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Reduction to Dimension Two of the Local Spectrum for an *AH* Algebra with the Ideal Property

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Abstract. A C^*-algebra A has the ideal property if any ideal I of A is generated as a closed two-sided ideal by the projections inside the ideal. Suppose that the limit C^*-algebra A of inductive limit of direct sums of matrix algebras over spaces with uniformly bounded dimension has the ideal property. In this paper we will prove that A can be written as an inductive limit of certain very special subhomogeneous algebras, namely, direct sum of dimension-drop interval algebras and matrix algebras over 2-dimensional spaces with torsion H^2 groups.

1 Introduction

An AH algebra is a nuclear C^*-algebra of the form $A = \lim_{\to}(A_n, \phi_{n,m})$ with

$$A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i},$$

where $X_{n,i}$ are compact metric spaces, t_n, $[n,i]$ are positive integers, $M_{[n,i]}(C(X_{n,i}))$ are algebras of $[n,i]\times[n,i]$ matrices with entries in $C(X_{n,i})$, the algebra of complex-valued functions on $X_{n,i}$, and finally, $P_{n,i} \in M_{[n,i]}(C(X_{n,i}))$ are projections (see [Bla]). If we further assume that $\sup_{n,i}\dim(X_{n,i}) < +\infty$ and A has the ideal property, *i.e.*, each ideal I of A is generated by the projections inside the ideal, then it is proved in [GJLP1, GJLP2] that A can be written as an inductive limit of

$$B_n = \bigoplus_{i=1}^{s_n} P'_{n,i} M_{[n,i]'}(C(Y_{n,i}))P'_{n,i}.$$

In this paper, we will further reduce the dimension of local spectra (that is, the spectra of A_n or B_n above) to 2 (instead of 3). Namely, the above A can be written as an inductive limit of a direct sum of matrix algebras over the $\{\mathrm{pt}\}$, $[0,1]$, S^1, $T_{II,k}$ (no $T_{III,k}$ and S^2) and $M_l(I_k)$, where I_k is the dimension-drop interval algebra

$$I_k = \left\{ f \in C([0,1], M_k(\mathbb{C})), f(0) = \lambda \mathbf{1}_k, f(1) = \mu \mathbf{1}_k, \lambda, \mu \in \mathbb{C} \right\}.$$

In this paper, we will also call $\bigoplus_{i=1}^{s} M_{l_i}(I_{k_i})$ a dimension-drop algebra.

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This result unifies the theorems of [DG, EGS] (for the rank zero case) and [Li4] (for the simple case). Note that Li's reduction theorem was not used in the classification of simple AH algebra, and Li's proof depends on the classification of simple AH algebra (see [Li4, EGL1]). For our case, the reduction theorem is an important step toward the classification (see [GJL]). The proof is more difficult than Li's case. For example, in the case of an AH algebra with the ideal property, one cannot remove the space S^2 without introducing $M_l(I_k)$ (for the simple case, the space S^2 is removed from the list of spaces in [EGL1] without introducing dimension-drop algebras). Another point is that, in the simple AH algebras, one can assume each partial map $\phi_{n,m}^{i,j}$ is injective, but in AH algebras with the ideal property, we cannot make such an assumption. For the classification of real rank zero AH algebras, we refer the readers to [Ell1, EG1, EG2, G3-4, DG, D1, D2, G1, G2]. For the classification simple AH algebra, we refer the readers to [Ell2, Ell3, Li1, Li2, Li3, EGL1, EGL2, G5].

The paper is organized as follows. In Section 2, we will do some necessary preparation. In Section 3, we will prove our main theorem.

2 Preparation

We will adopt all the notation from [GJLP2, section 2]. For example, we refer the reader to [GJLP2] for the concepts of G-δ multiplicative maps (see Definition 2.2 there), spectral variation $SPV(\phi)$ of a homomorphism ϕ (see 2.12 there) weak variation $\omega(F)$ of a finite set $F \subset QM_N(C(X))Q$ (see 2.16 there).

As in [GJLP2, 2.17], we will use • to denote any possible integer.

2.1 In this article, without lose of generality we will assume the AH algebras A are inductive limit of

$$A = \lim\Big(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m}\Big),$$

where $X_{n,i}$ are the spaces of $\{\mathrm{pt}\}$, $[0,1]$, S^1, $T_{II,k}$, $T_{III,k}$, and S^2. (Note that by the main theorem of [GJLP2], all AH algebras with the ideal property and with no dimension growth are corner subalgebras of the above form (see also [GJLP2, 2.7]).)

2.2 Recall that a projection $P \in M_k(C(X))$ is called a *trivial projection* if it is unitarily equivalent to $\begin{pmatrix} 1_{k_1} & 0 \\ 0 & 0 \end{pmatrix}$ for $k_1 = \operatorname{rank}(P)$. If P is a trivial projection and $\operatorname{rank}(P) = k_1$, then

$$PM_k(C(X))P \cong M_{k_1}(C(X)).$$

2.3 Let X be a connected finite simplicial complex, $A = M_k(C(X))$. A unital $*$ homomorphism $\phi: A \to M_l(A)$ is called a *(unital) simple embedding* if it is homotopic to the homomorphism $\mathrm{id} \oplus \lambda$, where $\lambda: A \to M_{l-1}(A)$ is defined by

$$\lambda(f) = \operatorname{diag}(\underbrace{f(x_0), f(x_0), \dots, f(x_0)}_{l-1})$$

for a fixed base point $x_0 \in X$.

The following two lemmas are special cases of [EGS, Lemma 2.15] (see also [EGS, 2.12]).

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Lemma 2.1 (*cf.* [EGS, 2.12 or case 2 of 2.15]) *For any finite set $F \subset A = M_n(C(T_{III,k}))$ and $\varepsilon > 0$, there is a unital simple embedding $\phi\colon A \to M_l(A)$ (for l large enough) and a C^*-algebra $B \subset A$, which is a direct sum of dimension-drop algebras and a finite dimensional C^*-algebra such that*

$$\operatorname{dist}(\phi(f), B) < \varepsilon, \quad \forall f \in F.$$

Lemma 2.2 (see [EGS, case 1 of 2.15]) *For any finite set $F \subset M_n(C(S^2))$ and $\varepsilon > 0$, there is a unital simple embedding $\phi\colon A \to M_l(A)$ (for l large enough) and a C^*-algebra $B \subset A$, which is a finite dimensional C^*-algebra such that*

$$\operatorname{dist}(\phi(f), B) < \varepsilon, \quad \forall f \in F.$$

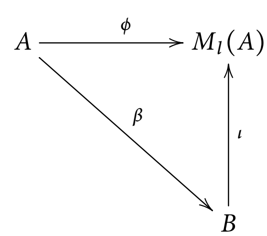
The following lemma is well known.

Lemma 2.3 (see [G5, 4.40]) *For any C^*-algebra A and finite set $F \subset A$, $\varepsilon > 0$, there is a finite set $G \subset A$ and $\eta > 0$ such that if $\phi\colon A \to B$ is a homomorphism and $\psi\colon A \to B$ is a completely positive linear map, satisfying*

$$\|\phi(g) - \psi(g)\| < \eta, \quad \forall g \in G,$$

then ψ is the F-ε multiplicative.

