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QUASI-INJECTIVE MODULES SATISFYING CERTAIN RELATIVE FINITENESS CONDITIONS

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Abstract

We study the endomorphism ring of a quasi-injective right R-module Q such that R satisfies certain finiteness conditions relative to Q. And we are concerned with a module ${}_S\mathrm{Hom}_R(M,Q)$, where S is the endomorphism ring of Q_R

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1. Introduction

Endomorphism rings of Σ (respectively Δ)-(quasi-)injective modules over an associative ring with identity have been studied mainly by Faith and Năstăsescu (refer to [4], [10], [2], and so on). An injective right R-module Q is said to be Σ (respectively Δ)-injective if the lattice of all Q-closed right ideals of R, that is, $\mathcal{C}_Q(R) = \{I_R \subseteq R_R | R/I \text{ is } Q\text{-torsionless}\}$ is noetherian (respectively artinian). Faith has shown in [4] that the endomorphism ring of a finitely generated Σ (respectively Δ)-injective right R-module is a right perfect (respectively a left artinian) ring. Moreover, Năstăsescu has shown in [10] that (1) the endomorphism ring $\mathrm{End}(Q_R)$ of a Σ (respectively Δ)-injective right R-module Q which has a finitely generated R-submodule Q' such that $\mathrm{Hom}_R(Q/Q', Q) = (0)$ (in particular, of a finitely generated Σ (respectively Δ)-injective right R-module

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Q), is a semi-primary (respectively a left artinian) ring, and (2) if Q is a noetherian or artinian, Δ-injective right R-module, then $\mathrm{End}(Q_R)$, the endomorphism ring of Q_R, is a left artinian ring and $\mathrm{Biend}(Q_R)$, the biendomorphism ring of Q_R, is a right artinian ring.

In the present paper we shall generalize those results to the case where Q_R is quasi-injective. For this purpose we shall introduce the concepts of Q-noetherian, Q-artinian and Q-finitely generated modules with respect to any right R-module Q. And we shall show that when Q is a quasi-injective right R-module with $S = \mathrm{End}(Q_R)$ and M is a Q-finitely generated (in particular, finitely generated) right R-module such that Q is M-injective, then (1) if R_R is Q-noetherian, ${}_S\mathrm{Hom}_R(M,Q)$ is coperfect (Theorem 4.1), (2) if R_R is Q-artinian, ${}_S\mathrm{Hom}_R(M,Q)$ is noetherian (Theorem 4.4), and (3) if R_R is both Q-noetherian and Q-artinian, ${}_S\mathrm{Hom}_R(M,Q)$ has finite length (Theorem 4.6). As these applications, we shall show that when Q is a quasi-injective, Q-finitely generated (in particular, finitely generated) right R-module with $S = \mathrm{End}(Q_R)$, then (1) if R_R is Q-noetherian, then S is a semi-primary ring (Theorem 4.2), (2) if R_R is Q-artinian, then S is a left noetherian ring (Corollary 4.5), and (3) if R_R is both Q-noetherian and Q-artinian, then S is a left artinian ring (Corollary 4.8). In addition, we shall show that if Q is a noetherian or artinian, quasi-injective right R-module such that R_R is Q-artinian, then $\mathrm{End}(Q_R)$ is a left artinian ring and $\mathrm{Biend}(Q_R)$ is a right artinian ring (Theorems 4.12 and 4.13). In the sequel, in Section 5 we shall be concerned with endomorphism rings of (quasi-)projective, quasi-injective modules satisfying some finiteness conditions.

