Can. J. Math., Vol. XXVII, No. 4, 1975, pp. 860-866

SOME CONSTRUCTIONS IN ABSTRACT MEASURE THEORY

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1. Introduction. In this paper we construct two examples which elucidate the relationships between several σ-algebras that arise in measure-theoretic constructions on locally compact spaces and groups. For any space X let $\mathscr{B}(X)$ be the *Borel σ-algebra* on X, i.e., the smallest σ-algebra of subsets of X which contains the family of all closed subsets of X. Let $\delta(X)$ be the smallest δ-ring of subsets of X which contains every compact subset of X, where by a *δ-ring* we mean a collection of subsets of X which is closed under the formation of countable intersections, finite unions and relative complements. Let $\sigma(X)$ be the smallest σ-ring of subsets of X which contains all compact subsets of X, where by a *σ-ring* we mean a collection of subsets of X which is closed under the formation of countable unions, finite intersections and relative complements. For any collection $\mathscr{C}$ of subsets of X we let

$$\mathscr{C}^{\text{loc}} = \{A \subseteq X: \text{for each } C \in \mathscr{C}, A \cap C \in \mathscr{C}\}.$$

The σ-algebras which we wish to compare are $\mathscr{B}(X)$, $\delta(X)^{\text{loc}}$ and $\sigma(X)^{\text{loc}}$ where X is a locally compact Hausdorff space. If X is actually a locally compact Abelian group with Haar measure m, we wish to compare those collections with $\mathscr{B}_m(X) = \{A \in \mathscr{B}(X) : m(A) < \infty\}$ and with $\mathscr{B}_m(X)^{\text{loc}}$.

For any space X one readily proves that $\delta(X) = \{A \in \mathscr{B}(X) : \text{cl}(A) \text{ is compact}\}$ and that $\mathscr{B}(X) \subseteq \delta(X)^{\text{loc}} = \sigma(X)^{\text{loc}}$; see § 3. The question (posed to the author by P. Masani) which we answer negatively in this paper asks whether $\mathscr{B}(X)$ will coincide with $\delta(X)^{\text{loc}}$ if X is a locally compact Hausdorff space or a locally compact Abelian group. (Indeed, in [**1**, § 14] the family of Borel sets in a locally compact space X was defined to be $\delta(X)^{\text{loc}}$.) We present two examples. The first, which involves a remarkable set of countable ordinals constructed by Mary Ellen Rudin, is easy to describe. To describe the second example, which is a metrizable, locally compact, locally connected Abelian group, we are forced to develop more machinery, including a structure theorem for $\delta(X)$.

Before describing the examples, let us pause to comment on the case where X is a locally compact Abelian group [**3**]. Let m be Haar measure on X. With notation as above, it is clear that

$$\delta(X) \subset \mathscr{B}_m(X) \subset \mathscr{B}(X) \subset \mathscr{B}_m(X)^{\text{loc}} \subset \delta(X)^{\text{loc}} = \sigma(X)^{\text{loc}},$$

Received January 11, 1974. This research was partially supported by NSF Grant GP-29401.

the last equality following from Theorem 3.4, below. In case X is σ-compact, the third and fourth containments are equalities. What our second example shows is that if a (metrizable) locally compact Abelian group is *not* σ-compact, then each of the first four containments may be proper.

Henceforth, all spaces will be assumed to be at least Hausdorff and the set of all countable ordinals (with the usual ordering) will be denoted by Ω.

Acknowledgement. The author would like to express his gratitude to Professors A. H. Stone and P. R. Meyer for some helpful conversations about Borel sets and to Professor Masani for suggesting the problem that is solved in this paper. The author is deeply indebted to M. Rice for pointing out the results appearing in § 5 of the paper.

