively, coray tube). A *quasi-tube* (in the sense of [24]) is a coil having the property that each admissible operation in the sequence defining it is of the form (ad 1), (ad 1*), (ad 2) or (ad 2*). The quasi-tubes occur frequently in the Auslander-Reiten quiver of selfinjective algebras (see [24]). Note that a coil $\Gamma$ in the Auslander-Reiten quiver $\Gamma_A$ of an arbitrary algebra $A$ is not necessarily standard. But for any coil $\Gamma$ there exists a triangular algebra $\Lambda$ (and hence of finite global dimension) such that $\Gamma$ is a standard component of $\Gamma_\Lambda$. We shall show now that the additive categories of standard coils are closed under extensions.

PROPOSITION 3.5. *Let $B$ be an algebra, $\Gamma$ a standard component of $\Gamma_B$, and assume that* add($\Gamma$) *is closed under extensions. Let $X$ be the pivot of an admissible operation,*

*$B'$ the modified algebra, and $\Gamma'$ the modified component. Then* add($\Gamma'$) *is closed under extensions.*

PROOF. We may assume, by duality, that the admissible operation leading from $\Gamma$ to $\Gamma'$ is one of (ad 1), (ad 2), or (ad 3). For a $B$-module $M$, we let $M_0$ denote its restriction to $B\times D$, if the operation is of type (ad 1) with $t \geq 1$, or to $B$ in the remaining cases. Denoting by $\omega$ the extension vertex of $B'$, we represent a $B'$-module $M$ as a triple $(M_\omega, M_0, \gamma_M)$, where $M_\omega$ is a finite dimensional $K$-vector space and $\gamma_M$ is a $K$-linear map from $M_\omega$ to $\mathrm{Hom}_{B\times D}(X\oplus Y_1, M_0)$ or to $\mathrm{Hom}_B(X, M_0)$, respectively. Let now

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0$$

be an exact sequence in mod $B'$ with $M$ and $N$ in add($\Gamma'$). Clearly, we may assume that this sequence is not splittable. We get an exact sequence

$$0 \longrightarrow M_0 \longrightarrow E_0 \longrightarrow N_0 \longrightarrow 0$$

in mod $B$ with $M_0$ and $N_0$ in add($\Gamma$). Since add($\Gamma$) is closed under extensions, we infer that $E_0 \in$ add($\Gamma$). From the description of admissible operations in (3.3) we know that the vector space category $\mathrm{Hom}_{B\times D}\big(X\oplus Y_1, \mathrm{add}(\Gamma)\big)$, if the admissible operation is of type (ad 1) and $t \geq 1$, and $\mathrm{Hom}_B\big(X, \mathrm{add}(\Gamma)\big)$ in the remaining cases, is given by a partially ordered set of width at most 2. Then, since $E_0 \in$ add($\Gamma$), the indecomposable direct summands of $E$ are of the form $(0, Z, 0)$ with $Z$ an indecomposable $B$-module lying in $\Gamma'$ (and hence in $\Gamma$), $(K, X_i \oplus Y_j, \binom{1}{1})$, $(K, X_i, 1)$ or $(K, Y_j, 1)$ (see [23, (2.4)] for details). Therefore, we must show that $E$ has no direct summand of the form $(K, Y_j, 1)$. Suppose this is not the case. Then there is a nonzero map from a module $(K, Y_j, 1)$ to an indecomposable direct summand, say $V$, of $N$. By our assumption, $V$ belongs to $\Gamma'$. Observe now that any indecomposable $B$-module $U$ in $\Gamma'$ with $\mathrm{Hom}_A(Y_j, U) \neq 0$ is isomorphic to $Y_l$ with $l \geq j$. Since the modules $(K, Y_l, 1)$ do not belong to $\Gamma'$, $V$ is isomorphic to a module of the form $(0, Y_l, 0)$ or $(K, X_i \oplus Y_l, \binom{1}{1})$. But it is easy to check that any map in mod $B'$ from $(K, Y_j, 1)$ to any of the modules $(0, Y_l, 0)$ or $(K, X_i \oplus Y_l, \binom{1}{1})$ is zero. Consequently, $E$ belongs to add($\Gamma'$). This shows that add($\Gamma'$) is closed under extensions.

THEOREM 3.6. *Let $A$ be an algebra and $\Gamma$ a standard coil of $\Gamma_A$. Then* add($\Gamma$) *is closed under extensions.*

PROOF. Let $I = \mathrm{ann}(\Gamma)$ be the annihilator of $\Gamma$ in $A$, that is, the intersection of the annihilators ann $X$ of the modules $X$ in $\Gamma$, and $B = A/I$. Clearly, $\Gamma$ is a standard coil in $\Gamma_B$. Moreover, if $0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0$ is an exact sequence in mod $A$ with $M$ and $N$ in add($\Gamma$) then $MI = 0, NI = 0$, and so $EI = 0$. Therefore, we may assume that $B = A$, that is, $\Gamma$ is a faithful standard coil of $\Gamma_A$. Repeating now the arguments from [4, (5.4)] we infer that there exists a sequence of algebras $C = A_0, A_1, \ldots, A_m = A$ and a standard faithful stable tube $\mathcal{T}$ in $\Gamma_C$ such that, for each $0 \leq i < m$, $A_{i+1}$ is obtained from the algebra $A_i$ by an admissible operation with pivot in the coil $\Gamma_i$ of $\Gamma_{A_i}$, obtained from the stable tube $\mathcal{T}$ by the sequence of admissible operations done so far, and $\Gamma$ is the modified coil $\Gamma_m = \Gamma'_{m-1}$. Hence, by Proposition 3.5, it is sufficient to show that add($\mathcal{T}$) is closed under extensions in mod $C$. Since $\mathcal{T}$ is a faithful standard (hence generalized standard) stable tube of $\Gamma_C$, we infer that $\mathrm{pd}_C X \leq 1$ for any $X$ in $\mathcal{T}$ (see [27, (5.9)]). Let $E_1, \ldots, E_r$ be a complete set of modules lying on the mouth of $\mathcal{T}$. Then the modules $E_1, \ldots, E_r$ are pairwise orthogonal with endomorphism rings isomorphic to $K$ (because $\mathcal{T}$ is standard), and $\mathrm{Ext}^2_C(E_i, E_j) = 0$ for all $1 \leq i, j \leq r$. Then by [22, (3.1)], add($\mathcal{T}$) is a serial abelian category consisting of all $C$-modules $X$ having a filtration

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_s = 0, \quad s \geq 1,$$

with $X_{i-1}/X_i$ being isomorphic to one of $E_1, \ldots E_r$, for any $1 \leq i \leq s$. But then add($\mathcal{T}$) is closed under extensions, and we are done.

## 4. **Exact sequences in quasi-tubes.**

4.1. Throughout this section $\Gamma$ denotes a standard quasi-tube in the Auslander-Reiten quiver $\Gamma_A$ of an algebra $A$. We shall investigate short exact sequences in the additive category add($\Gamma$) in mod $A$ given by $\Gamma$. Since $\Gamma$ is standard, add($\Gamma$) is equivalent to the additive category $\mathrm{add}\big(K(\Gamma)\big)$ of the mesh-category $K(\Gamma)$ of $\Gamma$. Hence we may assume that $\Gamma$ is a sincere quasi-tube in $\Gamma_A$, $A$ is obtained from an algebra $C$ by a sequence of admissible operations of type (ad 1), (ad 1*), (ad 2), (ad 2*), and $\Gamma$ is obtained from a sincere standard stable tube $\mathcal{T}$ of $\Gamma_C$ by the same sequence of admissible operations. By $\bar{\Gamma}$ we denote the translation quiver obtained from $\Gamma$ by removing all projective-injective vertices. Hence, $\bar{\Gamma}$ is a tube. A vertex $X$ of $\bar{\Gamma}$ will be said to belong to the mouth of $\bar{\Gamma}$ if $X$ is starting, or ending, vertex of a mesh in $\bar{\Gamma}$ with a unique middle term. The arrows of $\bar{\Gamma}$ may be subdivided into two classes: arrows pointing to the mouth and arrows pointing to infinity (from the mouth). Denote by $\bar{\Gamma}_0$ the set of vertices in $\bar{\Gamma}$. We define two functions

$$\varphi, \psi \colon \bar{\Gamma}_0 \cup \{0\} \longrightarrow \bar{\Gamma}_0 \cup \{0\}$$

such that: $\varphi(0) = 0$, $\psi(0) = 0$, and for $X \in \bar{\Gamma}_0$:

- $\varphi(X)$ is the starting vertex of a (unique) arrow with end vertex $X$ and pointing to the mouth, if such an arrow exists, and $\varphi(X) = 0$ otherwise;
- $\psi(X)$ is the ending vertex of a (unique) arrow with starting vertex $X$ and pointing to infinity, if such an arrow exists, and $\psi(X) = 0$ otherwise.

In an obvious way we define also partial inverse functions

$$\varphi^-, \psi^- : \bar{\Gamma}_0 \cup \{0\} \longrightarrow \bar{\Gamma}_0 \cup \{0\}$$

such that for $X \in \bar{\Gamma}_0$ we have:

- $\varphi^-(X) = Y$ if $\varphi(Y) = X$, and $\varphi^-(X) = 0$ otherwise;
- $\psi^-(X) = Y$ if $\psi(Y) = X$, and $\psi^-(X) = 0$ otherwise.

Recall also that an infinite sectional path in $\bar{\Gamma}$ starting from a module lying on the mouth of $\bar{\Gamma}$ and consisting of arrows pointing to infinity is called a *ray*. Dually, an infinite path in $\bar{\Gamma}$ with the ending module lying on the mouth of $\bar{\Gamma}$ and consisting of arrows pointing to the mouth is called a *coray* (see [22]). Then one associates two numerical invariants $\big(p(\Gamma), q(\Gamma)\big)$ such that $p(\Gamma)$ is the number of rays in $\bar{\Gamma}$ and $q(\Gamma)$ is the number of corays in $\bar{\Gamma}$. We shall use the abbreviation $p = p(\Gamma)$ and $q = q(\Gamma)$. Finally, observe that a module $X \in \bar{\Gamma}_0$ lies on a ray (respectively, coray) in $\bar{\Gamma}$ if and only if $\psi^i(X) \neq 0$ (respectively, $\varphi^i(X) \neq 0$) for all $i \geq 0$.