2. Preliminaries

Let R be an associative ring with identity and Mod-R the category of all unital right R-modules. For $M, Q \in$ Mod-R, M is said to be Q-torsion if $\mathrm{Hom}_R(M,Q) = (0)$, and said to be Q-torsionless if M is embeddable in a direct product of copies of Q. An R-submodule L of M is said to be a Q-closed submodule of M if M/L is Q-torsionless. The set of all Q-closed submodules of M is denoted by $\mathcal{C}_Q(M)$ throughout this paper. It is well known that $L \in \mathcal{C}_Q(M)$ if and only if $L = \mathrm{Ann}_M(\mathrm{Ann}_{M^*}(L))$, where $M^* = \mathrm{Hom}_R(M,Q)$. We set $\tau_Q(M) = \mathrm{Ann}_M(M^*) = \{x \in M \mid f(x) = 0 \text{ for all } f \in M^* = \mathrm{Hom}_R(M,Q)\}$ for $M, Q \in$ Mod-R. Clearly, $\tau_Q(M)$ is the smallest Q-closed submodule of M. By setting $L \wedge N = L \cap N$ and $(L \vee N)/(L+N) = \tau_Q(M/(L+N))$ for $L, N \in \mathcal{C}_Q(M)$, we can give a lattice structure to $\mathcal{C}_Q(M)$. We set $\Psi(Q) = \{M \in \text{Mod-}R \mid Q$ is M-injective$\}$ for any $Q \in$ Mod-R. If $Q \in \Psi(Q)$, Q is said to be quasi-injective, and if $\Psi(Q) =$ Mod-R, Q is injective. The following result is well known.

LEMMA 2.1. *$\Psi(Q)$ is closed under taking submodules, homomorphic images and direct sums.*

If $C_Q(R) = \{I_R \subseteq R_R | I = \mathrm{Ann}_R(X)$ for some subset X of $Q\}$ satisfies the ACC (respectively DCC), then Q_R is said to be a Σ (respectively Δ)-module for any $Q \in$ Mod-R. If an (a quasi-)injective right R-module Q is a Σ (respectively Δ)-module, Q is said to be Σ (respectively Δ)-(quasi-)injective. For $M \in$ Mod-R, let $\mathrm{End}(M_R)$ denote the endomorphism ring of M_R and $\mathrm{Biend}(M_R)$ the biendomorphism ring of M_R, that is, $\mathrm{Biend}(M_R) = \mathrm{End}({}_SM)$, where $S = \mathrm{End}(M_R)$. Any homomorphism will be written on the side opposite to the scalars. For $M \in$ Mod-R, M^n denotes the direct sum of n-copies of M_R. The ACC (respectively DCC) denotes the ascending (respectively descending) chain condition.

3. Q-noetherian modules and Q-artinian modules

Let E be an injective right R-module and $\mathcal{F} = \{I_R \subseteq R_R | \mathrm{Hom}_R(R/I, E) = (0)\}$. Then $\mathcal{F}$ is a Gabriel topology on R associated with a hereditary torsion theory defined by E. In [11], [8] and [1], Năstăsescu-Niţă-Albu have defined and studied $\mathcal{F}$-noetherian and $\mathcal{F}$-artinian modules and rings. In this section we shall define and study Q-noetherian and Q-artinian modules in case Q_R is not necessarily an injective module.

DEFINITIONS. Let $M, Q \in$ Mod-R.

(1) M is said to be *Q-noetherian* (respectively *Q-artinian*) if, for each ascending (respectively descending) chain

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \quad (\text{respectively } M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots)$$

of R-submodules of M, there exists an integer $k \geq 1$ such that M_{i+1}/M_i (respectively M_i/M_{i+1}) is Q-torsion for all $i \geq k$. A ring R is said to be Q-noetherian (respectively Q-artinian) if R_R is Q-noetherian (respectively Q-artinian).

(2) If $\mathcal{A}$ is a non-empty set of R-submodules of M, $N \in \mathcal{A}$ is said to be a *Q-maximal* (respectively *Q-minimal*) element in $\mathcal{A}$ if, for each $N' \in \mathcal{A}$ such that $N \subseteq N'$ (respectively $N' \subseteq N$), N'/N (respectively N/N') is Q-torsion.

(3) M is said to be *Q-finitely generated* if there exists a finitely generated R-submodule M' of M such that M/M' is Q-torsion.