2. The first example. In this section, Ω will denote the set of countable ordinals endowed with the usual open-interval topology. The following topological properties of Ω are well-known:

(a) Ω is locally compact and normal;

(b) Ω is sequentially compact (i.e., every sequence has a convergent subsequence) but Ω is not compact. The space Ω also contains some remarkable subsets: in [**5**] Mary Ellen Rudin proved

(c) there is a subset A of Ω such that neither A nor $B = \Omega \backslash A$ contains a closed uncountable subset of Ω. Note that Professor Rudin's set A cannot be a Borel subset of Ω since the collection $\mathscr{C} = \{C \subset \Omega : \text{either } C \text{ or } \Omega \backslash C \text{ contains an uncountable closed subset of } \Omega\}$ is a σ-algebra containing all closed subsets of Ω and hence also contains $\mathscr{B}(\Omega)$. On the other hand, it is easily proved that $\delta(\Omega) = \{D \subset \Omega : D \text{ is countable}\}$ so that $\delta(\Omega)^{\text{loc}}$ is the family of all subsets of Ω. Thus $A \in \delta(\Omega)^{\text{loc}} \backslash \mathscr{B}(\Omega)$.

While the space Ω does have a respectable topological structure, it is not a nice space from the point of view of analysis since it is neither a metric space nor a topological group.

3. Inductive constructions and structure theorems. With Ω denoting the set of countable ordinals, consider the following inductive construction of sets $\delta_\alpha(X)$ for each $\alpha \in \Omega$. Let $\delta_0(X)$ be the family of all compact subsets of X. If $\alpha \in \Omega$ is not a limit ordinal, say $\alpha = \beta + 1$, let $\delta_\alpha(X)$ be the collection of all subsets of X which can be represented as a finite union, a countable intersection, or a relative complement of members of the already defined collection $\delta_\beta(X)$. If α is a limit ordinal we define $\delta_\alpha(X) = \bigcup\{\delta_\beta(X) : 0 \leqq \beta < \alpha\}$. It is clear that the collections $\delta_\alpha(X)$ satisfy:

(a) if $0 \leqq \alpha < \beta$ are countable ordinals then $\delta_\alpha(X) \subseteq \delta_\beta(X)$;

(b) $\delta(X) = \bigcup\{\delta_\alpha(X) : \alpha \in \Omega\}$.

The results in this section are more general than analogous results appearing in [**1**, § 14] in that we do *not* assume that our spaces are locally compact.

3.1 LEMMA. *Suppose Y is a compact subset of a space X. Let*

$$\delta_Y(X) = \{D \cap Y: D \in \delta(X)\}.$$

Then $\delta(Y) = \{D \in \delta(X) : D \subseteq Y\} = \delta_Y(X)$.

Proof. Clearly $\delta(Y) \subseteq \{D \in \delta(X) : D \subset Y\} \subseteq \delta_Y(X)$, the first inclusion being valid because the middle collection is a δ-ring of subsets of Y which contains all compact subsets of Y. If the lemma is false, therefore, there must be a member of $\delta_Y(X)$ which is not a member of $\delta(Y)$. Then in the inductive construction of $\delta(X)$ there must be a first ordinal α such that some member D_0 of $\delta_\alpha(X)$ has the property that $D_0 \cap Y$ is not a member of $\delta(Y)$. Certainly $\alpha \neq 0$ since any $D \in \delta_0(X)$ is compact so that, X being Hausdorff, $D \cap Y$ is also compact and therefore is a member of $\delta(Y)$. Furthermore α cannot be a limit ordinal since, in that case, D_0 would belong to some $\delta_\beta(X)$ with $\beta < \alpha$. Write $\alpha = \beta + 1$ and express D_0 as an admissible combination of sets belonging to $\delta_\beta(X)$. There are three cases to consider. If $D = E_1 \cup \ldots \cup E_n$ where each E_i is in $\delta_\beta(X)$, then minimality of α forces us to conclude that $E_i \cap Y \in \delta(Y)$ for $1 \leqq i \leqq n$ so that $D \cap Y$ is a finite union of members of $\delta(Y)$. If $D = \bigcap\{E_n : n \geqq 1\}$ where $E_n \in \delta_\beta(X)$ for each n then $E_n \cap Y \in \delta(Y)$, again by minimality of α, so that $D \cap Y = \bigcap\{E_n \cap Y : n \geqq 1\}$ is a countable intersection of members of $\delta(Y)$. Finally, if $D = E\backslash F$ where $E, F \in \delta_\beta(Y)$ then $D \cap Y = (E \cap Y)\backslash(F \cap Y)$ is the relative complement of two members of $\delta(Y)$. Therefore, in any case, we are forced to conclude that $D \cap Y \in \delta(Y)$ and that contradiction is sufficient to establish (3.1).