4.2 Following [20] by a *short cycle* in add($\Gamma$) we mean a cycle $X \longrightarrow Y \longrightarrow X$ of nonzero nonisomorphisms between modules $X$ and $Y$ from $\Gamma$. We collect now the following properties of $\varphi$ and $\psi$, needed in our proofs.

LEMMA. *Let $X$ be an indecomposable module in $\bar{\Gamma}$. Then the following statements hold:*

*(i) $X$ lies on a short cycle in* add($\Gamma$) *if and only if $X$ lies on a ray and on a coray in $\bar{\Gamma}$. Moreover, if this is the case, then $\varphi^p X = \psi^q X$ and there is a cycle $X \longrightarrow \psi X \longrightarrow \cdots \longrightarrow \psi^q X = \varphi^p X \longrightarrow \cdots \longrightarrow \varphi X \longrightarrow X$.*

*(ii) $X$ lies on a short cycle in* add($\Gamma$) *if and only if $\varphi^{p-1}X \neq 0$ and $\psi^{q-1}X \neq 0$.*

*(iii) If $X$ lies on a short cycle in* add($\Gamma$) *then, for any integers $i, j, k \geq 0$, $\varphi^i\psi^j X = \psi^j\varphi^i X = \varphi^{i-kp}\psi^{j+kq}X$ lies on a short cycle.*

*(iv) If $\varphi^i\psi^j X = X$ or $\psi^j\varphi^i X = X$ then there is an integer $k$ such that $i = kp$ and $j = (-k)q$.*

Assume that $U$ is a module in $\bar{\Gamma}$ and $s, t$ are two positive integers such that the modules $\varphi^{-i}\psi^j U, 0 \leq i < s, 0 \leq j < t$, are nonzero. Then

$$\mathcal{R}(U,s,t) = \{\varphi^{-i}\psi^j U; 0 \leq i < s, 0 \leq j < t\}$$

is called a *rectangle of size* $(s,t)$ in $\bar{\Gamma}$ determined by $U$.

4.3. Let $\Gamma_0$ be the set of vertices in $\Gamma$. For any noninjective vertex $U \in \Gamma_0$ we have in the notation of (2.4) an Auslander-Reiten sequence

$$\Sigma(U): 0 \longrightarrow U \longrightarrow E(U) \longrightarrow \tau^- U \longrightarrow 0$$

where $E(U) = \pi(U) \oplus \psi(U) \oplus \varphi^-(U)$, and $\psi(U) \neq 0$.

LEMMA. *Let $U \in \Gamma_0$, $s, t \geq 1$ be integers, and assume that there exists in $\Gamma$ a rectangle $\mathcal{R} = \mathcal{R}(U,s,t)$ consisting of nonzero and noninjective modules. Then*

*(i) There exists a nonsplittable exact sequence*

$$\Sigma(U,s,t): 0 \longrightarrow U \longrightarrow E(U,s,t) \longrightarrow \varphi^{-s}\psi^t U \longrightarrow 0,$$

*where*

$$E(U,s,t) = \psi^t U \oplus \varphi^{-s} U \oplus \left( \bigoplus_{0 \leq i < s} \bigoplus_{0 \leq j < t} \pi(\varphi^{-i}\psi^j U) \right).$$

*(ii)* $\delta_{\Sigma(U,s,t)} = \sum_{0 \leq i < s} \sum_{0 \leq j < t} \delta_{\Sigma(\varphi^{-i}\psi^j U)}$.

*(iii)* $\delta_{\Sigma(U,s,t)}(Z) \geq 1$ *for any* $Z \in \mathcal{R}$ *and* $\delta_{\Sigma(U,s,t)}(Z) = 0$ *for the remaining indecomposable A-modules Z. Moreover, if* $s \leq p(\Gamma) = p$ *or* $t \leq q(\Gamma) = q$*, then* $\delta_{\Sigma(U,s,t)}(Z) = 1$ *for any* $Z \in \mathcal{R}$.

PROOF. (i) From our assumptions we have for any $0 \leq i < s$ and $0 \leq j < t$ Auslander-Reiten sequences

$$0 \longrightarrow \varphi^{-i}\psi^{j}U \longrightarrow \varphi^{-i-1}\psi^{j}U \oplus \varphi^{-i}\psi^{j+1}U \oplus \pi(\varphi^{-i}\psi^{j}U) \longrightarrow \varphi^{-i-1}\psi^{j+1}U \longrightarrow 0.$$

Applying now [2, Corollary 2.2] we get the required short exact sequence

$$\Sigma(U,s,t)\colon 0 \longrightarrow U \longrightarrow E(U,s,t) \longrightarrow \varphi^{-s}\psi^{t}U \longrightarrow 0$$

with

$$E(U,s,t) = \psi^{t}U \oplus \varphi^{-s}U \oplus \left( \bigoplus_{0\leq i<s} \bigoplus_{0\leq j<t} \pi(\varphi^{-i}\psi^{j}U) \right).$$

(ii) Let

$$W = \left( \bigoplus_{0\leq i\leq s} \bigoplus_{0<j<t} \varphi^{-i}\psi^{j}U \right) \oplus \left( \bigoplus_{0<i<s} \bigoplus_{0\leq j\leq t} \varphi^{-i}\psi^{j}U \right).$$

Then

$$\bigoplus_{0\leq i<s} \bigoplus_{0\leq j<t} (\varphi^{-i}\psi^{j}U \oplus \varphi^{-i-1}\psi^{j+1}U) = W \oplus U \oplus \varphi^{-s}\psi^{t}U$$

and

$$\bigoplus_{0\leq i<s} \bigoplus_{0\leq j<t} E(\varphi^{-i}\psi^{j}U) = W \oplus E(U,s,t).$$

Hence, for each $X \in \operatorname{mod} A$, we get

$$\sum_{0\leq i<s} \sum_{0\leq j<t} ([\varphi^{-i}\psi^{j}U \oplus \varphi^{-i-1}\psi^{j+1}U, X] - [E(\varphi^{-i}\psi^{j}U), X]) = [U \oplus \varphi^{-s}\psi^{t}U, X] - [E(U,s,t), X]$$

Therefore, by Lemma 2.5(i), we get

$$\delta_{\Sigma(U,s,t)}(X) = \sum_{0\leq i<s} \sum_{0\leq j<t} \delta_{\Sigma(\varphi^{-i}\psi^{j}U)}(X) = \sum_{0\leq i<s} \sum_{0\leq j<t} \mu(X, \varphi^{-i}\psi^{j}U).$$

Since $\mathcal{R} = \{\varphi^{-i}\psi^{j}U; 0 \leq i < s, 0 \leq j < t\}$ we conclude that $\delta_{\Sigma(U,s,t)}(Z) \geq 1$ for all $Z \in \mathcal{R}$ and $\delta_{\Sigma(U,s,t)} = 0$ for the remaining indecomposable $A$-modules $Z$. Now, if $s \leq p = p(\Gamma)$ or $t \leq q = q(\Gamma)$ then any module $\varphi^{-i}\psi^{j}U \in \mathcal{R}$ is uniquely determined (up to isomorphism) by the pair $(i,j)$, because $\Gamma$ is obtained from a standard stable tube $\mathcal{T}$ by a sequence of admissible operations. This shows that $\delta_{\Sigma(U,s,t)}(Z) = \sum_{X\in\mathcal{R}} \delta_{\Sigma(X)}(Z)$ has value 1 on any module $Z \in \mathcal{R}$. This finishes the proof.

LEMMA 4.4. *Assume that there exists a short exact sequence* $\Sigma(U,p,kq)$ *for some* $k \geq 1$ *and* $U \in \bar{\Gamma}_0$. *Then there exists a short exact sequence* $\Sigma(W,kp,q)$ *for* $W = \varphi^{-p}\psi^{kq}U$. *Moreover,* $\delta_{\Sigma(U,p,kq)} = \delta_{\Sigma(W,kp,q)}$ *and* $E(U,p,kq) \simeq E(W,kp,q)$.

PROOF. First observe that $\varphi^{-p+1}U$ has the property: $\varphi^{p-1}(\varphi^{-p+1}U) = U \neq 0$ and $\psi^{q-1}(\varphi^{-p+1}U) = \varphi^{-(p-1)}\psi^{q-1}U \neq 0$, because $\Sigma(U,p,kq)$ exists. Hence, by 4.2(ii), $\varphi^{-p+1}U$ lies on a short cycle in add($\Gamma$). Then clearly the modules $\varphi^{-i}\psi^{j}U = \psi^{j}\varphi^{-i}U$ for $0 \leq i < p$, $0 \leq j < kq$, also lie on short cycles in add($\Gamma$), by 4.2(iii).