If Q is a cogenerator in Mod-R, each Q-noetherian (respectively Q-artinian) module is exactly a noetherian (respectively artinian) module. When Q is an injective right R-module cogenerating a hereditary torsion theory associated with a Gabriel topology $\mathcal{F}$, these definitions are identified with those of $\mathcal{F}$-noetherian, $\mathcal{F}$-artinian, $\mathcal{F}$-maximal, $\mathcal{F}$-minimal and $\mathcal{F}$-finitely generated modules in the sense of Năstăsescu-Niţa-Albu.

LEMMA 3.1. *Let $M \in \Psi(Q)$. If M is Q-torison, then M is both Q-noetherian and Q-artinian.*

PROOF. Let $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ be any ascending chain of R-submodules of M. Since M is Q-torsion, M/M_k is Q-torsion, too, for all $k \geq 1$. Since $M/M_k \in \Psi(Q)$ by Lemma 2.1, every R-submodule of M/M_i, in particular M_{k+1}/M_k is Q-torsion by [6, Lemma 2.1]. Hence M is Q-noetherian. Similarly, M is Q-artinian.

LEMMA 3.2. *Let us consider the following conditions.*

(1) M is Q-noetherian.

(2) Each non-empty set of R-submodules of M has a Q-maximal element.

(3) $\mathcal{C}_Q(M)$ is a noetherian lattice.

(4) Each R-submodule of M is Q-finitely generated. Then we have the implications, (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). In addition, if $M \in \Psi(Q)$, all four conditions are equivalent.

PROOF. The implications, (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) can be proved in the same manner as in the proof of [2, Proposition 6.1]. Next, assume that Q is M-injective.

(3) $\Rightarrow$ (4). Suppose that M has a submodule N which is not Q-finitely generated. Choose $x_1 \in N$ with $x_1 \neq 0$. Then N/x_1R is not Q-torsion, and so in particular $N/x_1R \neq (0)$. Hence there exists $x_2 \in N$ such that $x_1R \subsetneqq x_1R + x_2R$ and $(x_1R + x_2R)/x_1R$ is not Q-torsion. For, since $N/x_1R \in \Psi(Q)$ by Lemma 2.1, $\tau_Q(N/x_1R)$ is Q-torsion by [6, (2) of Lemma 2.1], and hence $\tau_Q(N/x_1R) \subsetneqq N/x_1R$. Hence for each $x_2 + x_1R \notin \tau_Q(N/x_1R)$, there exists $f \in \mathrm{Hom}_R(N/x_1R, Q)$ such that $f(x_2 + x_1R) \neq 0$. So the restriction of f onto $(x_1R + x_2R)/x_1R$ is not a zero map. Hence $(x_1R + x_2R)/x_1R$ is not Q-torsion. And then, $N/(x_1R + x_2R)$ is not Q-torsion. Continuing the same argument, we are able to find a strictly ascending chain of R-submodules of M,

$$x_1R \subsetneqq x_1R + x_2R \subsetneqq x_1R + x_2R + x_3R \subsetneqq \cdots$$

such that N_{k+1}/N_k is not Q-torsion for all $k \geq 1$, where $N_k = x_1R + x_2R + \cdots + x_kR$. Let us put $N_i'/N_i = \tau_Q(M/N_i)$ for each integer i. Then we get an ascending chain of elements of $\mathcal{C}_Q(M), N_1' \subseteq N_2' \subseteq N_3' \subseteq \cdots$. Suppose $N_i' = N_{i+1}'$ for some i. Then $N_{i+1}/N_i \subseteq N_{i+1}'/N_i = N_i'/N_i$. By using Lemma 2.1 and [6, Lemma 2.1], since N_i'/N_i is Q-torsion, so is also N_{i+1}/N_i. This is a contradiction. Consequently, we have $N_i' \subsetneqq N_{i+1}'$ for all $i \geq 1$, and which contradicts the assumption (3).

(4) $\Rightarrow$ (1). Let $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ be an ascending chain of R-submodules of M. Since $N = \bigcup_{i=1}^{\infty} M_i$ is Q-finitely generated, then there exist

$x_1, x_2, \ldots, x_n \in N$ such that $N/(x_1R + \cdots + x_nR)$ is Q-torsion. On the other hand, there exists an integer k such that $x_1R + \cdots + x_nR \subseteq M_k$, so N/M_{k+j} is Q-torsion for all $j \geq 0$. Since $N/M_{k+j} \in \Psi(Q)$ by Lemma 2.1, M_{k+j+1}/M_{k+j} is Q-torsion according to [6, (1) of Lemma 2.1]. Hence M is Q-noetherian.

