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FINITE DIMENSIONAL H-INVARIANT SPACES

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A subset V of $M(G)$ is left H-invariant if it is invariant under left translation by the elements of H, a subset of a locally compact group G. We establish necessary and sufficient conditions on H which ensure that finite dimensional subspaces of $M(G)$ when G is compact, or of $L^\infty(G)$ when G is locally compact Abelian, which are invariant in this weaker sense, contain only trigonometric polynomials. This generalises known results for finite dimensional G-invariant subspaces. We show that if H is a subgroup of finite index in a compact group G, and the span of the H-translates of μ is a weak*-closed subspace of $L^\infty(G)$ or $M(G)$ (or is closed in $L^p(G)$ for $1 \leqslant p < \infty$), then μ is a trigonometric polynomial.

We also obtain some results concerning functions that possess the analogous weaker almost periodic condition relative to H.

1. Introduction

A linear space V of functions or measures on a topological group G is left-invariant if it contains all left translates of its elements by elements of G. In this paper we shall be concerned with a weaker translation-invariance property of V, left H-invariance, which means that V is invariant under left translation by elements of some subset H of G. Sets with this property have arisen naturally in the solution of problems discussed in [**3, 6, 8**].

For the case G compact (or locally compact Abelian), we prove that finite dimensional left H-invariant subspaces of $M(G)$ (respectively $L^\infty(G)$) contain only trigonometric polynomials precisely when the closed subgroup generated by H has finite index in G. We also show that if H is a subgroup of finite index in a compact group G, and the span of the H-translates of μ is a weak*-closed subspace of $L^\infty(G)$ or $M(G)$ (or is closed in $L^p(G)$ for $1 \leqslant p < \infty$), then μ is a trigonometric polynomial.

In proving this, we extend to left H-invariant spaces, results which have been obtained by a number of authors (see [**1, 5, 7**] and [**10, 11, 12, 13**]) for certain finite dimensional left-invariant spaces, and which have been used to study difference and differential operators [**5, 10**], and the convolution induced topology on $L^\infty(G)$ [**2**].

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We also obtain results concerning functions that possess the analogous weaker almost periodic condition relative to H. For a locally compact group G, we show that there exists a non-zero left H-almost periodic function in $L^p(G)$, where $1 \leqslant p < \infty$, if and only if H is relatively compact, extending results of [**4**]. It follows that if G is not compact then the zero subspace is the only such subspace of $L^p(G)$, where $1 \leqslant p < \infty$, when H generates a closed subgroup of finite index in G.

2. Finite dimensional H-invariant spaces of measures

Let G be a locally compact group and H a subset of G. We say that a subspace V of measurable functions is *left H-invariant* if it is invariant under left translation by elements of H, that is, if $f \in V$ and $h \in H$ then the left translate $\tau_h(f)$ belongs to V, where $\tau_h(f) = f(hx)$. Similarly, a subspace V of $M(G)$ is *left H-invariant* if whenever $\mu \in V$ and $h \in H$ then $\tau_h(\mu) \in V$, where $\tau_h(\mu)(f) = \mu(\tau_{h^{-1}}(f))$ for each $f \in C_0(G)$. If $\tau_h(\mu) = \mu$ for all $h \in H$ we say that μ is *left H-fixed.* Our first result shows that for many problems concerning H-invariant subspaces we may assume that H is a closed subgroup.

Proposition 1. *Let H be a subset of a locally compact group G. Suppose that V is either a closed left H-invariant subspace of $C(G)$ or $L^p(G)$ for $1 \leqslant p < \infty$, or a weak*-closed, left H-invariant subspace of $M(G)$ or $L^\infty(G)$. If H' denotes the semigroup generated by H, then V is left $cl(H')$-invariant. If G is compact and H^* denotes the closed subgroup generated by H then V is left H^*-invariant.*

Proof: Clearly V is invariant under the semigroup H' generated by H. To see that it is invariant under $cl(H')$, let $\mu \in V$, $x \in cl(H')$ and let (h_α) be a net in H' which converges to x. Since $\tau_{h_\alpha}\mu \in V$ and the translation operation $a \to \tau_a\mu$ is continuous from G into $C(G)$ or $L^p(G)$ for $p < \infty$, and weak*-continuous from G into $M(G)$ or $L^\infty(G)$, $\tau_x\mu$ is also in V.