Recall that a collection $\mathscr{K}$ of subsets of X is *directed by inclusion* if, given K_1 and K_2 in $\mathscr{K}$, there is a $K_3 \in \mathscr{K}$ having $K_1 \cup K_2 \subseteq K_3$. For example, a collection $\mathscr{K}$ is directed by inclusion if it is closed under the formation of finite unions.

3.2 THEOREM. *Suppose $\mathscr{K}$ is a collection of compact subsets of X which is directed by inclusion and which has the property that every compact subset of X is contained in some member of $\mathscr{K}$. Then $\delta(X) = \bigcup\{\delta(K) : K \in \mathscr{K}\}$.*

Proof. Let $\mathscr{D} = \bigcup\{\delta(K) : K \in \mathscr{K}\}$. According to (3.1) each $\delta(K)$ is a subcollection of $\delta(X)$. Furthermore $\mathscr{D}$ certainly contains every compact subset of X. Therefore we may complete the proof by showing that $\mathscr{D}$ is a delta-ring. To that end, suppose $S_1, \ldots, S_n \in \mathscr{D}$. Then there are members $K_i \in \mathscr{K}$ such that $S_i \in \delta(K_i)$. Since $\mathscr{K}$ is directed by inclusion, some $K \in \mathscr{K}$ contains $K_1 \cup \ldots \cup K_n$ whence $S_i \in \delta(K)$ for $1 \leqq i \leqq n$. But then $S_1 \cup \ldots \cup S_n \in \delta(K)$ and so $S_1 \cup \ldots \cup S_n \in \mathscr{D}$. Similarly $\mathscr{D}$ is closed under relative complementation. Suppose $S_n \in \mathscr{D}$ for each $n \geqq 1$ and consider the set $T = \bigcap\{S_n : n \geqq 1\}$. We show that $T \in \mathscr{D}$. Choose sets $K_n \in \mathscr{K}$ such that $S_n \in \delta(K_n)$. Let $C_n = K_1 \cap K_n$ for each $n \geqq 1$. Then each C_n is a compact subset of both K_1 and K_n so that, by Lemma 3.1, $S_n \cap C_n \in \delta(C_n) \subseteq \delta(K_1)$. Therefore $T = \bigcap\{S_n \cap C_n : n \geqq 1\}$ belongs to $\delta(K_1)$ and hence to $\mathscr{D}$, as required to complete the proof of (3.2).

It is well-known (and easily proved) that if Y is a compact space then $\mathscr{B}(Y) = \delta(Y) = \delta(Y)^{\text{loc}}$. This fact yields our next theorem.

3.3 THEOREM. *Suppose $\mathscr{K}$ is a family of compact subsets of a space X which is directed by inclusion and which has the property that every compact subset of X is contained in some member of $\mathscr{K}$. Then*

$$\delta(X) = \bigcup\{\mathscr{B}(K) : K \in \mathscr{K}\}$$

and

$$A \in \delta(X)^{\text{loc}} \textit{ if and only if } A \cap K \in \mathscr{B}(K) \textit{ for each } K \in \mathscr{K}.$$

Proof. That $\delta(X) = \bigcup\{\mathscr{B}(K) : K \in \mathscr{K}\}$ follows directly from (3.2). Suppose $A \in \delta(X)^{\text{loc}}$ and let $K \in \mathscr{K}$. Then $K \in \delta(X)$ so that $K \cap A \in \delta(X)$. According to (3.1), $K \cap A \in \delta(K) = \mathscr{B}(K)$. Conversely, suppose $A \cap K \in \mathscr{B}(K)$ for each $K \in \mathscr{K}$. Let $D \in \delta(X)$. Then $D \in \delta(K_0)$ for some $K_0 \in \mathscr{K}$. By assumption $A \cap K_0 \in \mathscr{B}(K_0) = \delta(K_0)$ so that

$$A \cap D = (A \cap K_0) \cap D \in \delta(K_0) \subseteq \delta(X)$$

as required to show that $A \in \delta(X)^{\text{loc}}$.