LEMMA 3.3. *Let us consider the following conditions.*

(1) *M is Q-artinian.*

(2) *Each non-empty set of R-submodules of M has a Q-minimal element.*

(3) *$\mathcal{C}_Q(M)$ is an artinian lattice.*

Then the implications (1) $\Rightarrow$ (2) *and* (2) $\Rightarrow$ (3) *hold. In addition, if $M \in \Psi(Q)$, all three conditions are equivalent.*

PROOF. The implications, (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) can be proved in the same manner as in the proof of [2, Proposition 6.2]. Next, assume that Q is M-injective.

(3) $\Rightarrow$ (1). Let $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ be a descending chain of R-submodules of M. Put $L_i'/L_i = \tau_Q(M/L_i)$ for each integer i. Then we have the descending chain of elements of $\mathcal{C}_Q(M), L_1' \supseteq L_2' \supseteq L_3' \supseteq \cdots$. By the assumption (3), there exists an integer k such that $L_k' = L_{k+1}' = L_{k+2}' = \cdots$. For each $i \geq k$, $L_i/L_{i+1} \subseteq L_i'/L_{i+1} = L_{i+1}'/L_{i+1} = \tau_Q(M/L_{i+1})$. Since $M/L_{i+1} \in \Psi(Q)$ by Lemma 2.1, $\tau_Q(M/L_{i+1})$ is Q-torsion, and hence so is also L_i/L_{i+1} for all $i \geq k$, by using [6, Lemma 2.1].

PROPOSITION 3.4. *Let $M, Q \in \text{Mod-}R$, and let*

$$(0) \to M' \xrightarrow{\psi} M \xrightarrow{\varphi} M'' \to (0)$$

be an exact sequence of right R-modules. Then, if M is Q-noetherian (resp. Q-artinian), so are also M' and M''. If $M \in \Psi(Q)$, and if both M' and M'' are Q-noetherian (resp. Q-artinian), so is also M.

PROOF. (I) Q-noetherian case. The first part of the statement can be proved by the standard discussion. Next, suppose that $M \in \Psi(Q)$ and both M' and M'' are Q-noetherian. Let L be an R-submodule of M. $\varphi(L)$ has a finitely generated R-submodule $N = \sum_{i=1}^{n} z_iR$ such that $\varphi(L)/N$ is Q-torsion. Choose $x_i \in L$ such that $\varphi(x_i) = z_i$ for $i = 1, 2, \ldots, n$. Put $K = \sum_{i=1}^{n} x_iR$. On the other hand, $L \cap \psi(M')$ has a finitely generated R-submodule $H = \sum_{j=1}^{m} y_jR$ such that $(L \cap \psi(M'))/H$ is Q-torsion. Then, since $L \cap \varphi^{-1}(N) = (L \cap \psi(M')) + K$, we have the exact sequence as follows:

$$(0) \to ((L \cap \psi(M')) + K)/(H + K) \to L/(H + K) \to \varphi(L)/N \to (0).$$

Since $L/(H+K) \in \Psi(Q)$ by Lemma 2.1, we have the exact sequence,

$$(0) \to \mathrm{Hom}_R(\varphi(L)/N, Q) \to \mathrm{Hom}_R(L/(H+K), Q) \\ \to \mathrm{Hom}_R(((L \cap \psi(M')) + K)/(H+K), Q) \to (0).$$

Since $((L \cap \psi(M')) + K)/(H+K)$ is a homomorphic image of $(L \cap \psi(M'))/H$, $((L \cap \psi(M')) + K)/(H+K)$ is Q-torsion, too. Hence

$$\mathrm{Hom}_R(\varphi(L)/N, Q) = \mathrm{Hom}_R(((L \cap \psi(M')) + K)/(H+K), Q) = (0);$$

so $\mathrm{Hom}_R(L/(H+K), Q) = (0)$. Therefore L is Q-finitely generated. Hence M is Q-noetherian.