Lemma 2.4 *Let $A = M_n(C(T_{III,k}))$ or $M_n(C(S^2))$, and let a finite set $F \subset A$ and $\varepsilon > 0$, there is a commutative diagram*



with the following conditions:

(i) *ϕ is a simple embedding;*

(ii) *if $A = M_n(C(S^2))$, then B is a finite dimensional C^*-algebra, and if $A = M_n(C(T_{III,k}))$, then B is a direct sum of dimension-drop C^*-algebras and a finite dimensional C^*-algebra, and ι is an inclusion;*

(iii) *$\|\iota \circ \beta(f) - \phi(f)\| < \varepsilon$, $\forall f \in F$, and β is F-ε multiplicative.*

Proof Let G and η be as Lemma 2.3 for F and ε. Apply Lemma 2.1 or Lemma 2.2 to A, $F \cup G \subset A$ and $\frac{1}{3}\min(\varepsilon, \eta)$. One can find a unital simple embedding $\phi\colon A \to M_l(A)$, and an sub-C^*-algebra $B \subset M_l(A)$ as required in condition (ii) such that

$$\operatorname{dist}(\phi(f), B) < \tfrac{1}{3}\min(\varepsilon, \eta), \quad \text{for all } f \in F.$$

Choose a finite $\widetilde{F} \subset B$ such that

$$\operatorname{dist}(\phi(f), \widetilde{F}) < \tfrac{1}{3}\min(\varepsilon, \eta), \quad \text{for all } f \in F.$$

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Since B is a nuclear C^*-algebra, there are two completely positive linear maps

$$\lambda_1: B \longrightarrow M_N(\mathbb{C}) \quad \text{and} \quad \lambda_2: M_N(\mathbb{C}) \longrightarrow B$$

such that

$$\|\lambda_2 \circ \lambda_1(g) - g\| < \tfrac{1}{3}\min(\varepsilon, \eta), \quad \text{for all } g \in \widetilde{F}.$$

Using Arveson's extension theorem, one can extend $\lambda_1: B \to M_N(\mathbb{C})$ to a map $\beta_1: M_l(A) \to M_N(\mathbb{C})$. Then it is straightforward to prove that

$$\beta = \lambda_2 \circ \beta_1 \circ \phi: A \longrightarrow B$$

is as desired. ∎

The following is a modification of [GJLP2, Theorem 3.8].

Proposition 2.5 *Let $\lim_{n\to\infty}(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ be AH inductive limit with the ideal property, with $X_{n,i}$ being $\{\mathrm{pt}\}, [0,1], S^1, T_{II,k}, T_{III,k}$, or S^2. Let $B = \bigoplus_{i=1}^{s} B^i$, where $B^i = M_{l_i}(C(Y_i))$, with Y_i being $\{\mathrm{pt}\}, [0,1], S^1$, or $T_{II,k}$, (no $T_{III,k}$ or S^2) or $B^i = M_{l_i}(I_{k_i})$ (a dimension-drop C^*-algebra). Suppose that*

$$\widetilde{G}(=\bigoplus \widetilde{G}^i) \subset G(=\bigoplus G^i) \subset B(=\bigoplus B^i),$$

is a finite set, ε_1 is a positive number with $\omega(\widetilde{G}^i) < \varepsilon_1$, if $Y_i = T_{II,k}$, and L is any positive integer. Let $\alpha: B \to A_n$ be any homomorphism. Denote

$$\alpha(\mathbf{1}_B) := R(=\bigoplus R^i) \in A_n(=\bigoplus A_n^i).$$

Let $F \subset RA_nR$ be any finite set and let $\varepsilon < \varepsilon_1$ be any positive number. It follows that there are A_m, and mutually orthogonal projections $Q_0, Q_1, Q_2 \in A_m$ with

$$\phi_{n,m}(R) = Q_0 + Q_1 + Q_2,$$

a unital map $\theta_0 \in \mathrm{Map}(RA_nR, Q_0A_mQ_0)_1$, two unital homomorphisms

$$\theta_1 \in \mathrm{Hom}(RA_nR, Q_1A_mQ_1)_1 \quad \textit{and} \quad \xi \in \mathrm{Hom}(RA_nR, Q_2A_mQ_2)_1$$

such that:

(i) $\| \phi_{n,m}(f) - (\theta_0(f) \oplus \theta_1(f) \oplus \xi(f)) \| < \varepsilon$, *for all* $f \in F$;

(ii) *there is a unital homomorphism*

$$\alpha_1: B \longrightarrow (Q_0 + Q_1)A_m(Q_0 + Q_1),$$

such that

$$\big\|\alpha_1(g) - (\theta_0 + \theta_1) \circ \alpha(g)\big\| < 3\varepsilon_1 \quad \forall g \in \widetilde{G}_i, \quad \text{if } B^i \text{ is of form } M_\bullet(T_{II,k}),$$
$$\big\|\alpha_1(g) - (\theta_0 + \theta_1) \circ \alpha(g)\big\| < \varepsilon, \quad \forall g \in G^i, \quad \text{if } B^i \text{ is not of the form }_\bullet(T_{II,k});$$

(iii) *θ_0 is F-ε multiplicative and θ_1 satisfies*

$$\theta_1^{i,j}([e]) \geqslant L \cdot [\theta_0^{i,j}(R^i)].$$

(iv) *ξ factors through a C^*-algebra C, which is a direct sum of matrix algebras over $C[0,1]$, as*

$$\xi: RA_nR \xrightarrow{\xi_1} C \xrightarrow{\xi_2} Q_2A_mQ_2.$$

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Proposition 2.6 *Let* $\lim_{n\to\infty}(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ *be an AH inductive limit with the ideal property, with* $X_{n,i}$ *being* $\{\mathrm{pt}\}, [0,1], S^1, T_{II,k}, T_{III,k}$, *or* S^2. *Let* $B = \bigoplus_{i=1}^{s} B^i$, *where* $B^i = M_{l_i}(C(Y_i))$, *with* Y_i *being* $\{\mathrm{pt}\}, [0,1], S^1$, *or* $T_{II,k}$, *(no* $T_{III,k}$ *or* S^2*) or* $B^i = M_{l_i}(I_{k_i})$ *(a dimension-drop* C^**-algebra). Suppose that*

$$\widetilde{G}(=\bigoplus \widetilde{G}^i) \subset G(=\bigoplus G^i) \subset B(=\bigoplus B^i),$$

is a finite set, ε_1 *is a positive number with* $\omega(\widetilde{G}^i) < \varepsilon_1$, *if* $Y_i = T_{II,k}$, *and* $L > 0$ *is any positive integer. Let* $\alpha\colon B \to A_n$ *be any homomorphism. Let* $F \subset A_n$ *be any finite set and* $\varepsilon < \varepsilon_1$ *be any positive number. It follows that there are* A_m *and mutually orthogonal projections* $P, Q \in A_m$ *with* $\phi_{n,m}(\mathbf{1}_{A_n}) = P + Q$, *a unital map* $\theta \in \mathrm{Map}(A_n, PA_mP)_1$, *and a unital homomorphism* $\xi \in \mathrm{Hom}(A_n, QA_mQ)_1$ *such that:*

