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COEFFICIENT INEQUALITIES FOR L^p-VALUED ANALYTIC FUNCTIONS

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ABSTRACT. A Hausdorff–Young theorem is given for L^p-valued analytic functions on the open unit disc and estimates on such functions and their derivatives are deduced.

Given a non-zero complex Banach space X and a holomorphic function $f:\Delta \to X$, where Δ denotes the open unit disc of the complex plane, define

$$M_\lambda(f) = \overline{\lim_{r\to 1^-}} \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(re^{i\theta})\|^\lambda \, d\theta\right)^{1/\lambda}$$

for $\lambda \geq 1$. By [7, p. 77], the power series $\sum_0^\infty a_n z^n$ converges uniformly to f on compact subsets of Δ when $\{a_n\}_0^\infty$ is the sequence in X given by $a_n = f^{(n)}(0)/n!$. Note that

$$\left(\sum_{n=0}^\infty \|a_n\|^2\right)^{1/2} = M_2(f) \tag{1}$$

when X is a Hilbert space since the classical proof for the case of complex-valued functions [10, p. 84] carries over without change (as is observed in [9]).

The following result is an extension of a variant of the Hausdorff–Young theorem [3, p. 94] and of (1) to functions with values in $X = L^p(S)$, where S is any positive measure space and $1 < p < \infty$. Throughout, for any given $1 < p < \infty$, p' is the conjugate index and $\bar{p} = \max\{p, p'\}$.

THEOREM 1. *If $f:\Delta \to L^p(S)$ is a holomorphic function with power series $f(z) = \sum_0^\infty a_n z^n$, then*

$$\left(\sum_{n=0}^\infty \|a_n\|^\lambda\right)^{1/\lambda} \leq M_{\lambda'}(f) \tag{2}$$

and

$$M_\lambda(f) \leq \left(\sum_{n=0}^\infty \|a_n\|^\lambda\right)^{1/\lambda'} \tag{3}$$

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for all $\lambda \geq \bar{p}$. In particular,

$$\left(\sum_{n=0}^{\infty} \|a_n\|^{\bar{p}}\right)^{1/\bar{p}} \leq M_2(f) \tag{4}$$

and equality holds in (4) *when both $p \geq 2$ and the coefficients $\{a_n\}$ have disjoint supports.*

COROLLARY 2. *If $f:\Delta \to L^p(s)$ is a holomorphic function satisfying $M_2(f) \leq 1$, then*

$$\|f(z)-f(0)\| \leq \frac{|z|}{(1-|z|^{p'})^{1/p'}}(1-\|f(0)\|^p)^{1/p} \tag{5}$$

$$\|f'(z)\| \leq \frac{\Gamma(p'+1)^{1/p'}}{(1-|z|^{p'})^{1+1/p'}}(1-\|f(0)\|^p)^{1/p} \tag{6}$$

for all $z \in \Delta$ when $2 \leq p < \infty$, and if $1 < p < 2$, the above inequalities hold with p and p' interchanged.

Proofs. Theorem 1 is a consequence of generalized Clarkson inequalities of L. R. Williams and J. H. Wells [11, Th. 2]. (These inequalities can be deduced easily from [6, Th. 3.1]. Note that a factor of $1/n$ is missing inside the first summation sign in the right hand side of [11, (9)]. The assumption of σ-finiteness is unnecessary by [2, p. 168].) Without loss of generality we may suppose that f is holomorphic in a neighborhood of $\bar{\Delta}$. Let $\alpha \geq 1$ and define $\varphi(\theta) = \|f(e^{i\theta})\|^\alpha$ and $\varphi_n(\theta) = \|f_n(e^{i\theta})\|^\alpha$, where $f_n(z) = \sum_0^{n-1} a_k z^k$. By [11, Th. 2], to prove (2) and (3), we need only show that $I_n \to 0$ as $n \to \infty$, where

$$I_n = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta)\, d\theta - \frac{1}{n}\sum_{j=1}^{n} \varphi_n\left(\frac{2\pi j}{n}\right);$$

but since f is uniformly continuous on $\bar{\Delta}$ and $f_n \to f$ uniformly on $\bar{\Delta}$, we have

$$|I_n| \leq \sup\left\{|\varphi(\theta') - \varphi_n(\theta)| : |\theta' - \theta| < 2\frac{\pi}{n}\right\} \to 0,$$

as required. Clearly (4) follows from (2) with $\lambda = \bar{p}$ since $M_\alpha(f)$ is an increasing function of α by [4, p. 143]. The remaining assertion of the theorem is easily verified.

There is an elementary proof of (4) for $p \geq 2$. Indeed, suppose that f is holomorphic in a neighborhood of $\bar{\Delta}$ and given $s \in S$, note that

$$\sum_{n=0}^{\infty} |a_n(s)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})(s)|^2\, d\theta$$

since the map $z \to f(z)(s)$ is holomorphic in a neighborhood of $\bar{\Delta}$ in the classical sense. Then by [4, p. 4] and Minkowski's inequality [2, p. 530, 13], we have

$$\sum_{n=0}^{\infty} \|a_n\|^p \le \int_S \left(\sum_{n=0}^{\infty} |a_n(s)|^2\right)^{p/2} d\mu(s)$$
$$\le \int_S \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})(s)|^2 \, d\theta\right)^{p/2} d\mu(s)$$
$$\le \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(e^{i\theta})\|^2 \, d\theta\right)^{p/2},$$

and clearly (4) follows.

Inequality (5) follows easily from Hölder's inequality and (4). Finally, (6) follows similarly from

$$\sum_{n=1}^{\infty} n^r x^{n-1} \le \frac{\Gamma(r+1)}{(1-x)^{r+1}}, \qquad 0<x<1,\ 1<r, \tag{7}$$

and this is a consequence of the binomial theorem and

$$(n+1)^r x^n \le \Gamma(r+1)(-1)^n \binom{-r-1}{n} x^n,$$

which holds since

$$\frac{(n+1)!\,(n+1)^{r-1}}{(r-1)r\cdots(r+n)} \le \Gamma(r-1)$$

by [8, p. 160]. (The number $\Gamma(r+1)$ is the best constant for which (7) holds for all $0<x<1$ by [1, p. 466].)

Other function-theoretic inequalities derived from interpolation theory are given in [5].

References

1. T. J. Bromwich, *Theory of Infinite Series*, Macmillan, 2nd ed., London 1926.
2. N. Dunford and J. T. Schwartz, *Linear Operators*, Interscience, New York, pt. I, 1958.
3. P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.
4. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, London, 1934.
5. L. Harris, *Bounds on the derivatives of holomorphic functions of vectors*, Proc. Colloq. Analysis, Rio de Janeiro, 1972, 145–163, L. Nachbin, Ed., Act. Sci. et Ind. Paris: Hermann, 1975.
6. T. Hayden and J. Wells, *On the extension of Lipschitz–Hölder maps of order α*, J. Math. Anal. Appl. **33** (1971), 627–640.
7. E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ. 31, Providence, 1957.
8. H. Kestelman, *Modern Theories of Integration*, Oxford University Press, London 1937.
9. A. Renaud, *Quelques propriétés des applications analytiques d'une boule de dimension infinie dans une autre*, Bull. Sci. Math. **97** (1973), 129–159.

10. E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 2nd ed., London 1939.
11. L. R. Williams and J. H. Wells, L^p *inequalities*, J. Math. Anal. Appl. **64** (1978), 518–529.

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