Take now nonnegative integers $i$, $c$, $d$ such that $i < p$, $c < k$, and $d < q$. Since $\varphi^{-i}\psi^{cq+d}U$ and $W = \varphi^{-p}\psi^{kq}$ lie on short cycles in add($\Gamma$), we get, again by 4.2(iii), that

$$\begin{aligned}\varphi^{-i}\psi^{cq+d}U &= \varphi^{-i-(k-c)p}\psi^{cq+d+(k-c)q}U\\ &= \varphi^{-i-(k-c)p}\psi^{d+kp}U\\ &= \varphi^{p-i-(k-c)p}\psi^{d}\varphi^{-p}\psi^{kq}U\\ &= \varphi^{-i-(k-c-1)p}\psi^{d}W.\end{aligned}$$

From the existence of $\Sigma(U,p,kq)$ we know that any module $X$ in the rectangle

$$\mathcal{R} = \mathcal{R}(U,p,kq) = \{\varphi^{-i}\psi^{j}U;\ 0 \leq i < p, 0 \leq j < kq\}$$

is nonzero and noninjective. Observe now that

$$\begin{aligned}\mathcal{R} &= \{\varphi^{-i}\psi^{cq+d}U;\ 0 \leq i < p,\ 0 \leq c < k,\ 0 \leq d < q\}\\ &= \{\varphi^{-i-(k-c-1)p}\psi^{d}W;\ 0 \leq i < p,\ 0 \leq c < k,\ 0 \leq d < q\},\end{aligned}$$

and so $\mathcal{R}$ coincides with the rectangle

$$\mathcal{R}' = \mathcal{R}(W,kp,q) = \{\varphi^{-e}\psi^{d}W;\ 0 \leq e < kp, 0 \leq d < q\}.$$

Consequently, we infer, by Lemma 4.3, that there exists a short exact sequence

$$\Sigma(W,kp,q)\colon 0 \longrightarrow W \longrightarrow E(W,kp,q) \longrightarrow \varphi^{-kp}\psi^{q}W \longrightarrow 0$$

and for any indecomposable $A$-module $X$ the equalities

$$\delta_{\Sigma(U,p,kq)}(X) = \sum_{Y\in\mathcal{R}} \delta_{\Sigma(Y)}(X) = \sum_{Y\in\mathcal{R}'} \delta_{\Sigma(Y)}(X) = \delta_{\Sigma(W,kp,q)}(X).$$

hold. This gives the equality

$$[U \oplus \varphi^{-p}\psi^{kq}U, X] - [E(U,p,kq), X] = [W \oplus \varphi^{-kp}\psi^{q}W, X] - [E(W,kp,q), X]$$

for any $X \in \operatorname{ind} A$. Since $U = \varphi^{-kp}\psi^{q}W$ and $\varphi^{-p}\psi^{kq}U = W$, we then obtain that

$$[E(U,p,kq), X] = [E(W,kp,q), X]$$

for all $X \in \operatorname{ind} A$. Therefore, $E(U,p,kq) \simeq E(W,kp,q)$, by the theorem of Auslander [6].

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LEMMA 4.5. *Let $M$ and $N$ be $A$-modules with $[M]=[N]$, and $W \in \bar{\Gamma}_0$. Then*

$$\mu(N,W)-\mu(M,W)=\delta_{M,N}(W)-\delta_{M,N}(\varphi W)-\delta_{M,N}(\psi^- W)+\delta_{M,N}(\psi^-\varphi W).$$

*Moreover, if $W$ is noninjective and $\pi(W)\neq 0$ then*

$$\mu\big(N,\pi(W)\big)-\mu\big(M,\pi(W)\big)=-\delta_{M,N}(W).$$

PROOF. We split the proof of the first formula into two cases. Assume first that $W$ is nonprojective. Then $\tau W=\psi^-\varphi W$ and $E(\tau W)=\varphi W\oplus\psi^- W\oplus\pi(\tau W)$. Applying 2.5(ii), we get the equalities

$$\begin{aligned}\mu(N,W)-\mu(M,W)&=\delta'_{\Sigma(\tau W)}(N)-\delta'_{\Sigma(\tau W)}(M)\\&=([N,\psi^-\varphi W\oplus W]-[N,\varphi W\oplus\psi^- W\oplus\pi(\tau W)])\\&\quad-([M,\psi^-\varphi W\oplus W]-[M,\varphi W\oplus\psi^- W\oplus\pi(\tau W)])\\&=\delta_{M,N}(W)+\delta_{M,N}(\psi^-\varphi W)-\delta_{M,N}(\varphi W)\\&\quad-\delta_{M,N}(\psi^- W)-\delta_{M,N}\big(\pi(\tau W)\big).\end{aligned}$$

Since $\pi(\tau W)$ is either zero or injective and $[M]=[N]$ we have $\delta_{M,N}\big(\pi(\tau W)\big)=0$. Hence the required formula is true. Assume now that $W$ is projective. Observe that then $W$ is noninjective, because $W\in\bar{\Gamma}_0$. Obviously, $\operatorname{rad} W=\varphi W\oplus\psi^- W$ and $\operatorname{Hom}_A(X,\operatorname{rad} W)\simeq\operatorname{rad}(X,W)$ as $K$-vector spaces. We then get that

$$\begin{aligned}\mu(N,W)-\mu(M,W)&=([N,W]-[N,\operatorname{rad} W])-([M,W]-[M,\operatorname{rad} W])\\&=\delta_{M,N}(W)-\delta_{M,N}(\operatorname{rad} W)\\&=\delta_{M,N}(W)-\delta_{M,N}(\varphi W)-\delta_{M,N}(\psi^- W).\end{aligned}$$

Since either $\psi^-\varphi W=0$ or $\psi^-\varphi W$ is injective we have $\delta_{M,N}(\psi^-\varphi W)=0$, and so the required formula is true.

Finally, assume that $W$ is noninjective and $\pi(W)\neq 0$. Then $W=\operatorname{rad}\pi(W)$, and we obtain that

$$\begin{aligned}\mu\big(N,\pi(W)\big)-\mu\big(M,\pi(W)\big)&=\big([N,\pi(W)]-[N,W]\big)-([M,\pi(W)]-[M,W])\\&=\delta_{M,N}\big(\pi(W)\big)-\delta_{M,N}(W)=-\delta_{M,N}(W)\end{aligned}$$

because $\pi(W)$ is injective and $[M]=[N]$. This finishes the proof.

LEMMA 4.6. *Let $M$ and $N$ be $A$-modules with $[M]=[N]$, and $U\in\bar{\Gamma}_0$. Assume that a rectangle $\mathcal{R}(U,s,t)$ consists of nonzero and noninjective modules. Then*

$$\begin{aligned}\sum_{0\le i<s}\sum_{0\le j<t}\big(\mu(N,\varphi^{-i}\psi^j U)-\mu\big(M,\varphi^{-i}\psi^j U\big)\big)&=\delta_{M,N}(\psi^-\varphi U)\\&\quad-\delta_{M,N}(\psi^-\varphi^{-s+1}U)-\delta_{M,N}(\varphi\psi^{t-1}U)+\delta_{M,N}(\varphi^{-s+1}\psi^{t-1}U).\end{aligned}$$

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PROOF. From Lemmas 2.5(i) and 4.3(ii) we get the equalities

$$\begin{aligned}
&\sum_{0\le i<s}\sum_{0\le j<t}\Big(\mu(N,\varphi^{-i}\psi^{j}U)-\mu\big(M,\varphi^{-i}\psi^{j}U\big)\Big)\\
&\quad=\sum_{0\le i<s}\sum_{0\le j<t}\Big(\delta_{\Sigma(\varphi^{-i}\psi^{j}U)}(N)-\delta_{\Sigma(\varphi^{-i}\psi^{j}U)}(M)\Big)\\
&\quad=\delta_{\Sigma(U,s,t)}(N)-\delta_{\Sigma(U,s,t)}(M)\\
&\quad=[U\oplus\varphi^{-s}\psi^{t}U,N]-[E(U,s,t),N]-[U\oplus\varphi^{-s}\psi^{t}U,M]+[E(U,s,t),M]\\
&\quad=\delta'_{M,N}(U\oplus\varphi^{-s}\psi^{t}U)-\delta'_{M,N}\big(E(U,s,t)\big)\\
&\quad=\delta_{M,N}(\tau U\oplus\tau\varphi^{-s}\psi^{t}U)-\delta_{M,N}\big(\tau E(U,s,t)\big)\\
&\quad=\delta_{M,N}(\tau U\oplus\varphi^{-s+1}\psi^{t-1}U)-\delta_{M,N}(\tau\varphi^{-s}U\oplus\tau\psi^{t}U)\\
&\quad=\delta_{M,N}(\psi^{-}\varphi U)+\delta_{M,N}(\varphi^{-s+1}\psi^{t-1}U)-\delta_{M,N}(\psi^{-}\varphi^{-s+1}U)-\delta_{M,N}(\varphi\psi^{t-1}U),
\end{aligned}$$

which is the required formula.

5. **Proofs of Theorems 1 and 2.** We shall divide our proof of Theorem 1 into several steps. We use the notations introduced in Sections 3 and 4.

5.1. Let $\mathcal{T}$ be a standard stable tube in $\Gamma_A$, and $E_1,\dots,E_r$ a complete set of modules lying on the mouth of $\mathcal{T}$. Then $\mathcal{T}$ consists of the modules $\psi^i E_j$, $i\ge 0$, $1\le j\le r$. For each $k$, $1\le k\le r$, we denote by $l_k:\operatorname{add}(\Gamma)\longrightarrow\mathbb{N}$ the additive function defined on modules $\psi^i E_j$ by

$$l_k(\psi^i E_j)=\#\{t\in\{j,j+1,\dots,j+i\};r\text{ divides }t-k\}.$$

Then it is easy to see that

$$[\psi^i E_j]=l_1(\psi^i E_j)[E_1]+\cdots+l_r(\psi^i E_j)[E_r]$$

for $i\ge 0$, $1\le j\le r$, and hence

$$[W]=l_1(W)[E_1]+\cdots+l_r(W)[E_r]$$

for any module $W$ in add($\Gamma$). Moreover, we have also the following lemma.

LEMMA. *For $i\ge m\ge 0$ and $1\le j,\ t\le r$, the following equality holds:*

$$[\psi^m E_t,\psi^i E_j]=l_j(\psi^m E_t).$$

PROOF. Straightforward because $\mathcal{T}$ is a standard stable tube.