(i) $\| \phi_{n,m}(f) - (\theta(f) \oplus \xi(f)) \| < \varepsilon$, *for all* $f \in F$;

(ii) *there is a homomorphism* $\alpha_1\colon B \to PA_mP$ *such that*

$$\left\| \alpha_1^{i,j}(g) - (\theta\circ\alpha)^{i,j}(g) \right\| < 3\varepsilon_1 \quad \forall g \in \widetilde{G}^i, \qquad \text{if } B^i \text{ is of the form } M_\bullet(C(T_{II,k})),$$
$$\left\| \alpha_1^{i,j}(g) - (\theta\circ\alpha)^{i,j}(g) \right\| < \varepsilon \quad \forall g \in G^i, \quad \text{if } B^i \text{ is not of the form } M_\bullet(C(T_{II,k}));$$

(iii) $\omega(\theta(F)) < \varepsilon$ *and* θ *is* F-ε *multiplicative;*

(iv) ξ *factors through a* C^**-algebra* C, *which is a direct sum of matrix algebras over* $C[0,1]$ *or* $\mathbb{C}$, *as*

$$\xi\colon A_n \xrightarrow{\xi_1} C \xrightarrow{\xi_2} QA_mQ.$$

The proof is similar to Proposition 2.5 and is omitted.

2.4 Let $\alpha\colon \mathbb{Z} \to \mathbb{Z}/k_1\mathbb{Z}$ be the group homomorphism defined by $\alpha(\mathbf{1}) = [\mathbf{1}]$, where the right-hand side is the equivalent class $[\mathbf{1}]$ of $\mathbf{1}$ in $\mathbb{Z}/k_1\mathbb{Z}$. Then it is well known from homological algebra that for the group $\mathbb{Z}/k\mathbb{Z}$, α induces a surjective map

$$\alpha_*\colon \mathrm{Ext}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z})(=\mathbb{Z}/k\mathbb{Z}) \longrightarrow \mathrm{Ext}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}/k_1\mathbb{Z})(=\mathbb{Z}/(k,k_1)\mathbb{Z}),$$

where (k, k_1) is the greatest common factor of k and k_1.

Recall, as in [DN], for two connected finite simplicial complexes X and Y, we use $kk(Y, X)$ to denote the group of equivalent classes of homomorphisms from $C_0(X\backslash\{\mathrm{pt}\})$ to $C_0(Y\backslash\{\mathrm{pt}\}) \otimes \mathcal{K}(H)$. Please see [DN] for details.

Lemma 2.7 (i) *Any unital homomorphism*

$$\phi\colon C(T_{II,k}) \longrightarrow M_\bullet(C(T_{III,k_1})),$$

is homotopy equivalent to unital homomorphism ψ *factor as*

$$C(T_{II,k}) \xrightarrow{\psi_1} C(S^1) \xrightarrow{\psi_2} M_\bullet(C(T_{III,k_1})).$$

(ii) *Any unital homomorphism* $\phi\colon C(T_{II,k}) \to PM_\bullet(C(S^2))P$ *is homotopy equivalent to unital homomorphism* ψ *factor as*

$$C(T_{II,k}) \xrightarrow{\psi_1} \mathbb{C} \xrightarrow{\psi_2} PM_\bullet(C(S^2))P.$$

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Proof Part (ii) is well known (see [EG2, chapter 3]). To prove part (i), we note that

$$KK\big(C_0(S^1\backslash\{1\}), C_0(T_{III,k_1}\backslash\{x_1\}) = kk(T_{III,k_1}, S^1) = \mathbb{Z}/k_1\mathbb{Z} = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/k_1\mathbb{Z})\big),$$

where $x_1 \in T_{III,k_1}$ is a base point. The map $\alpha: \mathbb{Z} \to \mathbb{Z}/k_1\mathbb{Z}$ in 2.4 can be induced by a homomorphism: $\psi_2: C(S^1) \to M_\bullet(C(T_{III,k}))$.

Let

$$[\phi] \in kk(T_{III,k_1}\ T_{II,k}) = \mathrm{Ext}(K_0(C_0(T_{II,k}\backslash\{x_0\})), K_1(C_0(T_{III,k}))),$$

be the element induced by homomorphism ϕ, where $\{x_0\}$ is the base point. By 2.4,

$$[\phi] = \beta \times [\psi_2], \text{ for } \beta \in kk(S^1, T_{II,k}) = \mathrm{Ext}\big(K_0(C(T_{II,k}\backslash\{x_0\})), K_1(C(S^1))\big),$$

on the other hand β can be realized by unital homomorphism

$$\psi_1: C(T_{II,k}) \longrightarrow C(S^1)$$

(see [EG2, section 3]). ∎

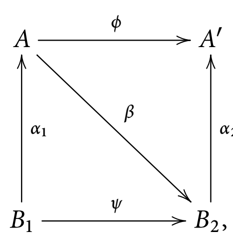
The following result is a modification of [GJLP2, Theorem 3.12].

Theorem 2.8 *Let $B_1 = \bigoplus_{i=1}^s B_1^i$, each B^i is either matrix algebras over $\{\mathrm{pt}\}, [0,1], S^1$ or $\{T_{II,k}\}_{k=2}^\infty$ or dimension-drop algebras. Let $\varepsilon_1 > 0$ and let*

$$\widetilde{G}_1(=\bigoplus \widetilde{G}_1^i) \subset G_1(=\bigoplus G_1^i) \subset B_1(=\bigoplus B_1^i)$$

be a finite set with $\omega(G_1^i) < \varepsilon_1$ for $B_1^i = M_\bullet(C(T_{II,k}))$.

Let $A = M_N(C(X))$, where X is one of $\{\mathrm{pt}\}, [0,1], S^1, \{T_{II,k}\}_{k=2}^\infty, \{T_{III,k}\}_{k=2}^\infty$, and S^2. Let $\alpha_1: B_1 \to A$ be a homomorphism. Let $F_1 \subset A$ be a finite set and let $\varepsilon(<\varepsilon_1)$ and δ be any positive number. Then there exists a commutative diagram



where $A' = M_K(A)$, and B_2 is as follows.

- *If $X = T_{III,k}$, then B_2 is a direct sum of a finite dimensional C^*-algebra and a dimension-drop algebra.*
- *If $X = S^2$, then B_2 is a finite dimensional algebra*
- *If Xi is one of $\{\mathrm{pt}\}, [0,1], S^1$, and $T_{II,k}$, then $B_2 = M_\bullet(A)$.*

Furthermore, the diagram satisfies the following conditions:

(i) *ψ is a homomorphism, α_2 is a unital injective homomorphism, and ϕ is a unital simple embedding;*
(ii) *$\beta \in \mathrm{Map}(A, B_2)_1$ is F_1-δ multiplicative;*
(iii) *if B_1^i is of the form $M_\bullet(C(T_{II,k}))$, then*

$$\| \psi(g) - \beta \circ \alpha_1(g) \| < 10\varepsilon_1, \quad \forall g \in \widetilde{G}_1^i;$$

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and if B_1^i *is not of the form* $M_\bullet(C(T_{II,k}))$*, then*

$$\| \psi(g) - \beta \circ \alpha_1(g) \| < \varepsilon, \quad \forall g \in G_1^i;$$

(iv) *if* $X = T_{II,k}$*, then* $\omega(\beta(F_1) \cup \psi(G_1)) < \varepsilon$.