We observe that if G is compact and $x \in H'$, then from (9.16) of [**9**] it follows that $x^{-1} \in cl(H')$ which completes the proof that V is H^*-invariant. ▯

Corollary 1. *If G is locally compact and V is a finite dimensional left H-invariant subspace of $M(G)$ or $L^\infty(G)$ then V is left invariant under the closed subgroup generated by H.*

Proof: Since finite dimensional subspaces are weak*-closed, we need only verify (in the locally compact, non-compact case) that if $x \in H$, and $\mu \in V$, then $\tau_{x^{-1}}\mu \in V$. To see why this is so, observe that since V is finite dimensional we can choose scalars $\alpha_0, \ldots, \alpha_N$, not all zero, such that $\sum_{n=0}^{N} \alpha_n \tau_x^n \mu = 0$. If N is chosen to be minimal with respect to this property, then $\alpha_0 \neq 0$, and hence

$$\tau_{x^{-1}}\mu = -\alpha_0^{-1} \sum_{n=1}^{N} \alpha_n \tau_x^{n-1} \mu \in V.$$ ▯

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We follow the convention of denoting the circle group by $\mathbf{T}$ which we identify with the interval $[0,1)$. The following examples illustrate that Proposition 1 may fail for subspaces of $L^1(\mathbf{T})$ that are not closed and for closed subspaces of $M(\mathbf{T})$ that are not weak*-closed. Similar examples can be constructed on $\mathbf{R}$ to show that the assumption of finite dimensionality is necessary in the locally compact case as well.

EXAMPLE 1. Choose an irrational number $\alpha \in (0,1)$ and let f be the characteristic function of the interval $(0,1/2)$. A finite linear combination of translates of f, say $\sum c_n \tau_{n\alpha} f$, is a step function with steps at $n\alpha$ mod 1 and $1/2 + n\alpha$ mod 1. The irrationality of α ensures that none of these steps coincide, so that if $\sum c_n \tau_{n\alpha} f = 0$, then $c_n = 0$ for all n. Thus any set of translates of f is linearly independent. Let $H = \{n\alpha : n \text{ is a positive integer}\}$. The subgroup generated by H is $\mathbf{Z}\alpha$ mod 1 and the closure of H is $\mathbf{T}$. Let V denote the smallest H-invariant linear space generated by $\tau_\alpha(f)$. Then V is an infinite dimensional H-invariant subspace of $L^1(\mathbf{T})$ which is not invariant under translation by the subgroup generated by H or by $cl(H)$ because it does not contain f. (The same proof works if f is the characteristic function of any subinterval of $(0,1)$ with rational endpoints.)

EXAMPLE 2. Let $H = \{n\alpha : n \text{ is a positive integer}\}$, where $\alpha \in (0,1)$ is an irrational number, and let V be the closure in $M(\mathbf{T})$ of the linear span of the set of point measures $\{\delta_{n\alpha} : n \in \mathbf{N}\}$. Then V is H-invariant, but not invariant under translation by $-\alpha(\text{mod } 1)$ which is in both the closure of H and the group generated by H.

It is well known that a closed translation-invariant subspace of $L^1(G)$ is a left ideal (under convolution). It follows easily from this and the orthogonality of the coordinate functions of representations that if G is compact and V is a finite dimensional left-invariant subspace of $M(G)$ then V contains only trigonometric polynomials. Our first theorem generalises this fact, as well as the analogous result known [7] for invariant subspaces of bounded measurable functions on locally compact Abelian groups, to finite dimensional H-invariant subspaces.