3.4 THEOREM. *For any space X, $\delta(X)^{\text{loc}} = \sigma(X)^{\text{loc}}$.*

Proof. The theorem will follow from (3.3), if $\mathscr{K}$ is taken to be the family of all compact subsets of X, and from the following characterization of $\sigma(X)^{\text{loc}}$:

$$\sigma(X)^{\text{loc}} = \{A \subseteq X : \text{for each compact } K \subseteq X,\ A \cap K \in \sigma(X)\}.$$

The characterization can be easily proved by applying a minimal counter-example argument (see (3.1)) to $\sigma(X)$, represented as an increasing union $\bigcup\{\sigma_\alpha(X) : \alpha \in \Omega\}$ where $\sigma_0(X)$ is the family of all compact subsets of X, where $\sigma_{\alpha+1}(X)$ is the collection of all subsets of X obtainable by taking countable unions, finite intersections, or relative complements of members of the already defined collection $\sigma_\alpha(X)$, and where $\sigma_\alpha(X) = \bigcup\{\sigma_\beta(X) : 0 \leqq \beta < \alpha\}$ whenever α is a limit ordinal.

Given that characterization, $\delta(X)^{\text{loc}} \subseteq \sigma(X)^{\text{loc}}$ is obvious. The reverse inclusion also follows, once it is observed that if $A \in \sigma(X)^{\text{loc}}$, then for each compact $K \subseteq X$ we have $A \cap K \in \sigma(X)$ and hence $A \cap K \in \sigma(K) = \mathscr{B}(K)$.

As a final item in this section we give an inductive construction of the Borel σ-algebra on a space X. The reader should be warned that our construction is not the usual one, described, for example, in [**4**]. We define $\mathscr{B}_0(X)$ to be the family of all closed subsets of X. If β is a countable ordinal for which $\mathscr{B}_\beta(X)$ has already been defined and if $\alpha = \beta + 1$, we define $\mathscr{B}_\alpha(X)$ to be the family of all subsets of X which may be obtained by forming countable unions of, or complements of, members of $\mathscr{B}_\beta(X)$. And if α is a limit ordinal we let $\mathscr{B}_\alpha(X) = \bigcup\{\mathscr{B}_\beta(X) | \beta < \alpha\}$. One can prove:

3.5 THEOREM. *If $\alpha < \beta$ are countable ordinals then $\mathscr{B}_\alpha(X) \subset \mathscr{B}_\beta(X)$. The collection $\mathscr{B}(X)$ satisfies $\mathscr{B}(X) = \bigcup\{\mathscr{B}_\alpha(X) : \alpha \in \Omega\}$. If Y is any infinite* *compact metric space having no isolated points then for each* $\alpha \in \Omega$ *the collection* $\mathscr{B}_{\alpha+1}(Y)\backslash\mathscr{B}_\alpha(Y)$ *is non-empty.*

The first two assertions are easily proved; the third is non-trivial and can be deduced from the analogous theorem about the usual Borel classification, roughly as follows. The usual classification of Borel sets (described, for example, in Section 30 of Kuratowski's text [**4**]) inductively defines classes $\mathscr{F}_\alpha$ for each $\alpha \in \Omega$, beginning with $\mathscr{F}_0 = \{S \subseteq X | S \text{ is a closed set}\}$. It is easily seen that $\mathscr{B}_\alpha \subset \mathscr{F}_{\alpha+1}$ for each $\alpha \in \Omega$. Now it is known that if X is an infinite compact metric space having no isolated points, then for each $\alpha \in \Omega$ the set $\mathscr{F}_{\alpha+1}\backslash\mathscr{F}_\alpha$ is non-empty [**2**, p. 276]. Therefore $\mathscr{B}_{\alpha+1}\backslash\mathscr{B}_\alpha$ must be non-empty.