(Note that we only require that the weak variation of finite sets in $M_\bullet(C(T_{II,k})$ to be small. In particular, we do not need to introduce the concept of weak variation for a finite subset of a dimension-drop algebra.)

Proof For $X = T_{II,k}, \{\text{pt}\}, [0,1]$ or S^1, one can choose $B_2 = M_K(A) = A'$ and let the homomorphism $\phi = \beta\colon A \to B_2$ be a simple embedding such that

$$\omega(\beta(F_1) \cup \alpha_1(G_1))) < \varepsilon.$$

This can be done by choosing K large enough. Choose $\psi = \beta \circ \alpha_1$, and $\alpha_2 = \text{id}\colon B_2 \to A'$.

For the case $X = T_{III,k}$, or S^2, requirement (iv) is an empty requirement.

We will deal with each block of B_1 separately. For the block B_1^i other than $M_\bullet(C(T_{II,k}))$, the construction can be done easily by using Lemma 2.4, since B_1^i is stably generated, which implies that any sufficiently multiplicative map from B_1^i is close to a homomorphism. So we assume that $B_1^i = M_\bullet(C(T_{II,k})$. Recall that we already assumed A is of the form $M_\bullet(C(T_{III,k}))$ or $M_\bullet(C(S^2)$. By Lemma 2.7, the homomorphism $\alpha_1\colon B_1^i \to A$ is a homotopy to $\alpha'\colon B_1^i \to A$ with $\alpha'(\mathbf{1}_{B_1^i}) = \alpha_1(\mathbf{1}_{B_1^i})$ and α' factor as

$$B_1^i \xrightarrow{\xi_1} C \xrightarrow{\xi_2} A,$$

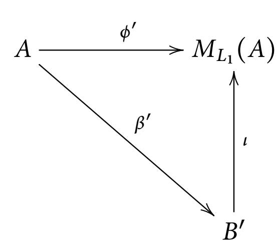
where C is a finite dimensional C^*-algebra for the case $X = S^2$ or $C = M_\bullet(C(S^1)$ for the case $X = T_{III,k}$ (note that $B_1^i = M_\bullet(C(T_{II,k}))$. Since C is stably generated, there is a finite set $E_1 \subset A$ and $\delta_1 > 0$ such that if a complete positive map $\beta\colon A \to D$ (for any C^*-algebra D) is E_1-δ_1 multiplicative, then the map $\beta \circ \xi_2\colon C \to D$ can be perturbed to a homomorphism $\widetilde{\xi}\colon C \to D$ such that

$$\|\widetilde{\xi}(g) - \beta \circ \xi_2(g)\| < \varepsilon_1, \quad \text{for all } g \in \xi_1(\widetilde{G}_1^i).$$

Apply [G5, Theorem 1.6.9] to two homotopic homomorphism

$$\alpha_1, \alpha'\colon B_1^i \longrightarrow A, \quad \text{and} \quad G_1^i \subset B_1^i,$$

which is approximately constant to within ε_1, to obtain a finite set $E_2 \subset A$, $\delta_2 > 0$ and positive integer $L' > 0$ (in places of G, δ and L in [G5, Theorem 1.6.9]). Apply Lemma 2.4 to the set $\widetilde{E} = E_1 \cup E_2 \cup F_1$ and $\widetilde{\delta} = \frac{1}{3}\min(\varepsilon, \delta, \delta_1, \delta_2)$ to obtain the commutative diagram



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with β' being $\widetilde{E}$-$\widetilde{\delta}$ multiplicative and

$$\| \iota \circ \beta'(f) - \phi'(f) \| < \widetilde{\delta}, \quad \text{for all } f \in \widetilde{E}.$$

Let $L = L' \cdot \operatorname{rank}(\mathbf{1}_A)$ and let $\beta_1 \colon A \to M_L(B')$ be any unital homomorphism defined by point evaluation. Then by [G5, Theorem 1.6.9], there is a unitary $u \in M_{L+1}(B)$ such that

$$\left\| u((\beta' \oplus \beta_1) \circ \alpha'(f))u^* - (\beta' \oplus \beta_1) \circ \alpha_1(f) \right\| < 8\varepsilon_1, \quad \forall f \in \widetilde{G}_1^i.$$

By the choice of E_1, there is a homomorphism $\widetilde{\xi} \colon C \to M_{L+1}(B')$, such that

$$\left\| \widetilde{\xi}(f) - u((\beta' \oplus \beta) \circ \xi_2(f))u^* \right\| < \varepsilon_1, \quad \text{for all } f \in \xi_1(\widetilde{G}_1^i).$$

Define $B_2 = M_{L+1}(B')$, $K = L_1(L+1)$, $A' = M_K(A) = M_{L+1}(M_{L_1}(A))$,

$$\begin{aligned} &\psi \colon B_1^i \longrightarrow B_2 && \text{by } \psi = \widetilde{\xi} \circ \xi_1 \colon B_1^i \xrightarrow{\xi_1} C \xrightarrow{\widetilde{\xi}} B_2, \\ &\beta \colon A \longrightarrow M_{L+1}(B') && \text{by } \beta = \beta' \oplus \beta_1, \\ &\phi \colon A \longrightarrow M_{L+1}(M_{L_1}(A)) && \text{by } \phi = \phi' \oplus ((\iota \otimes id_L) \circ \beta_1) \end{aligned}$$

(note that β_1 is a homomorphism) to finish the proof. ∎

2.5 Recall that for $A = \bigoplus_{i=1}^t M_{k_i}(C(X_i))$, where X_i are path connected simplicial complexes, we use the notation $r(A)$ to denote $\bigoplus_{i=1}^t M_{k_i}(\mathbb{C})$, which could be considered to be the subalgebra consisting of all t-tuples of constant function from X_i to $M_{k_i}(\mathbb{C})$ $(i = 1, 2, \dots, t)$. Fixed a base point $x_i^0 \in X_i$ for each X_i, one defines a map $r \colon A \to r(A)$ by

$$r(f_1, f_2, \dots, f_t) = \left(f_1(x_1^0), f_2(x_2^0), \dots, f_t(x_t^0) \right) \in r(A).$$

We have the following corollary.