THEOREM 1. *Let H be a subset of a locally compact group G. The closed subgroup generated by H is of finite index in G if and only if any finite dimensional left H-invariant subspace of $M(G)$ when G is compact, or of $L^\infty(G)$ if G is locally compact and Abelian, contains only trigonometric polynomials.*

PROOF: Suppose that H generates a closed subgroup H^* of finite index. If V is a finite dimensional left H-invariant subspace of $M(G)$ (or $L^\infty(G)$) then V is H^*-invariant by Corollary 1. Let $\{\mu_1, \mu_2, \dots, \mu_m\}$ be a basis for V and $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ be a set which contains one element from each coset of H^* in G. It is routine to verify that the set $\{\tau_{\zeta_j}(\mu_i) : i = 1, \dots, m \text{ and } j = 1, \dots, n\}$ spans a left-invariant subspace containing V. Being finite dimensional and invariant, this subspace, and hence V, contains only trigonometric polynomials.

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Conversely, if G is compact and the subgroup H^* is of infinite index, then $\lambda_G(H^*) = 0$ where λ_G is the normalised Haar measure on G. This means that the Haar measure on H^* is a singular measure on G, and hence is not a trigonometric polynomial. Being H-fixed, its H-translates obviously span a one dimensional subspace. Alternatively, if G is an Abelian group we then consider $A\big(\widehat{G}, H\big) \equiv \{\chi \in \Gamma : \chi(h) = 1 \text{ for all } h \in H\}$. Since $\left|A\big(\widehat{G}, H\big)\right| = [G : H]$ we can choose a countably infinite subset $\{\chi_n\} \subseteq A\big(\widehat{G}, H\big)$. The continuous function $\sum\limits_n \chi_n/n^2$ is H-fixed and not a trigonometric polynomial, a fact which can be easily seen from the orthogonality of the characters, viewed if necessary as characters on the Bohr compactification of G. ∎

Recall that if $\lambda_G(H) > 0$ then H generates an open subgroup. This is the key idea in our next corollary.

Corollary 2. *Let H be a closed subgroup of a compact group G. The following are equivalent:*

1. *$\lambda_G(H) > 0$;*
2. *any finite dimensional left H-invariant subspace of $M(G)$ consists of trigonometric polynomials;*
3. *there is no H-fixed, singular measure on G.*

Proof: $(1 \Rightarrow 2)$ Since H is both open and compact, G/H is both discrete and compact, and therefore finite. Now apply the theorem.

$(2 \Rightarrow 3)$ This is obvious.

$(3 \Rightarrow 1)$ If $\lambda_G(H) = 0$ then the Haar measure on H is a counterexample to (3). ∎

Remark 1. In [3] it is shown that if S is a weak*-closed subspace of $L^\infty(G)$ for G compact, then there is a unique normal, closed subgroup H such that S is the set of H-fixed functions in $L^\infty(G)$. Our work implies that if H has positive measure, then S contains only trigonometric polynomials.

Recall that a measure μ is central if and only if $\widehat{\mu}(\sigma)$ is a multiple of the identity I_σ for each $\sigma \in \widehat{G}$. Next we show when one can find a central, H-fixed measure.

Corollary 3. *Let H be a subset of the compact group G. There exists a central measure on G which is not a trigonometric polynomial, but which is left H-fixed, if and only if the smallest closed normal subgroup generated by H is of infinite index in G.*

Proof: Let K denote the smallest closed normal subgroup generated by H and suppose that there is a central measure μ which is left H-fixed and is not a polynomial. Since $\widehat{\mu}(\sigma)$ is a multiple of the identity for each $\sigma \in \widehat{G}$, $\sigma(h) = I_\sigma$ whenever $h \in H$ and $\widehat{\mu}(\sigma) \neq 0$. It follows that $\widehat{\tau_{x^{-1}hx}(\mu)} = \widehat{\mu}$ for each $x \in G$, and so μ is left K-fixed. By Theorem 1 K is of infinite index in G.