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LEMMA 5.2. *Let $\Gamma$ be a standard quasi-tube in $\Gamma_A$, and assume that $M$ and $N$ are two modules in* add($\Gamma$) *with* $[M] = [N]$ *and* $M \leq_\Gamma N$. *Then* $\delta_{M,N}(X) = 0$ *and* $\delta'_{M,N}(X) = 0$ *for all but finitely many modules $X$ in $\Gamma$.*

PROOF. Assume first that $\Gamma$ is a stable tube, say of rank $r$. Take $s \geq 0$ such that for any $i \geq s$ and $1 \leq j \leq r$, the module $\psi^i(E_j)$ is not a direct summand of $M \oplus N$. Then applying Lemma 5.1 we get that $[M, \psi^i E_j] = l_j(M)$ and $[N, \psi^i E_j] = l_j(N)$, which implies $l_j(N) - l_j(M) = \delta_{M,N}(\psi^i E_j) \geq 0$, because $M \leq_\Gamma N$. Hence, for $i \geq s$, we have

$$\begin{aligned}\sum_{1\leq j\leq r} \delta_{M,N}(\psi^i E_j)[E_j] &= \sum_{1\leq j\leq r} (l_j(N) - l_j(M))[E_j] \\ &= \left(\sum_{1\leq j\leq r} l_j(N)[E_j]\right) - \left(\sum_{1\leq j\leq r} l_j(M)[E_j]\right) \\ &= [N] - [M] = 0\end{aligned}$$

Therefore, $\delta_{M,N}(\psi^i E_j) = 0$ for any $i \geq s$ and $1 \leq j \leq r$, and so $\delta_{M,N}(X)$ for all but finitely many module $X$ in $\Gamma$. Since $\delta'_{M,N}(Y) = \delta_{M,N}(\tau Y)$ for all nonprojective modules $Y \in \mathrm{add}(\Gamma)$, we get that $\delta'_{M,N}(X) = 0$ for all but finitely many ann modules $X$ in $\Gamma$.

Assume now that $\Gamma$ is not a stable tube. Since $\Gamma$ is a standard tube in $\Gamma_{A/\mathrm{ann}(\Gamma)}$, where $\mathrm{ann}(\Gamma)$ is the annihilator of $\Gamma$ in $A$, we may assume that $\mathrm{ann}(\Gamma) = 0$. Then there exists (see [4, (5.4)]) a sequence of algebras $C = A_0, A_1, \ldots, A_{m-1}, A_m = A$ and a standard faithful stable tube $\mathcal{T}$ in $\Gamma_C$ such that, for each $0 \leq i < m$, $A_{i+1}$ is obtained from the algebra $A_i$ by an admissible operation with pivot in the quasi-tube $\Gamma_i$ of $\Gamma_{A_i}$, obtained from $\mathcal{T}$ by the sequence of admissible operations (of types (ad 1), (ad 1*), (ad 2), (ad 2*)) done so far, and $\Gamma = \Gamma_m$. Therefore, we may proceed by induction on $m$. The case $m = 0$ is discussed above. By duality, we may assume that $A$ is obtained from $B = A_{m-1}$ by an admissible operation of type (ad 1) or (ad 2). Clearly $B = eAe$ for some idempotent $e$ of $A$. Further, $\Gamma$ is the modified component $\mathcal{C}'$ of the standard quasi-tube $\mathcal{C} = \Gamma_{m-1}$ in $\Gamma_B$. From the description of $\mathcal{C}'$ given in Section 3, we infer that the $B$-modules $Me$ and $Ne$ belong to $\mathrm{add}(\mathcal{C})$. Moreover, $[M] = [N]$ implies that $[Me] = [Ne]$ in $K_0(B)$. Then, for any $X \in \mathcal{C}$, we get

$$\dim_K \mathrm{Hom}_B(X, Me) = [X, M] \leq [X, N] = \dim_K \mathrm{Hom}_B(X, Ne).$$

Thus $Me \leq_{\mathcal{C}} Ne$, and by induction we may assume that $\delta_{Me,Ne}(X) = 0$ and $\delta'_{Me,Ne}(X) = 0$ for all but finitely many modules $X$ in $\mathcal{C}$. Therefore, $\delta'_{M,N} = 0$ for all but finitely many indecomposable $B$-modules lying in $\Gamma$. From the shape of the modified component $\Gamma = \mathcal{C}'$ (see Section 3) we deduce that there exists $s \geq 1$ such that the modules $X_i$, $Z_{ij}$, $X'_i$, $i \geq s$, $1 \leq j \leq t$, are not direct summands of $M \oplus N$, and there are Auslander-Reiten sequences in mod $A$

$$0 \longrightarrow X_i \longrightarrow Z_{i1} \oplus X_{i+1} \longrightarrow Z_{i+1,1} \longrightarrow 0$$
$$0 \longrightarrow Z_{ij} \longrightarrow Z_{i,j+1} \oplus Z_{i+1,j} \longrightarrow Z_{i+1,j+1} \longrightarrow 0$$
$$0 \longrightarrow Z_{it} \longrightarrow X'_i \oplus Z_{i+1,t} \longrightarrow X'_{i+1} \longrightarrow 0$$

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for $s \leq i$, $1 \leq j < t$. Observe also that all but finitely many modules $L$ in $\Gamma$ with $L(1-e) \neq 0$ are of the above form $Z_{ij}, X'_i$. Applying now Lemma 2.6, we get, for $i \geq s$, $1 \leq j < t$, the equalities

$$[X_i, M] - [Z_{i1}, M] = \sum_{k \geq i} \mu(M, X_k) = 0,$$
$$[Z_{ij}, M] - [Z_{i,j+1}, M] = \sum_{k \geq i} \mu(M, Z_{kj}) = 0,$$
$$[Z_{it}, M] - [X'_i, M] = \sum_{k \geq i} \mu(M, Z_{kt}) = 0,$$

and similar ones if we replace $M$ by $N$. Hence $\delta'_{M,N}(Z_{ij}) = \delta'_{M,N}(X_i) = \delta'_{M,N}(X'_i)$ for $i \geq s$ and $1 \leq j \leq t$. But the modules $X_i$ belong to mod $B$, and so, by the above considerations, $\delta'_{M,N}(X_i) = 0$ for all but finitely many $i$. Therefore, $\delta'_{M,N}(X) = 0$, and hence also $\delta_{M,N}(X) = 0$, for all but finitely many modules $X$ in $\Gamma$. This finishes the proof.

LEMMA 5.3. *Let $\Gamma$ be a standard quasi-tube in $\Gamma_A$, and $M$, $N$ be modules in* add$(\Gamma)$ *such that $[M] = [N]$ and $M \leq_\Gamma N$. Assume that $\delta_{M,N}(Z) \neq 0$ for some module $Z$ in $\Gamma$. Then there exists a nonsplittable exact sequence*

$$\Sigma(U,s,t)\colon 0 \longrightarrow U \longrightarrow E(U,s,t) \longrightarrow \varphi^{-s}\psi^t U \longrightarrow 0,$$

*for some $U \in \bar{\Gamma}_0$, with $1 \leq s \leq p(\Gamma)$ or $1 \leq t \leq q(\Gamma)$, such that $E(U,s,t)$ is a direct summand of $M$ and $\delta_{M,N}(X) \geq \delta_{\Sigma(U,s,t)}(X)$ for all modules $X$ in $\Gamma$.*

PROOF. Take a module $W \in \bar{\Gamma}_0$ for which $\delta_{M,N}(W) > 0$. We may assume that $\delta_{M,N}(\varphi^- W) = 0$, $\delta_{M,N}(\psi^- W) = 0$ and $\delta_{M,N}(\varphi^- \psi^- W) = 0$. Since $\delta_{M,N}(W) \neq 0$ and $[M] = [N]$, we infer that $W$ is not injective. We put $\delta = \delta_{M,N}$. Observe first that $\varphi^- W$ is a direct summand of $M$. It is clear if $\varphi^- W = 0$. Assume $\varphi^- W \neq 0$. Then by Lemma 4.5 we get that

$$\begin{aligned}\mu(N, \varphi^- W) - \mu(M, \varphi^- W) &= \delta(\varphi^- W) - \delta\big(\psi^-(\varphi^- W)\big) - \delta\big(\varphi(\varphi^- W)\big) \\ &\quad + \delta\big(\psi^- \varphi(\varphi^- W)\big) \\ &= \delta(\varphi^- W) - \delta(\psi^- \varphi^- W) - \delta(W) + \delta(\psi^- W) \\ &= -\delta(W) < 0\end{aligned}$$

by our assumption on $W$. Hence $\mu(M, \varphi^- W) \neq 0$, and so $\varphi^- W$ is a direct summand of $M$.

Take now $a > 0$ minimal such that $\delta(\varphi^a W) = 0$. Observe that such $a$ exists because $\delta(X) = 0$ for all but finitely many $X \in \Gamma$, by the above lemma. Further, take a pair $(b, c)$ with $0 \leq c < a$ and $b > 0$ minimal such that $\delta(\psi^b \varphi^c W) = 0$. Then $\delta(\psi^i \varphi^j W) > 0$ for $0 \leq i < b, 0 \leq j < a$. Hence, for $Z = \psi\varphi^{a-1} W$, we get that $\varphi^{-(a-1-j)}\psi^{i-1} Z = \psi^i \varphi^j W \neq 0$, for $0 \leq j < a$, $0 \leq i < b$, and is noninjective, because $[M] = [N]$. Applying now

Lemma 4.6 we get

$$\sum_{1\le i\le b}\sum_{c\le j<a}\Big(\mu(N,\psi^i\varphi^j W)-\mu\big(M,\psi^i\varphi^j W\big)\Big)$$
$$=\sum_{0\le i<b}\sum_{0\le j<a-c}\Big(\mu(N,\varphi^{-j}\psi^i Z)-\mu\big(M,\varphi^{-j}\psi^i Z\big)\Big)$$
$$=\delta(\psi^-\varphi Z)-\delta(\psi^-\varphi^{-(a-c-1)}Z)-\delta(\varphi\psi^{b-1}Z)+\delta(\varphi^{-(a-c-1)}\psi^{b-1}Z)$$
$$=\delta(\psi^-\varphi Z)-\delta(\varphi^c W)-\delta(\psi^b\varphi^a W)+\delta(\psi^b\varphi^c W).$$

Observe that $\delta(\psi^-\varphi Z)=0$. Indeed, if $Z$ is projective then either $\psi^-\varphi Z=0$ or $\psi^-\varphi Z$ is injective, and hence in the both cases $\delta(\psi^-\varphi Z)=0$. Assume $Z$ is not projective. Then $\psi^-\varphi Z=\varphi\psi^- Z=\varphi\psi^-\psi\varphi^{a-1}W=\varphi^a W$, and so $\delta(\psi^-\varphi Z)=\delta(\varphi^a W)=0$ by our choice of $a$. Since $\delta(\psi^-\varphi Z)=0$, $\delta(\psi^b\varphi^c W)=0$ and $\delta(\varphi^c W)>0$, we obtain that

$$\sum_{1\le i\le b}\sum_{c\le j<a}\Big(\mu(N,\psi^i\varphi^j W)-\mu\big(M,\psi^i\varphi^j W\big)\Big)<0.$$

Thus there is a pair $(s,t)$ such that $c\le s-1<a$, $1\le t\le b$ and $\psi^t\varphi^{s-1}W$ is a direct summand of $M$. We set $U=\varphi^{s-1}W$. From Lemma 4.3 we infer that there exists a nonsplittable exact sequence

$$\Sigma(U,s,t)\colon 0\longrightarrow U\longrightarrow E(U,s,t)\longrightarrow \varphi^{-s}\psi^t U\longrightarrow 0.$$

Moreover, $\varphi^{-s}U\oplus\psi^t U=\varphi^- W\oplus\psi^t\varphi^{s-1}W$ is a direct summand of $M$.