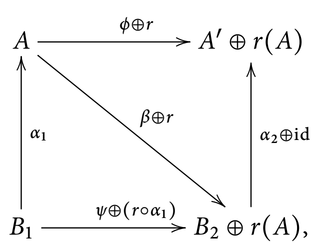
Corollary 2.9 *Let $B_1 = \bigoplus B_1^j$, where B_1^j is either of the form $M_{k(j)}(C(X_j))$, with X_j being one of $\{\mathrm{pt}\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^\infty$ or $B_1^j = M_{k(j)}(I_{l(j)})$. Let $\alpha_1; B_1 \to A$ be a homomorphism, where A is a direct sum of matrix algebras over $\{\mathrm{pt}\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^\infty$, $\{T_{III,k}\}_{k=2}^\infty$, and S^2. Let $\varepsilon_1 > 0$ and let*

$$\widetilde{E}(= \bigoplus \widetilde{E}^i) \subset E(= \bigoplus E^i) \subset B_1(= \bigoplus B_1^i)$$

be two finite subsets with the condition

$$\omega(\widetilde{E}^i) < \varepsilon_1, \text{ if } B_1^i = M_\bullet(C(Y_i)) \text{ with } Y_i \in \{T_{II,k}\}_{k=2}^\infty.$$

Let $F \subset A$ be any finite set, $\varepsilon_2 > 0$, $\delta > 0$. Then there exists a commutative diagram



where $A' = M_L(A)$, and B_2 is a direct sum of matrix algebras over spaces $\{\mathrm{pt}\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^{\infty}$, and dimension-drop algebras, with the following properties:

(i) *ψ is a homomorphism, α_2 is a injective homomorphism, and ϕ is a unital simple embedding;*

(ii) *$\beta \in \mathrm{Map}(A, B_2)_1$ is F_1-δ multiplicative;*

(iii) *for $g \in \widetilde{E}^i$ with $B_1^i = M_\bullet(C(X_i))$, $X_i \in \{T_{II,k}\}_{k=2}^{\infty}$, we have*

$$\|(\beta \oplus r)(g) - (\psi \oplus (r \circ \alpha_1))(g)\| \leqslant 10\varepsilon_1,$$

for $g \in E^i (\supset \widetilde{E}^i)$ where B_1^i is not of the form $M_\bullet(C(T_{II,k}))$, we have

$$\|(\beta \oplus r)(g) - (\psi \oplus (r \circ \alpha_1))(g)\| < \varepsilon_1,$$

and for $f \in F$, we have

$$\|(\alpha_2 \oplus id) \circ (\beta \oplus r)(f) - (\phi \oplus r)(f)\| < \varepsilon_1;$$

(iv) *for B_2^i of the form $M_\bullet(C(T_{II,k}))$,*

$$\omega(\pi_i(\beta(F) \cup \psi(E))) < \varepsilon_2,$$

where π_i is the canonical projection from B_2 to B_2^i.

Remark In the application of this corollary, we will denote the map $\beta \oplus r$ by β and $\psi \oplus (r \circ \alpha_1)$ by ψ.

3 Proof of the Main Theorem

In this section, we prove the following main theorem.

Theorem 3.1 *Suppose $\lim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ is an AH inductive limit with $X_{n,i}$ being among the spaces $\{\mathrm{pt}\}, [0,1], S^1, \{T_{II,k}\}_{k=2}^{\infty}$, and $\{T_{III,k}\}_{k=2}^{\infty}$, such that the limit algebra A has the ideal property. Then there is another inductive system, $B_n = \bigoplus B_n^i, \psi_{n,m}$, with same limit algebra, where each B_n^i is either $M_{[n,i]'}(C(Y_{n,i}))$ with $Y_{n,i}$ being one of $\{\mathrm{pt}\}, [0,1], S^1, \{T_{II,k}\}_{k=2}^{\infty}$ (but without $T_{III,k}$ and S^2), or B_n^i is the dimension-drop algebra $M_{[n,i]'}(I_{k(n,i)})$.*

Proof Let $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots$ be a sequence of positive numbers with $\sum \varepsilon_n < +\infty$. We need to construct the intertwining commutative diagram

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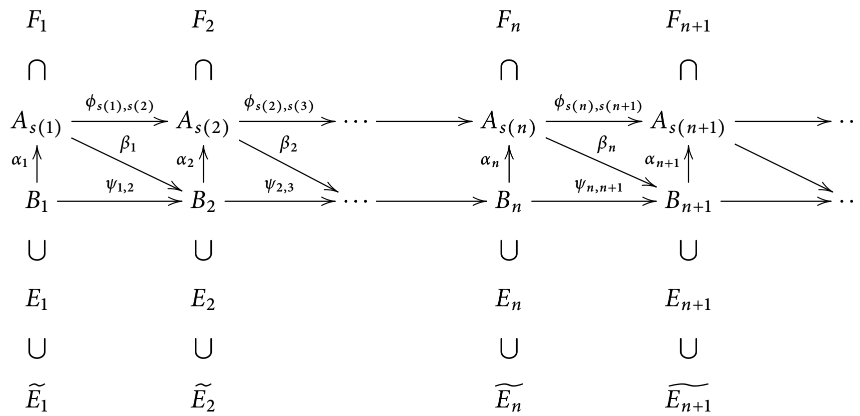

satisfying the following conditions.

(a) $(A_{s(n)}, \phi_{s(n),s(m)})$ is a sub-inductive system of $(A_n, \phi_{n,m}), (B_n, \psi_{n,m})$ is an inductive system of direct sum of matrix algebras over the spaces $\{\text{pt}\}, [0,1], S^1, T_{II,k}$ and dimension drop algebra $M_\bullet(I_{k(n,i)})$.

(b) Choose $\{a_{i,j}\}_{j=1}^{\infty} \subset A_{s(i)}$ and $\{b_{i,j}\}_{j=1}^{\infty} \subset B_i$ to be countable dense subsets of unit balls of $A_{s(i)}$ and B_i, respectively. F_n are subsets of unit balls of $A_{s(n)}$, and $\widetilde{E_n} \subset E_n$ are both subsets of unit balls of B_n satisfying

$$\phi_{s(n),s(n+1)}(F_n) \cup \alpha_{n+1}(E_{n+1}) \cup \bigcup_{i=1}^{n+1} \phi_{s(i),s(n+1)}(\{a_{i1}, a_{i2}, \dots, a_{in+1}\}) \subset F_{n+1},$$
$$\psi_{n,n+1}(E_n) \cup \beta_n(F_n) \subset \widetilde{E}_{n+1} \subset E_{n+1},$$
$$\bigcup_{i=1}^{n+1} \psi_{i,n+1}(\{b_{i1}, b_{i2}, \dots, b_{in+1}\}) \subset E_{n+1}.$$

(Here $\phi_{n,n}: A_n \to A_n$, and $\psi_{n,n}: B_n \to B_n$ are understood as identity maps.)

(c) β_n are F_n-$2\varepsilon_n$ multiplicative and α_n are homomorphism.

(d) For all $g \in \widetilde{E}_n$,

$$\|\psi_{n,n+1}(g) - \beta_n \circ \alpha_n(g)\| < 14\varepsilon_n,$$

and for all $f \in F_n$,

$$\|\phi_{s(n),s(n+1)}(f) - \alpha_{n+1} \circ \beta_n(f)\| < 14\varepsilon_n.$$

(e) For any block B_n^i with spectrum $T_{II,k}$, we have $\omega(\widetilde{E}_n^i) < \varepsilon_n$, where $\widetilde{E}_n^i = \pi_i(\widetilde{E}_n)$ for $\pi_i: B_n \to B_n^i$ the canonical projections.