(II) Q-artinian case. The first part can be proved by the standard discussion. Next, suppose that $M \in \Psi(Q)$ and both M' and M'' are Q-artinian. let $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ be a descending chain of R-submodules of M. Then for the descending chain, $L_1 \cap \psi(M') \supseteq L_2 \cap \psi(M') \supseteq L_3 \cap \psi(M') \supseteq \cdots$, there exists an integer k such that $(L_i \cap \psi(M'))/(L_{i+1} \cap \psi(M'))$ is Q-torsion for all $i \geq k$. And, for the descending chain $\varphi(L_1) \supseteq \varphi(L_2) \supseteq \varphi(L_3) \supseteq \cdots$, there exists an integer k' such that $\varphi(L_i)/\varphi(L_{i+1})$ is Q-torsion for all $i \geq k'$. Let $n = \max\{k, k'\}$. Then for all $i \geq n$, let us consider the exact sequence as follows:

$$(0) \to ((L_i \cap \psi(M')) + L_{i+1})/L_{i+1} \to L_i/L_{i+1} \to \varphi(L_i)/\varphi(L_{i+1}) \to (0).$$

Then we have the exact sequence,

$$(0) \to \mathrm{Hom}_R(\varphi(L_i)/\varphi(L_{i+1}), Q) \to \mathrm{Hom}_R(L_i/L_{i+1}, Q) \\ \to \mathrm{Hom}_R(((L_i \cap \psi(M')) + L_{i+1})/L_{i+1}, Q) \to (0),$$

because $L_i/L_{i+1} \in \Psi(Q)$. Since

$$((L_i \cap \psi(M')) + L_{i+1})/L_{i+1} \cong (L_i \cap \psi(M'))/((L_i \cap \psi(M')) \cap L_{i+1}) \\ = (L_i \cap \psi(M'))/(L_{i+1} \cap \psi(M')),$$

$\mathrm{Hom}_R(\varphi(L_i)/\varphi(L_{i+1}), Q) = \mathrm{Hom}_R(((L_i \cap \psi(M')) + L_{i+1})/L_{i+1}, Q) = (0)$. Hence $\mathrm{Hom}_R(L_i/L_{i+1}, Q) = (0)$, that is, L_i/L_{i+1} is Q-torsion for all $i \geq n$. Therefore M is Q-artinian.

COROLLARY 3.5. *Let $M \in \Psi(Q)$. If R is a Q-noetherian (resp. Q-artinian) ring, and if M is a Q-finitely generated right R-module, then M is Q-noetherian (resp. Q-artinian).*

PROOF. Since R is Q-noetherian (resp. Q-artinian), every cyclic right R-module is Q-noetherian (resp. Q-artinian) by Proposition 3.4. By the assumption there exists a finitely generated R-submodule M' of M such that M/M' is Q-torsion. Put $M' = \sum_{i=1}^{n} x_i R$. Then $x_i R$ is Q-noetherian (resp. Q-artinian).

Since Q is M-injective, $\bigoplus_{i=1}^{k} x_iR \in \Psi(Q)$ for all integer k such that $1 \leq k \leq n$, by Lemma 2.1. Let us consider the exact sequence,

$$0 \to x_1R \to x_1R \oplus x_2R \to x_2R \to (0).$$

Since x_1R and x_2R both are Q-noetherian (resp. Q-artinian), $x_1R \oplus x_2R$ is Q-noetherian (resp. Q-artinian) by Proposition 3.4. By the similar discussion, if $x_1R + \cdots + x_{k-1}R$ is Q-noetherian (resp. Q-artinian), the exact sequence

$$(0) \to x_1R \oplus \cdots \oplus x_{k-1}R \to x_1R \oplus \cdots \oplus x_{k-1}R \oplus x_kR \to x_kR \to (0)$$