Suppose now that $s>p(\Gamma)=p$ and $t>q(\Gamma)=q$. Then $\varphi^{p-1}W\neq 0$, $\psi^{q-1}W\neq 0$, and so $W$ lies on a short cycle in $\mathrm{add}(\Gamma)$, by Lemma 4.2. Then $\varphi^{a-p}W$ lies on a short cycle, and $\psi^q(\varphi^{a-p}W)=\varphi^p(\varphi^{a-p}W)=\varphi^a W$. But then $\delta(\psi^q\varphi^{a-p}W)=\delta(\varphi^a W)=0$, which contradicts the minimality of $b$, since $0\le s-p\le a-p<a$ and $0<q<t\le b$. Consequently, $1\le s\le p(\Gamma)$ or $1\le t\le q(\Gamma)$. Consider now the rectangle

$$\mathcal{R}=\mathcal{R}(U,s,t)=\{\varphi^{-j}\psi^i U; 0\le j<s,\ 0\le i<t\}.$$

By Lemma 4.3(iii) we have that $\delta_{\Sigma(U,s,t)}(Z)=1$ for $Z\in\mathcal{R}$ and $\delta_{\Sigma(U,s,t)}(Z)=0$ for the remaining indecomposable $A$-modules $Z$. Our choice of $b$ and the inequalities $s\le a$, $t\le b$, imply that $\delta(X)>0$ for all $X\in\mathcal{R}$. Hence $\delta=\delta_{M,N}(X)\ge\delta_{\Sigma(U,s,t)}(X)$ for all modules $X$ in $\Gamma$. Further, by Lemma 4.5, if $\pi(X)\neq 0$ for some $X\in\mathcal{R}$, then

$$\mu\big(N,\pi(X)\big)-\mu\big(M,\pi(X)\big)=-\delta_{M,N}(X)<0$$

and so $\pi(X)$ is a direct summand of $M$. Finally, since $s\le p(\Gamma)$ or $t\le q(\Gamma)$, then

$$E(U,s,t)=\varphi^{-s}U\oplus\psi^t U\oplus\left(\bigoplus_{X\in\mathcal{R}}\pi(X)\right)$$

is a direct summand of $M$. This finishes the proof.

PROPOSITION 5.4. *Let $\Gamma$ be a standard quasi-tube in $\Gamma_A$ and $M$, $N$ two modules in* add($\Gamma$) *with* $[M] = [N]$. *If $M \leq_\Gamma N$ then $M \leq_{\text{ext}} N$.*

PROOF. We shall proceed by induction on $\sum_{X \in \Gamma_0} \delta_{M,N}(X) \geq 0$. Observe that, by Lemma 5.2, this sum is finite. If $\sum_{X \in \Gamma_0} \delta_{M,N}(X) = 0$ then $\delta_{M,N}(X) = 0$ for all $X \in \Gamma_0$, and so also $N \leq_\Gamma M$. Hence, $M \simeq N$ by Corollary 2.8, and this implies $M \leq_{\text{ext}} N$.

Assume that $\sum_{X \in \Gamma_0} \delta_{M,N}(X) > 0$. Applying Lemma 5.3 we infer that there exists a nonsplittable exact sequence

$$\Sigma : 0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0$$

and $M' \in \text{add}(\Gamma)$ such that $M = E \oplus M'$ and $\delta_{M,N}(X) \geq \delta_\Sigma(X)$ for all $X \in \Gamma_0$. Then, for any $X \in \Gamma_0$, we get that

$$\begin{aligned} \delta_{M' \oplus D \oplus F, N}(X) &= [N, X] - [M' \oplus D \oplus F, X] \\ &= ([N, X] - [M' \oplus E, X]) - ([D \oplus F, X] - [E, X]) \\ &= \delta_{M' \oplus E, N}(X) - \delta_{E, D \oplus F}(X) = \delta_{M,N}(X) - \delta_\Sigma(X) \geq 0. \end{aligned}$$

Thus $M' \oplus D \oplus F \leq_\Gamma N$, because $[M' \oplus D \oplus F] = [M' \oplus E] = [M] = [N]$. Observe that $E <_{\text{ext}} D \oplus F$ implies $E <_\Gamma D \oplus F$, and hence $\delta_\Sigma(X) \geq 0$ for all $X \in \Gamma_0$ and $\delta_\Sigma(D) > 0$, because $\Sigma$ is not splittable. Hence we get

$$\sum_{X \in \Gamma_0} \delta_{M' \oplus D \oplus F, N}(X) = \sum_{X \in \Gamma_0} \big(\delta_{M,N}(X) - \delta_\Sigma(X)\big) < \sum_{X \in \Gamma_0} \delta_{M,N}(X).$$

Therefore, $M' \oplus D \oplus F \leq_{\text{ext}} N$ by our inductive assumption. Since $M = M' \oplus E$ and $M' \oplus E \leq_{\text{ext}} M' \oplus D \oplus F$, we have $M \leq_{\text{ext}} N$. This finishes the proof.

LEMMA 5.5. *Let $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$ be a family of pairwise orthogonal standard quasi-tubes in $\Gamma_A$ and $M$, $N$ modules in* add($\mathcal{C}$) *such that $[M] = [N]$ and $[X, M] \leq [X, N]$ for all modules $X$ in $\mathcal{C}$. Moreover, let $M = \bigoplus_{i \in I} M_i$ and $N = \bigoplus_{i \in I} N_i$, for $M_i, N_i \in$* add($\mathcal{C}_i$). *Then $[M_i] = [N_i]$ and $M_i \leq_{\mathcal{C}_i} N_i$ for all $i \in I$.*

PROOF. Assume first that $\mathcal{C}_i$ is a stable tube, say of rank $r$. From the orthogonality of quasi-tubes in $\mathcal{C} = (\mathcal{C}_i)$, we deduce that $[M, X] = [M_i, X]$ and $[N, X] = [N_i, X]$ for all $X \in \mathcal{C}_i$, and hence $[N_i, X] \geq [M_i, X]$ for all $X \in \text{add}(\mathcal{C}_i)$. Let $E_1, \ldots, E_r$ be a complete set of modules lying on the mouth of $\mathcal{C}_i$. Take now $n \geq 0$ such that if $\psi^s E_k$ is a direct summand of $M_i \oplus N_i$, for some $1 \leq k \leq r$, then $s \leq n$. Applying Lemma 5.1 we obtain that

$$[M_i, \psi^n E_k] = l_k(M_i) \text{ and } [N_i, \psi^n E_k] = l_k(N_i),$$

and so $l_k(M_i) \leq l_k(N_i)$, for any $1 \leq k \leq r$. Since

$$[M_i] = \sum_{1 \leq k \leq r} l_k(M_i)[E_k] \text{ and } [N_i] = \sum_{1 \leq k \leq r} l_k(N_i)[E_k]$$

we infer that $[M_i] \leq [N_i]$.

Assume now that $\mathcal{C}_i$ is not a stable tube. As in (5.2) we may assume that there exists an algebra $B$ and a standard quasi-tube $\Gamma_i$ in $\Gamma_B$ such that $A$ is obtained from $B$ by one of the admissible operations of type (ad 1), (ad 1*), (ad 2) or (ad 2*) with pivot in $\Gamma_i$, and $\mathcal{C}_i$ is the modified component $\Gamma_i'$ of $\Gamma_i$. By duality we may assume that $A$ is obtained from $B$ by one of the admissible operations (ad 1) or (ad 2). Let $e$ be an indempotent of $A$ such that $B = eAe$. Observe that $[Xe, Y] = \dim_K \operatorname{Hom}_B(Xe, Ye)$. Moreover, from the description of $\mathcal{C}_i = \Gamma_i'$ we know that $M_ie, N_ie \in \operatorname{add}(\Gamma_i)$. Since $\Gamma_i$ has less projective modules than $\mathcal{C}_i$, by induction, we get that $[M_ie] \leq [N_ie]$. Further, we have $M_i(1-e) = M(1-e) = N(1-e) = N_i(1-e)$, and hence $[M_i] = [M_ie]+[M_i(1-e)] \leq [N_ie]+[N_i(1-e)] = [N_i]$. From the equality $\sum_{i\in I}[M_i] = [M] = [N] = \sum_{i\in I}[N_i]$ we then conclude that $[M_i] = [N_i]$ for all $i \in I$. Moreover, $M_i \leq_{\mathcal{C}_i} N_i$ for any $i \in I$, because the quasi-tubes in $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ are pairwise orthogonal. This proves our lemma.