The diagram will be constructed inductively. First, let $B_1 = \{0\}, A_{s(1)} = A_1, \alpha_1 = 0$. Let $b_{1j} = 0 \in B_1$ for $j = 1, 2, \dots$, and let $\{a_{1j}\}_{j=1}^{\infty}$ be a countable dense subset of the unit ball of $A_{s(1)}$. And let $\widetilde{E}_1 = E_1 = \{b_{11}\} = B_1$ and $F_1 = \bigoplus_{i=1}^{t_1} F_1^i$, where $F_1^i = \pi_i(\{a_{11}\}) \subset A_1^i$.

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As inductive assumption, assume that we already have the commutative diagram

$$
\begin{array}{ccccccc}
F_1 & & F_2 & & & & F_n \\
\cap & & \cap & & & & \cap \\
A_{s(1)} & \xrightarrow{\phi_{s(1),s(2)}} & A_{s(2)} & \xrightarrow{\phi_{s(2),s(3)}} & \cdots & \longrightarrow & A_{s(n)} \\
{\scriptstyle\alpha_1}\uparrow & \searrow^{\beta_1} & {\scriptstyle\alpha_2}\uparrow & \searrow^{\beta_2} & & \searrow^{\beta_{n-1}} & {\scriptstyle\alpha_n}\uparrow \\
B_1 & \xrightarrow{\psi_{1,2}} & B_2 & \xrightarrow{\psi_{2,3}} & \cdots & \longrightarrow & B_n \\
\cup & & \cup & & & & \cup \\
E_1 & & E_2 & & & & E_n \\
\cup & & \cup & & & & \cup \\
\widetilde{E}_1 & & \widetilde{E}_2 & & & & \widetilde{E_n}
\end{array}
$$

and for each $i = 1, 2, \dots, n$, we have dense subsets $\{a_{ij}\}_{j=1}^{\infty}$ of the unit ball of $A_{s(i)}$ and $\{b_{ij}\}_{j=1}^{\infty}$ of the unit ball of B_i, satisfying conditions (a)–(e) above. We have to construct the next piece of the diagram

$$
\begin{array}{ccc}
F_n \subset A_{s(n)} & \xrightarrow{\phi_{s(n),s(n+1)}} & A_{s(n+1)} \supset F_{n+1} \\
{\scriptstyle\alpha_n}\uparrow & \searrow^{\beta_n} & \uparrow{\scriptstyle\alpha_{n+1}} \\
\widetilde{E}_n \subset E_n \subset B_n & \xrightarrow[\psi_{n,n+1}]{} & B_{n+1} \supset E_{n+1} \supset \widetilde{E}_{n+1}
\end{array}
$$

to satisfy conditions (a)–(e).

Among the conditions for induction assumption, we will only use the conditions that α_n is a homomorphism and (e) above.

Step 1. We enlarge $\widetilde{E}_n$ to $\bigoplus_i \pi_i(\widetilde{E}_n^i)$ and enlarge E_n to $\bigoplus_i \pi_i(E_n)$. Then we have $\widetilde{E}_n(= \bigoplus \widetilde{E}_n^i) \subset E_n(= \bigoplus E_n)$, and for each B_n^i with spectrum $T_{II,k}$, we have $\omega(E_n^i) < \varepsilon_n$ from induction assumption (e). By Proposition 2.6 applied to $\alpha_n\colon B_n \to A_{s(n)}, \widetilde{E}_n \subset E_n \subset B_n, F_n \subset A_{s(n)}$ and $\varepsilon_n > 0$, there are $A_{m_1}(m_1 > s(n))$, two orthogonal projections $P_0, P_1 \in A_{m_1}$ with $\phi_{s(n),m_1}(\mathbf{1}_{A_{s(n)}}) = P_0 + P_1$ and P_0 trivial, a C^*-algebra C, that is, a direct sum of matrix algebras over $C[0,1]$ or $\mathbb{C}$, and a unital map $\theta \in \mathrm{Map}(A_{s(n)}, P_0 A_{m_1} P_0)_1$, a unital homomorphism $\xi_1 \in \mathrm{Hom}(A_{s(n)}, C)_1$, a unital homomorphism $\xi_2 \in \mathrm{Hom}(C, P_1 A_{m_1} P_1)_1$ such that

(1.1) $\|\phi_{s(n),m_1}(f) - \theta(f) \oplus (\xi_2 \circ \xi_1)(f)\| < \varepsilon_n$ for all $f \in F_n$.
(1.2) θ is F_n-ε multiplicative and $F := \theta(F_n)$ satisfies $\omega(F) < \varepsilon_n$.
(1.3) $\|\alpha(g) - \theta \circ \alpha_n(g)\| < 3\varepsilon_n$ for all $g \in \widetilde{E}_n$.

Let all the blocks of C be parts of the C^*-algebra B_{n+1}. That is,

$$B_{n+1} = C \oplus \text{ (some other blocks)}.$$

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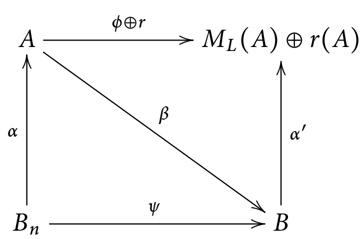
The map $\beta_n : A_{s(n)} \to B_{n+1}$, and the homomorphism $\psi_{n,n+1} : B_n \to B_{n+1}$ are defined by $\beta_n = \xi_1 : A_{s(n)} \to C(\subset B_{n+1})$ and $\psi_{n,n+1} = \xi_1 \circ \alpha_n : B_n \to C(\subset B_{n+1})$ for the blocks of $C(\subset B_{n+1})$. For this part, β_n is also a homomorphism.

Step 2. Let $A = P_0 A_{m_1} P_0$, $F = \theta(F_n)$. Since P_0 is a trivial projection,

$$A \cong \bigoplus M_{l_i}(C(X_{m_1,i})).$$

Let $r(A) := \bigoplus M_{l_i}(\mathbb{C})$ and $r: A \to r(A)$ be as in 2.13. Applying Corollary 2.9 and its remark to $\alpha: B_n \to A$, $\widetilde{E}_n \subset E_n \subset B_n$ and $F \subset A$, we obtain the commutative diagram



such that

(2.1) B is a direct sum of matrix algebras over $\{\mathrm{pt}\}, [0,1], S^1, T_{II,k}$ and dimension-drop algebras;
(2.2) α' is an injective homomorphism and β is F-ε_n multiplicative;
(2.3) $\phi: A \to M_L(A)$ is a unital simple embedding and $r: A \to r(A)$ is as in 2.13;
(2.4) $\|\beta \circ \alpha(g) - \psi(g)\| < 10\varepsilon_n$ for all $g \in \widetilde{E}_n$ and $\|(\phi \oplus r)(f) - \alpha' \circ \beta(f)\| < \varepsilon_n$ for all $f \in F(:= \theta(F_n))$;
(2.5) $\omega(\pi_i(\psi(E_n)) \cup \beta(F)) < \varepsilon_{n+1}$ (note that $\beta(F) = \beta \circ \theta(F_n)$), for B_n^i being of the form $M_\bullet(C(X))$ with $X \in \{T_{II,k}\}_{k=2}^{\infty}$.