implies that $x_1R \oplus \cdots \oplus x_kR$ is Q-noetherian (resp. Q-artinian) by Proposition 3.4. Thus, we can conclude that $\bigoplus_{i=1}^{k} x_iR$ for each k, in particular, $\bigoplus_{i=1}^{n} x_iR$ is Q-noetherian (resp. Q-artinian). Next, since the map $\psi \colon \bigoplus_{i=1}^{n} x_iR \to M'$ defined by $\psi(x_1r_1, x_2r_2, \ldots, x_nr_n) = \sum_{i=1}^{n} x_ir_i$, is an R-epimorphism, M' is Q-noetherian (resp. Q-artinian) by Proposition 3.4. On the other hand, M/M' is Q-noetherian (resp. Q-artinian) by Lemma 3.1. Hence the exact sequence

$$(0) \to M' \to M \to M/M' \to (0)$$

implies that M is Q-noetherian (resp. Q-artinian) by Proposition 3.4, as desired.

4. Quasi-injective module Q such that R is Q-noetherian (Q-artinian)

Let $M \in \text{Mod-}R$. If $\mathcal{C}_M(R) = \{I_R \subseteq R_R \mid I = \text{Ann}_R(X)$ for some subset X of $M\}$ is noetherian (respectively artinian), then M_R is said to be a Σ (respectively Δ)-module. If an (a quasi-)injective right R-module Q is a Σ (respectively Δ)-module, then Q_R is said to be a Σ (respectively Δ)-(quasi-)injective module. According to Lemmas 3.2 and 3.3, an injective right R-module Q such that R_R is Q-noetherian (respectively Q-artinian) is exactly a Σ (respectively Δ)-injective module. And, a quasi-injective right R-module Q such that R is Q-noetherian (respectively Q-artinian) is Σ (respectively Δ)-quasi-injective. However, a Σ (respectively Δ)-quasi-injective right R-module Q does not necessarily satisfy the condition for R to be Q-noetherian (respectively Q-artinian). In this section we are concerned with a quasi-injective right R-module Q such that R is Q-noetherian or Q-artinian.

THEOREM 4.1. *Let Q be a quasi-injective right R-module such that R is Q-noetherian and $S = \text{End}(Q_R)$. If M_R is Q-finitely generated (in particular, finitely generated) and $M \in \Psi(Q)$, then ${}_S\text{Hom}_R(M, Q)$ is coperfect.*

PROOF. Since M is a Q-finitely generated right module over a Q-noetherian ring R, M is Q-noetherian by Corollary 3.5. Hence $\mathcal{C}_Q(M)$ is a noetherian lattice by Lemma 3.2. Therefore ${}_S\text{Hom}_R(M, Q)$ is coperfect by [6, Theorem 4.1] or [2, Corollary 4.3].

THEOREM 4.2. *Let Q be a quasi-injective, Q-finitely generated (in particular, finitely generated) right R-module such that R is Q-noetherian. Then $S = \text{End}(Q_R)$ is a semi-primary ring.*

PROOF. In this case $\mathcal{C}_Q(Q)$ is a noetherian lattice. Hence S is a semi-primary ring by [2, Corollary 4.5].

COROLLARY 4.3. *If Q is a quasi-injective, Q-finitely generated (in particular, finitely generated) right module over a right noetherian ring R, then $S = \text{End}(Q_R)$ is a semi-primary ring.*

THEOREM 4.4. *Let Q be a quasi-injective right R-module such that R is Q-artinian and $S = \text{End}(Q_R)$. If M is a Q-finitely generated (in particular, finitely generated) right R-module and $M \in \Psi(Q)$, then ${}_S\text{Hom}_R(M, Q)$ is noetherian.*

PROOF. Since M_R is a Q-finitely generated module over a Q-artinian ring R, M_R is Q-artinian by Corollary 3.5. Hence $\mathcal{C}_Q(M)$ is an artinian lattice by Lemma 3.3. According to [6, Theorem 4.3] or [2, Corollary 4.3], ${}_S\text{Hom}_R(M, Q)$ is noetherian if and only if $\mathcal{C}_Q(M)$ is artinian, as desired.