5.6 *Proof of Theorem 1.* Let $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ be a family of pairwise orthogonal standard quasi-tubes in $\Gamma_A$ and $M, N$ modules in $\operatorname{add}(\mathcal{C})$ with $[M] = [N]$. Clearly, $M \leq_{\text{ext}} N \Rightarrow M \leq N \Rightarrow M \leq_{\mathcal{C}} N$. Assume that $[X, M] \leq [X, N]$ for all modules $X$ in $\mathcal{C}$. Then, by (2.8), we get that $[M, X] \leq [N, X]$ for all $X \in \operatorname{add}(\mathcal{C})$. Consider decompositions $M = \bigoplus_{i\in I} M_i$ and $N = \bigoplus_{i\in I} N_i$, with $M_i, N_i \in \operatorname{add}(\mathcal{C}_i)$, for $i \in I$. It follows from Lemma 5.5 that, for any $i \in I$, $[M_i] = [N_i]$ and $M_i \leq_{\mathcal{C}_i} N_i$. Then, by Proposition 5.4, we get $M_i \leq_{\text{ext}} N_i$ for any $i \in I$, which clearly implies that $M \leq_{\text{ext}} N$.

5.7 *Proof of Theorem 2.* Let $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ be a family of pairwise orthogonal standard quasi-tubes in $\Gamma_A$. Assume that, for $M, N \in \operatorname{add}(\mathcal{C})$ and $V \in \operatorname{mod} A$, we have $[M] = [V] = [N]$ and $M \leq_{\text{deg}} V \leq_{\text{deg}} N$. Clearly, then $M \leq N$. We first show that $\delta_{M,N}(X) = 0$ for all indecomposable $A$-modules $X$ which are not in $\mathcal{C}$. Let $M = \bigoplus_{i\in I} M_i$ and $N = \bigoplus_{i\in I} N_i$, with $M_i, N_i \in \operatorname{add}(\mathcal{C}_i)$ for any $i \in I$. Then, by Lemma 5.5, we get $[M_i] = [N_i]$ and $M_i \leq_{\mathcal{C}_i} N_i$ for any $i \in I$. Observe that

$$\delta_{M,N}(X) = [N, X] - [M, X] = \sum_{i\in I}([N_i, X] - [M_i, X]) = \sum_{i\in I} \delta_{M_i,N_i}(X).$$

Therefore we may assume that $M$ and $N$ belong to the additive category of a quasi-tube $\Gamma = \mathcal{C}_{i_0}$. Applying now (5.3) and (5.4), we infer that there exists an exact sequence

$$\Sigma(U, s, t)\colon 0 \longrightarrow U \longrightarrow E(U, s, t) \longrightarrow \varphi^{-s}\psi^t U \longrightarrow 0$$

such that $M = E(U, s, t) \oplus M'$ and $\delta_{M,N}(X) \geq \delta_{\Sigma(U,s,t)}(X)$ for all $X$ in $\Gamma$. Moreover,

$$\begin{aligned}\delta_{\Sigma(U,s,t)}(X) &= [U \oplus \varphi^{-s}\psi^t U, X] - [E(U, s, t), X] \\ &= [U \oplus \varphi^{-s}\psi^t U \oplus M', X] - [E(U, s, t) \oplus M', X] = \delta_{Z_0,Z_1}(X)\end{aligned}$$

for any $X \in \operatorname{mod} A$ and $Z_0 = M = E(U, s, t) \oplus M'$ and $Z_1 = U \oplus \varphi^{-s}\psi^t U \oplus M'$. In particular, $\delta_{M,N}(X) \geq \delta_{Z_0,Z_1}(X)$ for all $X \in \Gamma$, which gives $Z_1 \leq_\Gamma N$. By Theorem 1 we then get $Z_1 \leq N$. Repeating these arguments we obtain a sequence $M = Z_0 \leq Z_1 \leq Z_2 \leq \cdots \leq Z_k = N$ such that, for each $0 \leq i \leq k-1$, $\delta_{Z_i,Z_{i+1}} = \delta_{\Sigma(U_i,s_i,t_i)}$ for the corresponding exact sequence $\Sigma(U_i, s_i, t_i)$. Observe also that $\delta_{M,N} = \sum_{0\leq j\leq k-1} \delta_{Z_j,Z_{j+1}}$.

Hence, in order to prove our claim, we may assume that $\delta_{M,N} = \delta_{\Sigma(U,s,t)}$ for a short exact sequence and some $s, t \geq 1$. Applying now Lemma 4.3(iii), we get that $\delta_{\Sigma(U,s,t)}(X) = 0$ for any indecomposable module $X$ which is not in $\Gamma$. Consequently, $\delta_{M,N}(X) = 0$ for all indecomposable modules $X$ which are not in $\Gamma$. Let now $\Gamma'' = \mathcal{C} = (\mathcal{C}_i)_{i\in I}$ and $\Gamma'$ be the union of the remaining connected components of $\Gamma_A$. Since $M \leq V \leq N$ we have $\delta_{M,N} = \delta_{M,V} + \delta_{V,N}$ and $\delta_{M,V}(X) \geq 0, \delta_{V,N}(X) \geq 0$ for all $A$-modules $X$. From the first part of our proof we know that $\delta_{M,N}(X) = 0$ for all $X$ in $\Gamma'$. Clearly, then $\delta_{M,V}(X) = 0$ for all $X$ in $\Gamma'$. Applying now Lemma 2.7(ii), we conclude that $V \in \operatorname{add}(\Gamma'') = \operatorname{add}(\mathcal{C})$. This finishes the proof.

## 6. **Proof of Theorem 3.**

6.1. Let $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ be a family of pairwise orthogonal standard quasi-tubes in $\Gamma_A$, and $M, N$ two modules in $\operatorname{add}(\mathcal{C})$ with $[M] = [N]$. From Theorem 3.6 we know that $\operatorname{add}(\mathcal{C})$ is closed under isomorphism classes, extensions and direct summands. Moreover, by Theorem 1, the partial orders $\leq_{\text{ext}}$ and $\leq$ coincide on isomorphism classes of modules in $\operatorname{add}(\mathcal{C})$ with the same composition factors. Therefore, by [11, Theorem 4], $N$ is a minimal degeneration of $M$ if and only if there exist an exact sequence $0 \longrightarrow U \longrightarrow E \longrightarrow V \longrightarrow 0$ and integers $m, r \geq 1$ with the following properties:

($\alpha$) $U$ and $V$ are indecomposable such that $M = E \oplus U^{m-1} \oplus V^{r-1} \oplus X$ and $N = U^m \oplus V^r \oplus X$, and $U \oplus V$ and $E \oplus X$ have no common nonzero direct summands.

($\beta$) $U \oplus V$ is a minimal degeneration of $E$.

($\gamma$) Any common indecomposable direct summand $W \not\simeq V$ of $M$ and $N$ satisfies $[W, N] = [W, M]$.

($\delta$) Any common indecomposable direct summand $W \not\simeq U$ of $M$ and $N$ satisfies $[N, W] = [M, W]$.

Hence, in order to prove our theorem, it remains to show that the minimal degenerations $U \oplus V <_{\deg} E$ given by the exact sequences $0 \longrightarrow U \longrightarrow E \longrightarrow V \longrightarrow 0$, with $U$, $V$ indecomposable modules from $\mathcal{C}$, coincide with those described in (iii) of Theorem 3, and ($\gamma$), ($\delta$) are equivalent to (iv) and (v), respectively. Clearly, in our case, $U$ and $V$ must belong to the same quasi-tube in $\mathcal{C}$.

From now on let $\Gamma$ be a standard quasi-tube in $\Gamma_A$. We use the notations introduced in Section 4.

LEMMA 6.2. *Let $M$ and $N$ be two modules in* $\operatorname{add}(\Gamma)$ *with* $[M] = [N]$, *and assume* $M <_{\deg} N$. *Then there exists a nonsplittable exact sequence*

$$\Sigma(U, s, t)\colon 0 \longrightarrow U \longrightarrow E(U, s, t) \longrightarrow \varphi^{-s}\psi^{t}U \longrightarrow 0$$

*in* $\operatorname{add}(\Gamma)$ *such that* $N = U \oplus \varphi^{-s}\psi^{t}U \oplus N'$ *and* $M \leq_{\deg} N' \oplus E(U, s, t) <_{\deg} N$.

PROOF. Since any chain of neighbours $M = M_0 < M_1 < \cdots < M_r = N$ has at most $[N, N] - [M, M]$ members (see [10, (2.1)]) there exists a module $W \in \operatorname{add}(\Gamma)$ such that

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$[M] = [W] = [N]$, $M \leq_{\deg} W <_{\deg} N$ and $W <_{\deg} N$ is minimal. Applying Lemma 5.3, we infer that there exists an exact sequence

$$\Sigma(U,s,t)\colon 0 \longrightarrow U \longrightarrow E(U,s,t) \longrightarrow \varphi^{-s}\psi^t U \longrightarrow 0$$

in $\operatorname{add}(\Gamma)$ such that $W = E(U,s,t) \oplus N'$ and $\delta_{\Sigma(U,s,t)}(X) = \delta_{W,N}(X)$ for all modules $X$ in $\Gamma$, because $W <_{\deg} N$ is minimal, and $<_{\deg}$ and $<_{\Gamma}$ coincide on $\operatorname{add}(\Gamma)$, by Theorem 1. Hence, for $X$ in $\operatorname{add}(\Gamma)$, we get the equality

$$[U \oplus \varphi^{-s}\psi^t U, X] - [E(U,s,t), X] = [N,X] - [E(U,s,t) \oplus N', X].$$

This gives that

$$[U \oplus \varphi^{-s}\psi^t U \oplus N', X] = [N,X]$$

for all $X \in \operatorname{add}(\Gamma)$, and finally $N = U \oplus \varphi^{-s}\psi^t U \oplus N'$ by Corollary 2.8. This finishes the proof.