Let all the blocks B be also part of B_{n+1}, that is,

$$B_{n+1} = C \oplus B \oplus \text{ (some other blocks)}.$$

The maps $\beta_n : A_{s(n)} \to B_{n+1}, \psi_{n,n+1} : B_n \to B_{n+1}$ are defined by

$$\beta_n := \beta \circ \theta : A_{s(n)} \xrightarrow{\theta} A \xrightarrow{\beta} B(\subset B_{n+1}),$$
$$\psi_{n,n+1} := \psi : B_n \to B(\subset B_{n+1}),$$

for the blocks of $B(\subset B_{n+1})$. This part of β_n is F_n-$2\varepsilon_n$ multiplicative, since θ is F_n-ε_n multiplicative, β is F-ε_n multiplicative, and $F = \theta(F_n)$.

Step 3. By [GJLP2, Lemma 3.15] applied to $\phi \oplus r: A \to M_L(A) \oplus r(A)$, there is an A_{m_2} and there is a unital homomorphism

$$\lambda: M_L(A) \oplus r(A) \longrightarrow R A_{m_2} R,$$

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where $R = \phi_{m_1,m_2}(P_0)$ (write R as $\bigoplus_j R^j \in \bigoplus_j A^j_m$) such that the diagram

$$\begin{array}{ccc} A(=P_0A_mP_0) & \xrightarrow{\phi_{m_1,m_2}} & RA_{m_2}R \\ & \searrow^{\phi\oplus r} & \uparrow \lambda \\ & & M_L(A)\oplus r(A) \end{array}$$

satisfies the following condition:

(3.1) $\lambda\circ(\phi\oplus r)$ is homotopy equivalent to $\phi' := \phi_{m_1,m_2}|_A$.

Step 4. Applying [G5, Theorem 1.6.9] to finite set $F\subset A$ (with $\omega(F) < \varepsilon_n$) and to two homotopic homomorphisms ϕ' and $\lambda\circ(\phi\oplus r)\colon A\to RA_{m_2}R$ (with $RA_{m_2}R$ in place of C in [G5, Theorem 1.6.9]), we obtain a finite set $F'\subset RA_{m_2}R$, $\delta>0$ and $L>0$ as in Theorem 3.1.

Let $G = \bigoplus \pi_i(\psi(E_n)\cup\beta(F)) = \bigoplus G^i$. Then by (2.5), we have $\omega(G^i) < \varepsilon_{n+1}$, if B^i is of the form $M_\bullet(C(T_{II,k}))$. By Proposition 2.5 applied to $RA_{m_2}R$ and

$$\lambda\circ\alpha'\colon B\longrightarrow RA_{m_2}R,$$

finite set $G\subset B$, $F'\cup(\phi_{m_1m_2}|_A(F))\in RA_{m_2}R$, $\min(\varepsilon_n,\delta)>0$ (in place of ε) and $L>0$, there are $A_{s(n+1)}$, mutually orthogonal projections $Q_0, Q_1, Q_2\in A_{s(n+1)}$ with $\phi_{m_2,s(n+1)}(R) = Q_0\oplus Q_1\oplus Q_2$, a C^*-algebra D,a direct sum of matrix algebras over C[0,1] or $\mathbb{C}$, a unital map $\theta_0\in \mathrm{Map}(RA_{m_2}R, Q_0A_{s(n+1)}Q_0)$, and four unital homomorphisms

$$\begin{aligned} &\theta_1\in \mathrm{Hom}(RA_{m_2}R, Q_1A_{s(n+1)}Q_1)_1, && \xi_3\in\mathrm{Hom}(RA_{m_2}R, D)_1,\\ &\xi_4\in\mathrm{Hom}(D, Q_2A_{s(n+1)}Q_2)_1, && \alpha''\in\mathrm{Hom}(B,(Q_0+Q_1)A_{s(n+1)}(Q_0+Q_1))_1 \end{aligned}$$

such that the following are true:

(4.1) $\|\phi_{m_2,s(n+1)}(f) - ((\theta_0+\theta_1)\oplus\xi_4\circ\xi_3)(f)\| < \varepsilon_n$, for all $f\in\phi_{m_1,m_2}|_A(F)\subset RA_{m_2}R$.

(4.2) $\|\alpha''(g) - (\theta_0+\theta_1)\circ\lambda\circ\alpha'(g)\| < 3\varepsilon_{n+1} < 3\varepsilon_n,\ \forall\, g\in G$.

(4.3) θ_0 is F'-$\min(\varepsilon_n,\delta)$ multiplicative and θ_1 satisfies that

$$\theta_1^{i,j}([q]) > L\cdot[\theta_0^{i,j}(R^i)],$$

for any non zero projection $q\in R^iA_{m_1}R^i$.

By [G5, Theorem 1.6.9], there is a unitary $u\in(Q_0\oplus Q_1)A_{s(n+1)}(Q_0+Q_1)$ such that

$$\|(\theta_0+\theta_1)\circ\phi'(f) - \mathrm{Ad}\,u\circ(\theta_0+\theta_1)\circ\lambda\circ(\phi\oplus r)(f)\| < 8\varepsilon_n,$$

for all $f\in F$.

Combining with the second inequality of (2.4), we have

(4.4) $\|(\theta_0+\theta_1)\circ\phi'(f) - \mathrm{Ad}\,u\circ(\theta_0+\theta_1)\circ\lambda\circ\alpha'\circ\beta(f)\| < 9\varepsilon_n$ for all $f\in F$.

Step 5. Finally let all blocks of D be the rest of B_{n+1}. Namely, let

$$B_{n+1} = C\oplus B\oplus D,$$

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where C is from Step 1, B is from Step 2, and D is from Step 4.

We already have the definition of $\beta_n\colon A_{s(n)} \to B_{n+1}$ and $\psi_{n,n+1}\colon B_n \to B_{n+1}$ for those blocks of $C \oplus B \subset B_{n+1}$ (from Step 1 and Step 2). The definition of β_n and $\psi_{n,n+1}$ for blocks of D and the homomorphism $\alpha_{n+1}\colon C \oplus B \oplus D \to A_{s(n+1)}$ will be given below.

The part of $\beta_n\colon A_{s(n)} \to D(\subset B_{n+1})$ is defined by

$$\beta_n = \xi_3 \circ \phi' \circ \theta\colon A_{s(n)} \xrightarrow{\theta} A \xrightarrow{\phi} RA_{m_2}R \xrightarrow{\xi_3} D.$$

(Recall that $A = P_0 A_{m_2} P_0$ and $\phi' = \phi_{m_1,m_2}|_A$.) Since θ is F_n-ε_n multiplicative, and ϕ' and ξ_3 are homomorphism, we know this part of β_n is F_n-ε_n multiplicative.