For the converse, we just take the Haar measure on K. ∎

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EXAMPLE 3. We can strengthen Corollary 3 as follows. If K has infinite index in G then there exists a central element f in $A(G)$ which is H-fixed and is not a trigonometric polynomial. One way to see this is to follow the method of proof of Theorem 1: choose an infinite subset $\{\sigma_n\}$ of $A\left(\widehat{G}, K\right)$ and then set

$$f = \sum \frac{1}{n^2 \deg \sigma_n} \operatorname{Tr} \sigma_n.$$

Alternatively, observe that λ_K is a central, H-fixed, singular measure and therefore its spectrum contains a countably infinite subset, say X. If $g \in A(G)$ is chosen with $\widehat{g}(\sigma)$ a non-zero multiple of the identity for each $\sigma \in X$, then $f = \lambda_K * g$ has the desired properties.

Notice that if we only assume that H^*, the closed subgroup generated by H, is of infinite index then for an appropriate choice of g the function $\lambda_{H^*} * g$ is H-fixed, belongs to $A(G)$ and is not a trigonometric polynomial.

In [**13**] it was shown that if $sp\{\tau_g f : g \in G\}$, the linear span of the set of G-translates of f, is a closed subspace of $C(G)$ then f is a trigonometric polynomial. This generalises to H-spans as well.

THEOREM 2. *Suppose that H generates a closed subgroup of finite index in the compact group G, and that $V = sp\{\tau_h \mu : h \in H\}$ is a closed subspace of $C(G)$ or $L^p(G)$ for $1 \leqslant p < \infty$, or is a weak*-closed subspace of $M(G)$ or $L^\infty(G)$. Then μ is a trigonometric polynomial.*

PROOF: By Proposition 1 we may assume (in any of the settings) that H is itself a closed subgroup of finite index in G.

First we consider the cases $V \subseteq C(G)$ or $L^p(G)$ for $1 \leqslant p < \infty$. We let

$$S_N = \left\{\sum_{i=1}^{N} a_i \tau_{h_i} \mu : |a_i| \leqslant N,\ h_i \in H\right\},$$

so that $V = \bigcup_{N=1}^{\infty} S_N$. If $V \subseteq C(G)$ then each set S_N is compact because it is closed, bounded and equicontinuous. To show compactness when S_N is in $L^p(G)$ we consider a net $\left\{\sum_{i=1}^{N} a_i^{(\alpha)} \tau_{h_i^{(\alpha)}} \mu\right\}$ in S_N. Since H is compact, by passing to a subnet, not renamed, we may assume that for each $i = 1, 2, \ldots, N$, $h_i^{(\alpha)} \to h_i \in H$ and $a_i^{(\alpha)} \to a_i$ with $|a_i| \leqslant N$. By continuity of translation one sees that the net converges in L^p norm to $\sum_{i=1}^{N} a_i \tau_{h_i} \mu \in S_N$.

By the Baire Category Theorem some set S_N has interior, and as this set is compact, the subspace V is finite dimensional. Theorem 1 implies that μ is a trigonometric polynomial.

For $V \subseteq M(G)$ it appears we have to work harder. We choose a set of right coset representatives of H in G, $\{g_1, \ldots, g_k\}$, and for each $i = 1, \ldots, k$ define $v_i \in M(G)$ by

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$v_i(E) = \mu(Hg_i \cap Eg_i)$ for each measurable subset E of G. Because H is closed we may also view v_i as belonging to $M(H)$.

Being a subgroup of finite index, H is also open. Thus if $f \in C(G)$ then $f_i(x) = f(xg_i)$ defines a continuous function on H for each $i = 1, \ldots, k$, while if $f_1, \ldots, f_k \in C(H)$ then f defined by

$$f(x) = f_i\left(xg_i^{-1}\right) \text{ if } x \in Hg_i$$

is a continuous function on G. Clearly we have

$$\int_H f_i\,dv_i = \int_G 1_{Hg_i} f\,d\mu \text{ and } \sum_{i=1}^{k} \int_H f_i\,dv_i = \int_G f\,d\mu.$$

Let

$$V' = \left\{ \left(\sum_{i=1}^{n} a_i \tau_{h_i} v_1, \ldots, \sum_{i=1}^{n} a_i \tau_{h_i} v_k \right) : n \in \mathbf{N},\ a_i \in \mathbf{C},\ h_i \in H \right\}.$$