PROPOSITION 6.3. *Let $\Sigma(U,s,t)$ be an exact sequence*

$$0 \longrightarrow U \longrightarrow E(U,s,t) \longrightarrow \varphi^{-s}\psi^t U \longrightarrow 0$$

*with $U$ in the quasi-tube $\Gamma$ and $s,t \geq 1$. Then the degeneration $E(U,s,t) <_{\deg} U \oplus \varphi^{-s}\psi^t U$ induced by $\Sigma(U,s,t)$ is minimal if and only if the pair $(s,t)$ satisfies one of the conditions:*

*(a) $s < p(\Gamma)$.*

*(b) $t < q(\Gamma)$.*

*(c) $s = p(\Gamma)$ and $t = kq(\Gamma)$ for some $k \geq 1$.*

*(d) $s = kp(\Gamma)$ and $t = q(\Gamma)$ for some $k \geq 1$.*

PROOF. We set $p = p(\Gamma)$ and $q = q(\Gamma)$. Assume first that one of the above conditions (a)–(d) is satisfied. Suppose that there is a chain of degenerations $E(U,s,t) <_{\deg} E' <_{\deg} U \oplus \varphi^{-s}\psi^t U$ for some $E'$ in $\operatorname{mod} A$ with $[E'] = [E(U,s,t)]$. Since $E(U,s,t)$ and $U \oplus \varphi^{-s}\psi^t U$ belong to $\operatorname{add}(\Gamma)$ we infer by Theorem 2 that $E' \in \operatorname{add}(\Gamma)$. Then by Lemma 6.2, applied to $E' <_{\deg} U \oplus \varphi^{-s}\psi^t U$, we conclude that there exists an exact sequence

$$\Sigma(X,m,r)\colon 0 \longrightarrow X \longrightarrow E(X,m,r) \longrightarrow \varphi^{-m}\psi^r X \longrightarrow 0$$

such that $U \oplus \varphi^{-s}\psi^t U \simeq X \oplus \varphi^{-m}\psi^r X$ and $E' \leq_{\deg} E(X,m,r)$. Hence we get $E(U,s,t) <_{\deg} E(X,m,r)$, $\delta_{\Sigma(U,s,t)} \geq \delta_{\Sigma(X,m,r)}$ but $\delta_{\Sigma(U,s,t)} \neq \delta_{\Sigma(X,m,r)}$. We have two cases to consider:

1° Assume $U \simeq X$ and $\varphi^{-s}\psi^t U \simeq \varphi^{-m}\psi^r X$. Then $p$ divides $m-s$, and $q$ divides $r-t$. Since $s \leq p$ and $t \leq q$, we get $s \leq m$ and $t \leq r$. Hence, by Lemma 4.3, we have

$$\delta_{\Sigma(X,m,r)} = \sum_{0 \leq i < r} \sum_{0 \leq j < m} \delta_{\Sigma(\varphi^{-j}\psi^i X)} \geq \sum_{0 \leq i < t} \sum_{0 \leq j < s} \delta_{\Sigma(\varphi^{-j}\psi^i U)} = \delta_{\Sigma(U,s,t)},$$

and consequently $\delta_{\Sigma(X,m,r)} = \delta_{\Sigma(U,s,t)}$, a contradiction.

2° Assume $U \simeq \varphi^{-m}\psi^r X$ and $X \simeq \varphi^{-s}\psi^t U$. Then $U \simeq \varphi^{-m}\psi^r\varphi^{-s}\psi^t U = \varphi^{-(m+s)}\psi^{r+t} U$ and there exists $l \geq 1$ such that $m+s = lp$ and $r+t = lq$. If $s < p$ or $t < q$ then, by Lemma 4.3(iii), we get $\delta_{\Sigma(U,s,t)}(X) = \delta_{\Sigma(U,s,t)}(\varphi^{-s}\psi^t U) = 0$ while

$\delta_{\Sigma(X,m,r)}(X) \geq 1$. But this gives a contradiction because $\delta_{\Sigma(X,m,r)} \leq \delta_{\Sigma(U,s,t)}$. Assume that $s = p$ and $t = kq$ for some $k \geq 1$. Then $l > k, m \geq kp, r \geq q$, and applying Lemma 4.3(ii) we have

$$\delta_{\Sigma(X,m,r)} = \sum_{0\leq i<r} \sum_{0\leq j<m} \delta_{\Sigma(\varphi^{-i}\psi^{j}X)} \geq \sum_{0\leq i<q} \sum_{0\leq j<kp} \delta_{\Sigma(\varphi^{-i}\psi^{j}X)} = \delta_{\Sigma(X,kp,q)}.$$

But by Lemma 4.4 $\delta_{\Sigma(U,p,kq)} = \delta_{\Sigma(X,kp,q)}$. This implies $\delta_{\Sigma(X,m,r)} = \delta_{\Sigma(U,s,t)}$, a contradiction. We get a similar contradiction in case $s = kp$ and $t = q$ for some $k \geq 1$. Therefore, the degeneration $E(U,s,t) <_{\deg} U \oplus \varphi^{-s}\psi^{t}U$ induced by $\Sigma(U,s,t)$ is minimal.

Assume now that the pair $(s,t)$ does not satisfy any of the conditions (a)–(d). We shall show that there exists an $A$-module $E'$ with the properties $[E(U,s,t)] = [E']$ and $E(U,s,t) <_{\deg} E' <_{\deg} U \oplus \varphi^{-s}\psi^{t}U$. By our assumption we know that $s \geq p$ and $t \geq q$, and hence applying Lemma 4.2, we infer that $\varphi^{-(s-1)}U$ lies on a short cycle in add($\Gamma$), and $\varphi^{-i}\psi^{j}U$, for any $0 \leq i < s$, $0 \leq j < t$, also lies on a short cycle in add($\Gamma$). We have three cases to consider:

1° Assume $s > p$ and $t > q$. Then by Lemma 4.3 there exists a nonsplittable short exact sequence $\Sigma(U, s-p, t-q)$ and

$$\delta_{\Sigma(U,s-p,t-q)} = \sum_{0\leq i<s-p} \sum_{0\leq j<t-q} \delta_{\Sigma(\varphi^{-i}\psi^{j}U)} \leq \sum_{0\leq i<s} \sum_{0\leq j<t} \delta_{\Sigma(\psi^{-i}\psi^{j}U)} = \delta_{\Sigma(U,s,t)}.$$

Since $\varphi^{-s}\psi^{t-q}U$ lies on a short cycle, we have $\varphi^{p}(\varphi^{-s}\psi^{t-q}U) = \psi^{q}(\varphi^{-s}\psi^{t-q}U)$, and hence, by (4.2), $\varphi^{-(s-p)}\psi^{t-q}U = \varphi^{-s}\psi^{t}U$. Then $\delta_{\Sigma(U,s-p,t-q)} \leq \delta_{\Sigma(U,s,t)}$ and $\delta_{\Sigma(U,s-p,t-q)} \neq \delta_{\Sigma(U,s,t)}$ imply that $E(U,s,t) < E(U,s-p,t-q)$, and so $E(U,s,t) <_{\deg} E(U,s-p,t-q)$. Moreover, $E(U,s-p,t-q) <_{\deg} U \oplus \varphi^{-(s-p)}\psi^{t-q}U = U \oplus \varphi^{-s}\psi^{t}U$. Hence, in this case we may take $E' = E(U,s-p,t-q)$.

2° Assume $s = p$ and $t = kq+m$ for some $m$, $1 \leq m < q$. We set $V = \varphi^{-s}\psi^{t}U$. Then

$$\varphi^{-kp}\psi^{q-m}V = \varphi^{-kp}\psi^{q-m}\varphi^{-s}\psi^{t}U = \varphi^{-(k+1)p}\psi^{(k+1)q}U = U.$$

Applying Lemma 4.3(ii), we get

$$\begin{aligned}\delta_{\Sigma(U,s,t)} &= \sum_{0\leq i<p} \sum_{0\leq j<kq+m} \delta_{\Sigma(\varphi^{-i}\psi^{j}U)} \\ &\geq \sum_{0\leq i<p} \sum_{0\leq j<kq} \delta_{\Sigma\left(\psi^{-i}\psi^{j}(\psi^{m}U)\right)} = \delta_{\Sigma(\psi^{m}U,p,kq)}.\end{aligned}$$

Further, by Lemma 4.4, we have

$$\begin{aligned}\delta_{\Sigma(\psi^{m}U,p,kq)} &= \delta_{\Sigma(\varphi^{-p}\psi^{kq}(\psi^{m}U),kp,q)} = \delta_{\Sigma(V,kp,q)} \\ &\geq \sum_{0\leq i<kp} \sum_{0\leq j<q-m} \delta_{\Sigma(\varphi^{-i}\psi^{j}V)} = \delta_{\Sigma(V,kp,q-m)}.\end{aligned}$$

Hence, $\delta_{\Sigma(U,s,t)} \geq \delta_{\Sigma(V,kp,q-m)} \neq 0$, and $\delta_{\Sigma(U,s,t)} \neq \delta_{\Sigma(V,kp,q-m)}$. Observe that $U \oplus \varphi^{-s}\psi^{t}U = V \oplus \varphi^{-kp}\psi^{q-m}V$. Consequently, $E(U,s,t) < E(V,kp,q-m)$ and so $E(U,s,t) <_{\deg} E(V,kp,q-m) <_{\deg} U \oplus \varphi^{-s}\psi^{t}U$. Thus we may take $E' = E(V,kp,q-m)$.

3° In case $s = kp + r$, for $1 \leq r < p$, and $t = q$, the proof of the existence of the required $E'$ is similar.