The part of $\psi_{n,n+1}\colon B_n \to D(\subset B_{n+1})$ is defined by

$$\psi_{n,n+1} = \xi_3 \circ \phi' \circ \alpha\colon B_n \xrightarrow{\alpha} A \xrightarrow{\phi'} RA_m R \xrightarrow{\xi_3} D,$$

which is a homomorphism.

The homomorphism $\alpha_{n+1}\colon C \oplus B \oplus D \to A_{s(n+1)}$ is defined as follows.

Let $\phi'' = \phi_{m_1,s(n+1)}|_{P_1 A_{m_1} P_1}\colon P_1 A_{m_1} P_1 \to \phi_{m_1,s(n+1)}(P_1) A_{s(n+1)} \phi_{m_1,s(n+1)}(P_1)$, where P_1 is from Step 1. Define

$$\alpha_{n+1}|_C = \phi'' \circ \xi_2\colon C \xrightarrow{\xi_2} P_1 A_{m_1} P_1 \xrightarrow{\phi''} \phi_{m_1,s(n+1)}(P_1) A_{s(n+1)} \phi_{m_1,s(n+1)}(P_1),$$

where ξ_2 is from Step 1, and define

$$\alpha_{n+1}|_B = \operatorname{Ad} u \circ \alpha''\colon B \xrightarrow{\alpha''} (Q_0 \oplus Q_1) A_{s(n+1)} (Q_0 + Q_1) \xrightarrow{Adu} (Q_0 \oplus Q_1) A_{s(n+1)} (Q_0 + Q_1)$$

where α'' is from Step 4, and define

$$\alpha_{n+1}|_D = \xi_4\colon D \to Q_2 A_{s(n+1)} Q_2.$$

Finally choose $\{a_{n+1,j}\}_{j=1}^{\infty} \subset A_{s(n+1)}$ and $\{b_{n+1,j}\}_{j=1}^{\infty} \subset B_{n+1}$ to be countable dense subsets of the unit balls of $A_{s(n+1)}$ and B_{n+1}, respectively, and choose

$$F'_{n+1} = \phi_{s(n),s(n+1)}(F_n) \cup \alpha_{n+1}(E_{n+1}) \cup \bigcup_{i=1}^{n+1} \phi_{s(i),s(n+1)}(\{a_{i1}, a_{i2}, \dots, a_{in+1}\}),$$

$$E'_{n+1} = \psi_{n,n+1}(E_n) \cup \beta_n(F_n) \cup \bigcup_{i=1}^{n+1} \psi_{i,n+1}(\{b_{i1}, b_{i2}, \dots, b_{in+1}\}),$$

$$\widetilde{E}_{n+1}{}' = \psi_{n,n+1}(E_n) \cup \beta_n(F_n) \subset E_{n+1}{}'.$$

Define $F_{n+1}^i = \pi_i(F_{n+1}{}')$ and $F_{n+1} = \bigoplus_i F_{n+1}^i$, $E_{n+1}^i = \pi_i(E'_{n+1})$ and $E_{n+1} = \bigoplus_i E_{n+1}^i$. For those blocks B_{n+1}^i inside the algebra B define $\widetilde{E}_{n+1}^i = \pi_i(\widetilde{E}_{n+1})$. For those blocks inside C and D, define $\widetilde{E}_{n+1}^i = E_{n+1}^i$. And finally let $E_{n+1} = \bigoplus_i \widetilde{E}_{n+1}^i$. Note all the blocks with spectrum $T_{II,k}$ are in B, and (2.5) tells us that for those blocks $\omega(\widetilde{E}_{n+1}^i) < \varepsilon_{n+1}$. Thus we obtain the commutative diagram

$$\begin{array}{ccc}
F_n \subset A_{s(n)} & \xrightarrow{\phi_{s(n),s(n+1)}} & A_{s(n+1)} \supset F_{n+1} \\
\uparrow{\scriptstyle \alpha_n} & \searrow^{\beta_n} & \uparrow{\scriptstyle \alpha_{n+1}} \\
\widetilde{E}_n \subset E_n \subset B_n & \xrightarrow[\psi_{n,n+1}]{} & B_n \supset E_{n+1} \supset \widetilde{E}_{n+1}.
\end{array}$$

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Step 6. Now we need to verify conditions (a)–(e) for the above diagram.

From the end of Step 5, we know that (e) holds; (a)–(b) hold from the construction (see the construction of B, C, D in Steps 1, 2 and 4, and $\widetilde{E}_{n+1} \subset E_{n+1}$, F_{n+1} is the end of Step 5); (c) follows from the end of Step 1, the end of Step 2 and the part of definition of β_n for D from Step 5.

So we only need to verify (d).

Combining (1.1) with (4.1), we have

$$\|\phi_{s(n),s(n+1)}(f) - [(\phi'' \circ \xi_2 \circ \xi_1) \oplus (\theta_0 + \theta_1) \circ \phi' \circ \theta \oplus (\xi_4 \circ \xi_3 \circ \phi' \circ \theta)(f)](f)\| < \varepsilon_n + \varepsilon_n = 2\varepsilon_n$$

for all $f \in F_n$ (recall that $\phi'' = \phi_{m_1,s(n+1)}|_{P_1 A_{m_1} P_1}$, $\phi' := \phi_{m_1,m_2}|_{P_0 A_{m_1} P_o}$).

Combined with (4.2), (4.4), and the definitions of β_n and α_{n+1}, the above inequality yields

$$\left\| \phi_{s(n),s(n+1)}(f) - (\alpha_{n+1} \circ \beta_{n+1})(f) \right\| < 9\varepsilon_n + 3\varepsilon_n + 2\varepsilon_n = 14\varepsilon_n, \quad \forall f \in F_n.$$

Combining (1.3), the first inequality of (2.4), and the definition of β_n and $\psi_{n,n+1}$, we have

$$\|\psi_{n,n+1}(g) - (\beta_n \circ \alpha_n)(g)\| < 10\varepsilon_n + 3\varepsilon_n < 14\varepsilon_n, \quad \forall\, g \in \widetilde{E}_n.$$

So we obtain (d). The theorem follows from [GJLP2, Proposition 4.1]. ■

Note that if $q \in M_l(I_k)$, then $qM_k(I_k)q$ isomorphic to $M_{l_1}(I_k)$. Combining with the main theorem of [GJLP2] (see [GJLP2, Theorem 4.2, and 2.7]) we have the following theorem.

Theorem 3.2 *Suppose that* $A = \lim(A_n = \bigoplus P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i})$ *is an AH inductive limit with* $\dim(X_{n,i}) \leqslant M$ *for a fixed positive integer* M *such that limit algebra* A *has the ideal property. Then* A *can be rewrite as inductive limit* $\lim(B_n = \bigoplus B_n^i, \psi_{n,m})$, *where either* $B_n^i = Q_{n,i} M_{[n,i]'}(C(Y_{n,i}))Q_{n,i}$ *with* $Y_{n,i}$ *being one of the spaces* $\{\mathrm{pt}\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^{\infty}$, *or* $B_n^i = M_{[n,i]'}(I_{l_{(n,i)}})$ *a dimension-drop algebra.*

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