Certainly V' is a subspace of $[M(H)]^k$. If

$$\left\{ \sum a_{h(\alpha)} \tau_{h(\alpha)} (v_1, \ldots, v_k) \right\}$$

is a weak*-convergent net in V', then one can check that $\{\sum a_{h(\alpha)} \tau_{h(\alpha)} \mu\}$ is weak*-convergent in $M(G)$ with limit $\sum a_h \tau_h \mu \in V$ say, and the original net has limit $\sum a_h \tau_h (v_1, \ldots, v_k) \in V'$. Thus V' is weak*-closed. Standard arguments can be used to prove that if X is any weak*-closed subspace of $[M(H)]^k$ and $\eta \in M(H)$, then $(\eta * \omega_1, \ldots, \eta * \omega_k) \in X$ for all $(\omega_1, \ldots, \omega_k) \in X$. In particular,

$$(\operatorname{Tr} \sigma * v_1, \ldots, \operatorname{Tr} \sigma * v_k) \in V' \cap [L^1(H)]^k$$

for all $\sigma \in \widehat{H}$.

Now $V' \cap [L^1(H)]^k = \bigcup S_N$ where

$$S_N = \left\{ \sum_{i=1}^{N} a_i \tau_{h_i} (v_1, \ldots, v_k) : |a_i| \leqslant N,\ h_i \in H \right\} \cap [L^1(H)]^k.$$

Let $(v_j)_a$ denote the absolutely continuous part of v. If $\sum\limits_{i=1}^{N} a_i \tau_{h_i} (v_1, \ldots, v_k)$ is in $[L^1(H)]^k$ then it must equal $\sum\limits_{i=1}^{N} a_i \tau_{h_i} ((v_1)_a, \ldots, (v_k)_a)$, and hence

$$S_N = \left\{ \sum_{i=1}^{N} a_i \tau_{h_i} ((v_1)_a, \ldots, (v_k)_a) : |a_i| \leqslant N,\ h_i \in H \right\}.$$

One can show that the sets S_N are compact in the norm topology of $[L^1(H)]^k$ by the same kind of arguments as those used for $L^1(G)$.

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Again, an application of the Baire Category Theorem allows us to conclude that $V' \cap [L^1(H)]^k$ is finite dimensional. Hence

$$\{(\operatorname{Tr} \sigma * v_1, \ldots, \operatorname{Tr} \sigma * v_k) : \sigma \in \widehat{H}\}$$

is contained in a finite dimensional subspace of $[L^1(H)]^k$, and an orthogonality argument proves that each v_j is a trigonometric polynomial on H. Viewed as measures on G, each v_j obviously belongs to $L^1(G)$, say $v_j = F_j \lambda_G$. But

$$\mu = \sum_{j=1}^{k} 1_H\left(xg_j^{-1}\right) F_j\left(xg_j^{-1}\right) \lambda_G,$$

so $\mu \in L^1(G)$. Thus V is a closed subspace of $L^1(G)$ and the first part of the proof shows that μ is a trigonometric polynomial.

Finally, observe that if V is a weak*-closed subspace of $L^\infty(G)$ then V is also a weak*-closed subspace of $M(G)$, and so the proof is complete. ∎

3. Left H-almost periodic functions in $L^p(G)$ where $1 \leqslant p < \infty$

It is easy to see that if G is compact then every function in $L^p(G)$, where $1 \leqslant p < \infty$, is left-almost periodic, while only the zero function is left almost periodic if G is not compact [**4**]. Since a norm-bounded subset of a finite dimensional subspace of $L^p(G)$ is relatively compact, each element of a finite dimensional left invariant subspace of $L^p(G)$ must be left-almost periodic. Consequently the zero subspace is the only finite dimensional left-invariant subspace of $L^p(G)$ when G is not compact.

We call $f \in L^p(G)$ *left H-almost periodic* if the set $\{\tau_h f : h \in H\}$ of left H-translates of f is relatively compact in $L^p(G)$. It is natural to ask whether there exist non-zero left H-almost periodic functions in $L^p(G)$ when G is not compact. As one might expect, we see that this occurs precisely when H is relatively compact.