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LEMMA 6.4. *Let $\Sigma\colon 0 \longrightarrow U \longrightarrow E \longrightarrow V \longrightarrow 0$ be a nonsplittable exact sequence in* $\mathrm{add}(\mathcal{C})$ *with $U$ and $V$ indecomposable. Assume that the induced degeneration $E <_{\deg} U \oplus V$ is minimal. Then there exists an exact sequence*

$$\Sigma(U,s,t)\colon 0 \longrightarrow U \longrightarrow E(U,s,t) \longrightarrow \varphi^{-s}\psi^{t}U \longrightarrow 0$$

*with $s,t \geq 1$ such that $V = \varphi^{-s}\psi^{t}U$ and $E = E(U,s,t)$.*

PROOF. Since the quasi-tubes in $\mathcal{C}$ are standard and pairwise orthogonal and the sequence is not splittable, we infer that $U$ and $V$ belong to one coil $\Gamma = \mathcal{C}_{i_0}$ of $\mathcal{C}$. Applying now Lemma 6.2 for $M = E$, $N = U \oplus V$, we get a nonsplittable exact sequence

$$\Sigma(W,s,t)\colon 0 \longrightarrow W \longrightarrow E(W,s,t) \longrightarrow \varphi^{-s}\psi^{t}W \longrightarrow 0$$

in $\mathrm{add}(\Gamma)$, with $W$ indecomposable, such that $U \oplus V = W \oplus \varphi^{-s}\psi^{t}W \oplus N'$ and $E \leq_{\deg} N' \oplus E(W,s,t) <_{\deg} U \oplus V$. Hence $N' = 0$ and $U \oplus V \simeq W \oplus \varphi^{-s}\psi^{t}W$. Moreover, since $E <_{\deg} U \oplus V$ is minimal, we have $E = E(W,s,t)$ and $\delta_{\Sigma} = \delta_{\Sigma(W,s,t)}$. If $U = W$ and $V = \varphi^{-s}\psi^{t}W$ then $\Sigma(U,s,t)$ is the required sequence. Assume that $U = \varphi^{-s}\psi^{t}W$ and $V = W$. Then the exact sequence $\Sigma$ induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(V,U) \longrightarrow \mathrm{Hom}_A(E,U) \xrightarrow{g} \mathrm{Hom}_A(U,U).$$

Since $\Sigma$ is not splittable, we infer that $g$ is not epimorphism, and so we get

$$\delta_{\Sigma(W,s,t)}(\varphi^{-s}\psi^{t}W) = \delta_{\Sigma(W,s,t)}(U) = \delta_{\Sigma}(U) = [U \oplus V, U] - [E,U] > 0.$$

Applying now Lemma 4.3(ii) we obtain the inequality

$$\sum_{0\leq i<s}\sum_{0\leq j<t} \delta_{\Sigma(\varphi^{-i}\psi^{j}W)}(\varphi^{-s}\psi^{t}W) > 0.$$

Hence there exist $i$ and $j$ such that $0 \leq i < s$, $0 \leq j < t$, and $\delta_{\Sigma(\varphi^{-i}\psi^{j}W)}(\varphi^{-s}\psi^{t}W) > 0$. Then $\varphi^{-s}\psi^{t}W = \varphi^{-i}\psi^{j}W$, by Lemma 2.5(i). But then, by Lemma 4.2(iv), there exists a positive integer $l$ such that $s - i = lp$ and $t - j = lq$. Clearly then $s \geq p$ and $t \geq q$. The sequence $\Sigma(W,s,t)$ induces the same degeneration as the sequence $\Sigma$, and hence the pair $(s,t)$ satisfies one of the conditions (c) or (d) of Proposition 6.3. By duality, we may assume that $s = p$ and $t = kq$ for some $k \geq 1$. Now, applying Lemma 4.4, we infer that there exists an exact sequence $\Sigma(Y,kp,q)$ such that $Y = \varphi^{-s}\psi^{t}W = U$, $\varphi^{-kp}\psi^{q}Y = W = V$, $E(Y,kp,q) = E(U,p,kq) = E$. We see that $\Sigma(U,kp,q)$ is the required exact sequence. This finishes our proof.

6.5. The required fact that the degenerations $U \oplus V <_{\deg} E$ induced by the exact sequences $0 \longrightarrow U \longrightarrow E \longrightarrow V \longrightarrow 0$, with $U$ and $V$ indecomposable from $\mathcal{C}$, coincide with those described in (iii) of Theorem 3 is a direct consequence of Lemmas 6.3 and 6.4. Further, since $E = E(U,s,t)$ and $V = \varphi^{-s}\psi^{t}U$, we have that, for each indecomposable $A$-module $W$, $[N,W] = [M,W]$ if and only if $\delta_{M,N}(W) = \delta_{\Sigma(U,s,t)}(W) = 0$. But $\delta_{\Sigma(U,s,t)}(W) = 0$ if and only if $W \notin \mathcal{R}(U,s,t)$, by Lemma 4.3(iii). This shows that $(\delta)$ is equivalent to (v). Dually, for each indecomposable $A$-module $W$, we have that

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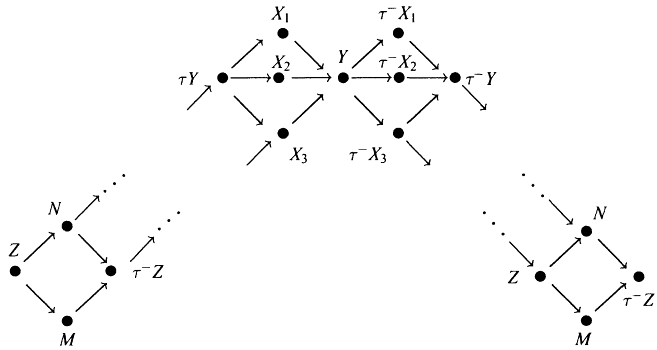
$[W,N] = [W,M]$ if and only if $\delta'_{M,N}(W) = \delta_{M,N}(\tau W) = 0$. Clearly, $W \in \mathcal{R}(\tau^- U, s, t)$ if and only if $\tau W \in \mathcal{R}(U, s, t)$. Therefore, the conditions ($\gamma$) and (iv) are also equivalent. This finishes the proof of Theorem 3.

7. **Proof of Theorem 4.**

7.1. Let $\mathcal{C}$ be a standard coil in $\Gamma_A$ which is not a quasi-tube. Then in any sequence of admissible operations leading from a stable tube $\mathcal{T}$ to $\mathcal{C}$, we need at last one of the admissible operations (ad 3) or (ad 3*). But then $\mathcal{C}$ admits a full translation subquiver of the form




where $M \not\simeq N$. Moreover, if $U$ is a module lying on the sectional path $Z \longrightarrow N \longrightarrow \cdots \longrightarrow \tau Y$ and different from $\tau Y$, then the middle term of the Auslander-Reiten sequence with left term $U$ is a direct sum of two indecomposable modules. Dually, if $V$ is a module lying on the sectional path $\tau^- Y \longrightarrow \cdots \longrightarrow N \longrightarrow \tau^- Z$ and different from $\tau^- Y$, then the middle term of the Auslander-Reiten sequence with right term $V$ is a direct sum of two indecomposable modules.

Applying now [2, Corollary 2.2] we get exact sequences

$$\Sigma_1 : 0 \longrightarrow Z \longrightarrow X_1 \oplus X_2 \oplus M \longrightarrow Y \longrightarrow 0$$

and

$$\Sigma_2 : 0 \longrightarrow Y \longrightarrow \tau^- X_1 \oplus \tau^- X_2 \oplus Z \longrightarrow N \longrightarrow 0.$$

Clearly, we have also exact sequences

$$\Sigma_3 : 0 \longrightarrow X_1 \longrightarrow Y \longrightarrow \tau^- X_1 \longrightarrow 0$$

and

$$\Sigma_4 : 0 \longrightarrow X_2 \longrightarrow Y \longrightarrow \tau^- X_2 \longrightarrow 0.$$

Applying now Lemma (3 + 3 + 2) in [2, (2.1)] to the exact sequences $\Sigma_1$ and $\Sigma_3$ we get an exact sequence

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$$0 \longrightarrow Z \longrightarrow X_2 \oplus M \longrightarrow \tau^{-}X_1 \longrightarrow 0.$$

Similarly, from the exact sequences $\Sigma_4$ and $\Sigma_2$ we get an exact sequence

$$0 \longrightarrow X_2 \longrightarrow \tau^{-}X_1 \oplus Z \longrightarrow N \longrightarrow 0.$$

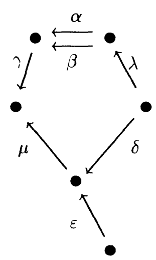
Further, applying again [2, (2.1)] to the above two sequences we obtain an exact sequence

$$0 \longrightarrow Z \longrightarrow Z \oplus M \longrightarrow N \longrightarrow 0.$$

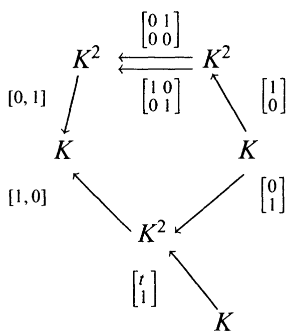
Observe that $[M] = [N]$. Finally, by [21, Proposition 3.4], we infer that $M \leq_{\deg} N$. Then $M <_{\deg} N$, since $M \not\simeq N$. This finishes the proof.

7.2 We end the paper with an example illustrating the situation described above. Let $A$ be the bound quiver algebra $KQ/I$ given by the quiver

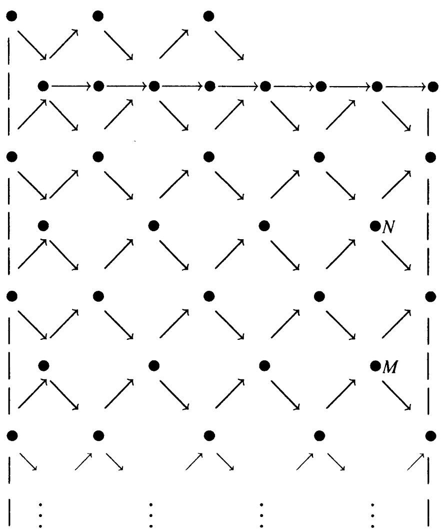
$Q$ :




and the ideal $I$ in the path algebra $KQ$ of $Q$ generated by $\lambda\alpha$, $\alpha\gamma$, $\lambda\beta\gamma - \delta\mu$ (see [4, (2.5)]). Consider the algebraic family $M_t$, $t \in K$, of indecomposable $A$-modules of dimension 9 defined by




Let $M = M_1$ and $N = M_0$. It is easy to see that $M_t \simeq M$ for any $t \in K \setminus \{0\}$ and $M \not\simeq N$. Clearly, $M <_{\deg} N$. Moreover, by [4, (2.5)], $M$ and $N$ lie in a standard coil in $\Gamma_A$ of the form




where one identifies along the vertical dotted lines. Hence, $M <_{\mathrm{deg}} N$ follows also from (7.1).

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