COROLLARY 4.5. *Let Q be a quasi-injective, Q-finitely generated (in particular, finitely generated) right R-module such that R is Q-artinian. Then $S = \text{End}(Q_R)$ is a left noetherian ring.*

THEOREM 4.6. *Let Q be a quasi-injective right R-module such that R is both Q-noetherian and Q-artinian and $S = \text{End}(Q_R)$. If M is a Q-finitely generated (in particular, finitely generated) right R-module and $M \in \Psi(Q)$, then ${}_S\text{Hom}_R(M, Q)$ has finite length.*

PROOF. By Theorems 4.1 and 4.4, ${}_S\text{Hom}_R(M, Q)$ is coperfect and noetherian. Therefore ${}_S\text{Hom}_R(M, Q)$ has finite length.

COROLLARY 4.7. *Let Q be a Δ-injective right R-module with $S = \text{End}(Q_R)$. If M is a Q-finitely generated (in particular, finitely generated) right R-module, then ${}_S\text{Hom}_R(M, Q)$ has finite length.*

COROLLARY 4.8. *Let Q be a quasi-injective, Q-finitely generated (in particular, finitely generated) right R-module such that R is both Q-noetherian and Q-artinian. Then $S = \text{End}(Q_R)$ is a left artinian ring. In particular, if Q is a Q-finitely generated (in particular, finitely generated) Δ-injective right R-module, then $S = \text{End}(Q_R)$ is a left artinian ring.*

REMARK. The latter part of Corollary 4.8 is due to Faith [3, Corollary 6.4] and Năstăsescu [10, Proposition 1.5].

THEOREM 4.9. *Let U be a right R-module such that R is both U-noetherian and U-artinian. If Q is a quasi-injective, U-torsionless, U-finitely generated right R-module such that U is Q-injective, then $S = \mathrm{End}(Q_R)$ is a left artinian ring.*

PROOF. Since Q is a U-finitely generated right module over a U-noetherian and U-artinian ring R and since U is Q-injective, then Q_R is both U-noetherian and U-artinian by Corollary 3.5. So $\mathcal{C}_U(Q)$ is a noetherian and artinian lattice by Lemmas 3.2 and 3.3. On the other hand, since Q_R is U-torsionless, every Q-closed submodule of Q is also a U-closed submodule of Q. Hence $\mathcal{C}_Q(Q)$ is a noetherian and artinian lattice, too. Thus, since Q_R is quasi-injective, Q_R has a Q-composition series by [6, Theorem 2.6]. Therefore, according to [6, Theorems 2.8 and 3.4], we have $\mathrm{len}_S S = Q - \mathrm{len}\, Q_R = n$ for some integer $n \geq 0$, as desired.

COROLLARY 4.10 (NĂSTĂSESCU [10, PROPOSITION 1.5]). *Let U be a Δ-injective right R-module. If Q is a quasi-injective, U-torsionless, U-finitely generated right R-module, then $S = \mathrm{End}(Q_R)$ is a left artinian ring.*

PROOF. Since U_R is Δ-injective, R is a both U-noetherian and U-artinian ring according to Miller-Teply's theorem in [7]. Hence the result follows directly from Theorem 4.9.

LEMMA 4.11. *Let Q be a quasi-injective right R-module such that R is Q-artinian, and let us put $T = \mathrm{Biend}(Q_R)$. Then T is a semi-primary ring and Q_T is a Δ-injective module.*

PROOF. In this case Q_R is Δ-quasi-injective by Lemma 3.3. Hence $_SQ$ has finite length according to [4, Proposition 8.1], where $S = \mathrm{End}(Q_R)$. Therefore $T = \mathrm{End}(_SQ)$ is a semi-primary ring. And, since Q_R is finendo and quasi-injective, Q_T is injective (refer to [3, Proposition 19.18]). Since $S = \mathrm{End}(Q_T)$, it follows that Q_T is Δ-injective by [4, Corollary 7.5].