4. The second example. Let S^1 be the unit circle in the plane, endowed with its usual topology and group operation. Let R_d be the additive group of real numbers endowed with the *discrete topology*. Let X be the product group $X = S^1 \times R_d$. Then X is an Abelian topological group which is metrizable, locally compact and locally connected. The components of X are the open, compact sets $S(x) = S^1 \times \{x\}$. For any finite subset F of R_d let $S(F) = \bigcup\{S(x) : x \in F\}$. Then the collection $\mathscr{K} = \{S(F) : F \text{ is a finite non-empty subset of } R_d\}$ is directed by inclusion and every compact subset of X is contained in some member of $\mathscr{K}$. Therefore the structure theorems of Section 3 may be applied.

There is a one-to-one function $f : \Omega \to R_d$. (If one invokes the Continuum Hypothesis, it can be assumed that f is also surjective; we do not need this added assumption.) The set $S(f(\alpha))$ is an infinite compact metric space having no isolated points so that there is a subset

$$T(\alpha) \in \mathscr{B}_{\alpha+1}(S(f(\alpha)))\backslash\mathscr{B}_\alpha(S(f(\alpha))).$$

Let $T = \bigcup\{T(\alpha) : \alpha \in \Omega\}$. Let F be a finite, non-empty subset of R_d. Then $T \cap S(F) = \bigcup\{T(\alpha) : f(\alpha) \in F\}$, being a finite union of Borel subsets of $S(F)$, is a Borel subset of $S(F)$. According to Theorem 3.3, $T \in \delta(X)^{\text{loc}}$. However T cannot be a Borel subset of X. For, define a collection $\mathscr{C}$ by

$$\mathscr{C} = \{A \subseteq X : \text{there is an ordinal } \gamma \in \Omega \text{ such that } A \cap S(x) \in \mathscr{B}_\gamma(S(x)) \text{ for each } x \in R_d\}.$$

Certainly every closed subset of X belongs to $\mathscr{C}$. Also, $\mathscr{C}$ is a σ-algebra. For suppose $A_n \in \mathscr{C}$ for each $n \geqq 1$. Let $\gamma_n \in \Omega$ have the property that $A_n \cap S(x) \in \mathscr{B}_{\gamma_n}(S(x))$ for each $x \in R_d$ and let $\gamma = \sup\{\gamma_n : n \geqq 1\}$. Then $A_n \cap S(x) \in \mathscr{B}_\gamma(S(x))$ for each $x \in R_d$ and each $n \geqq 1$ so that $(\bigcup\{A(n) : n \geqq 1\}) \cap S(x) = \bigcup\{A(n) \cap S(x) : n \geqq 1\} \in \mathscr{B}_{\gamma+1}(S(x))$ for each $x \in R_d$. Hence $\bigcup\{A_n : n \geqq 1\} \in \mathscr{C}$. Finally suppose $A \in \mathscr{C}$ and consider $X\backslash A$. Let $\gamma \in \Omega$ have the property that $A \cap S(x) \in \mathscr{B}_\gamma(S(x))$ for each $x \in R_d$. Then $S(x)\backslash(A \cap S(x)) \in \mathscr{B}_{\gamma+1}(S(x))$ for each $x \in R_d$. Since

$$(X\backslash A) \cap S(x) = S(x)\backslash(A \cap S(x)) \in \mathscr{B}_{\gamma+1}(S(x))$$

we see that $\mathscr{C}$ is indeed a σ-algebra of subsets of X. But then every Borel subset of X must belong to $\mathscr{C}$ so that, by construction, T cannot be a Borel set in X. Therefore, even in the metrizable, locally compact, locally connected abelian group X, $\delta(X)^{\text{loc}} \backslash \mathscr{B}(X) \neq \phi$.

Let us conclude with a result showing that the non-separability of the space $X = S^1 \times R_d$ was what made the example possible.

4.1 THEOREM. *If X is separable, paracompact and locally compact then* $\mathscr{B}(X) = \delta(X)^{\text{loc}}$.

Proof. The theorem follows from (3.3) since there is a sequence $U_1 \subseteq U_2 \subseteq \ldots$ of open subsets of X such that $X = \bigcup\{U_n : n \geqq 1\}$ and such that $\bar{U}_n$ is compact for each $n \geqq 1$.