Theorem 3. *Let H be a subset of the locally compact group G. There exists a non-zero left H-almost periodic function in $L^p(G)$, where $1 \leqslant p < \infty$, if and only if H is relatively compact. If the closed group generated by H is compact, then there exists a non-zero function in $A(G)$ which is left H-fixed and has compact support.*

Proof: The proof of necessity is a modification of that given in [**4**]. We leave the details for the reader.

Suppose that H is relatively compact. Let $(\tau_{h_\alpha}(f))$ be a net in the set $\{\tau_h(f) : h \in H\}$. Since H is relatively compact, the net (h_α) has a subnet that converges to some h in the closure of H, and so $(\tau_{h_\alpha}(f))$ has a subnet that converges to $\tau_h(f)$. Thus any $f \in L^p(G)$ is left H-almost periodic.

Suppose now that the closed subgroup H^* generated by H is compact. Since G/H^* is locally compact, it contains a non-empty open set U with compact closure. Let f denote

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the characteristic function of the preimage of U in G. Then f is compactly supported because this preimage is contained in the compact set $\{hx : h \in H^* \text{ and } H^*x \in cl(U)\}$, and is in $L^p(G)$ for all p since it is also bounded. Clearly f is non-zero as the preimage is open and non-empty, and f is left H-fixed. To obtain a function in $A(G)$ with the required properties, set $g = f * f'$ where $f'(x) = f(x^{-1})$ for each x. It is easy to check that g is left H-fixed, belongs to $A(G)$ and has compact support. It is non-zero because $g(e)$ equals the Haar measure of the preimage of U in G which is positive. ∎

We can now characterise those subsets H of a locally compact group G for which there are non-trivial finite dimensional left H-invariant subspaces of $L^p(G)$, where $1 \leqslant p < \infty$.

COROLLARY 4. *Let H be a subset of the locally compact G. There exists a non-trivial finite dimensional left H-invariant subspace of $L^p(G)$, where $1 \leqslant p < \infty$, if and only if the closed subgroup generated by H is compact.*

PROOF: Let H^* be the closed subgroup generated by H. It is a consequence of Corollary 1 that any finite dimensional left H-invariant subspace V of $L^p(G)$, where $1 \leqslant p < \infty$, is also left H^*-invariant, and therefore every $f \in V$ is left H^*-almost periodic. The result is now immediate from Theorem 3. ∎

COROLLARY 5. *Let G be a locally compact, non-compact group and let H be a subset which generates a closed subgroup of finite index in G. Then the only finite dimensional left H-invariant subspace of $L^p(G)$, where $1 \leqslant p < \infty$, is the zero subspace.*

PROOF: This follows since any compact subgroup of G must be of infinite index. ∎

The definition of left H-almost periodicity is easily extended to measures. Since elements of $M(G)$ are regular, arguments similar to those found in [3] can also be used to show that the existence of a non-zero left H-almost periodic measure implies that H is relatively compact. Consequently analogues of part of Corollaries 4 and 5 hold for left H-almost periodic measures as well.

It is known [**6**] that if G is locally compact and Abelian and if H is an integrable subset of G with positive Haar measure, then H-almost periodic measures in $M(G)$ are absolutely continuous with respect to Haar measure. Our final proposition gives examples of subsets H for which there exist singular H-almost periodic measures. Notice that Theorem 3 implies that if any such measures exist, then H must be relatively compact.

PROPOSITION 2. *Suppose that H is a relatively compact subset of the locally compact, Abelian group G which generates a closed subgroup of zero measure. Then there exists an H-almost periodic measure in $M(G)$ which is singular.*

PROOF: Let H^* denote the closed subgroup generated by H. By regularity, there is a compact subset K of H^* with $0 < \lambda_{H^*}(K) < \infty$. Let $\mu = \chi_K \lambda_{H^*}$, where χ_K is the characteristic function of K. Then μ is a non-zero, singular measure in $M(G)$ which

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is H-almost periodic since we can view it as belonging to $L^1(H^*)$ and H is relatively compact. ∎

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