5. Relations to descriptive set theory. Our terminology in this section generally follows that of [**4**]. In particular, a collection $\mathscr{D}$ of subsets of a space X is *discrete* if each point of X has a neighborhood that meets at most one member of $\mathscr{D}$ and a collection which is the union of countably many discrete subcollections is *σ-discrete*. The term "hyper-Borel" was suggested to the author by M. Rice.

Descriptive set theory provides another characterization of $\delta(X)^{\text{loc}}$. Recall that $\mathscr{H}\mathscr{B}(X)$, the collection of *hyper-Borel subsets* of X, is the smallest collection of subsets of X which is closed under complementation and under the formation of σ-discrete unions, and which contains all closed subsets of X. Certainly $\delta(X)^{\text{loc}}$ is closed under the formation of discrete unions of its members; being a σ-algebra, it is also closed under σ-discrete unions so that $\mathscr{H}\mathscr{B}(X) \subseteq \delta(X)^{\text{loc}}$. Let $\mathscr{L}\mathscr{B}(X)$ be the family of all subsets A of X with the property that for each $p \in A$ there is an open neighborhood $V(p)$ of p such that $A \cap V(p) \in \mathscr{B}(X)$. If X is locally compact, it follows from (3.3) that $\delta(X)^{\text{loc}} \subseteq \mathscr{L}\mathscr{B}(X)$. Therefore we have:

5.1 THEOREM. *Let X be a locally compact metric space. Then*

$$\mathscr{H}\mathscr{B}(X) = \delta(X)^{\text{loc}} = \mathscr{L}\mathscr{B}(X).$$

Proof. It will be enough to show that $\mathscr{L}\mathscr{B}(X) \subseteq \mathscr{H}\mathscr{B}(X)$. Let $A \in \mathscr{L}\mathscr{B}(X)$ and for each $p \in A$ let $V(p)$ be an open neighborhood of p such that $A \cap V(p) \in \mathscr{B}(X)$. Because X is metrizable, there is a σ-discrete collection $\mathscr{W} = \bigcup\{\mathscr{W}(n) : n \geqq 1\}$ of open sets which refines $\mathscr{V} = \{V(p) : p \in A\}$ and which covers A. For each $W \in \mathscr{W}(n)$, $A \cap W$ belongs to $\mathscr{B}(X)$. Therefore, the set $A = \bigcup_{n=1}^{\infty}(\bigcup\{W \cap A : W \in \mathscr{W}(n)\})$ is a σ-discrete union of Borel sets so that $A \in \mathscr{H}\mathscr{B}(X)$.

5.2 *Remark.* The proof of (5.1) actually yields a more general, but more technical, result: if X is a locally compact, paracompact space in which each closed set is a G_δ, then $\mathscr{H}\mathscr{B}(X) = \delta(X)^{\text{loc}} = \mathscr{L}\mathscr{B}(X)$. Furthermore, Theorem 5.1 points out the importance of Masani's original question about the coincidence of $\mathscr{B}(X)$ and $\delta(X)^{\mathrm{loc}}$ since in a locally compact metric space in which $\mathscr{B}(X) = \delta(X)^{\mathrm{loc}}$ we have $\mathscr{B}(X) = \mathscr{H}\mathscr{B}(X)$ and in such a space, Lusin's First Separation Theorem [**4**] is known to hold.

References

1. N. Dinculeanu, *Vector measures* (Pergamon Press, New York, 1966).

2. H. Hahn, *Reelle Funktionen* (Chelsea Publ. Co., New York, 1948).

3. E. Hewitt and K. Ross, *Abstract harmonic analysis*, Vol. 1 (Academic Press, New York, 1963).

4. K. Kuratowski, *Topology*, Vol. 1 (Adademic Press, New York, 1966).

5. Mary Ellen Rudin, *A subset of the countable ordinals*, Amer. Math. Monthly *64* (1957), p. 351.

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