THEOREM 4.12. *If Q is a noetherian, quasi-injective right R-module such that R is Q-artinian, then we have the following assertions.*
(1) $S = \mathrm{End}(Q_R)$ *is a left artinian ring.*
(2) $T = \mathrm{Biend}(Q_R)$ *is a right artinian ring.*

PROOF. (1) Since Q_R is noetherian, so is also Q_T. In particular, Q_T is finitely generated. On the other hand, Q_T is Δ-injective by Lemma 4.11, and $S = \mathrm{End}(Q_T)$. Hence S is a left artinian ring by Corollary 4.8.

(2) In this case Q_R is Δ-quasi-injective, and so ${}_SQ$ has finite length by [4, Proposition 8.1]. In particular, ${}_SQ$ is finitely gnerated. Thus, Q_T is a finendo, faithful, injective module. So Q_T is compactly faithful by [3, Proposition 19.15], that is $T_T \hookrightarrow Q_T^n$ for some integer $n \geq 1$. Since Q_T is noetherian, Q_T^n, and hence T_T is noetherian. On the other hand, since T is a semi-primary ring by Lemma 4.11, T is a right artinian ring.

THEOREM 4.13. *If Q is an artinian, quasi-injective right R-module such that R is Q-artinian, then we have the following assertions.*

(1) $S = \mathrm{End}(Q_R)$ *is a left artinian ring.*

(2) $T = \mathrm{Biend}(Q_R)$ *is a right artinian ring.*

PROOF. In this case $\mathcal{C}_Q(R)$ is an artinian lattice by Lemma 3.3. Since $\mathrm{Ann}_R(Q) = \bigcap_{x\in Q} \mathrm{Ann}_R(x)$, there exist a finite number of elements $x_1, x_2, \dots, x_n \in Q$ such that $\mathrm{Ann}_R(Q) = \bigcap_{i=1}^n \mathrm{Ann}_R(x_i)$. Hence, if we put $\overline{R} = R/\mathrm{Ann}_R(Q)$, $\overline{R}_R \hookrightarrow x_1R \oplus x_2R \oplus \cdots \oplus x_nR$. Since $x_1R \oplus \cdots \oplus x_nR$ is an artinian right R-module, $\overline{R}_R$ is artinian, too. Hence $\overline{R}$ is a right artinian ring. Since Q_R is artinian, $Q_{\overline{R}}$ is an artinian module over a right artinian ring $\overline{R}$. Hence $Q_{\overline{R}}$ is also noetherian by [9, Corollary 1.3]. Thus, Q_R is noetherian. Therefore the results follow directly from Theorem 4.12.

COROLLARY 4.14 (FAITH-NĂSTĂSESCU). *Let Q be a Δ-injective right R-module with $S = \mathrm{End}(Q_R)$ and $T = \mathrm{Biend}(Q_R)$. If Q_R is either noetherian or artinian, then S is a left artinian ring and T is a right artinian ring.*

COROLLARY 4.15. *Let R be a right artinian ring. If Q is a noetherian (or an artinian), quasi-injective right R-module, then $S = \mathrm{End}(Q_R)$ is a left artinian ring and $T = \mathrm{Biend}(Q_R)$ is a right artinian ring.*

5. Endomorphism rings of quasi-projective, quasi-injective modules

THEOREM 5.1. *If Q is a finitely generated projective, quasi-injective right R-module such that R is Q-artinian, then $S = \mathrm{End}(Q_R)$ is a left artinian ring.*

PROOF. According to Corollary 4.5, S is a left noetherian ring. On the other hand, since Q_R is Δ-quasi-injective in this case, ${}_SQ$ has finite length by [4, Theorem 8.1]. Hence $T = \mathrm{End}({}_SQ)$ is a semi-primary ring. And, since Q_T is finitely generated projective and $S = \mathrm{End}(Q_T)$, S is a semi-primary ring, too, by [5, Proposition 4.5]. Hence S is a left artinian ring.

THEOREM 5.2. *Let R be a left noetherian ring. If Q is a finitely generated projective, quasi-injective, finendo right R-module, then $S = \mathrm{End}(Q_R)$ is a left artinian ring.*

PROOF. In this case S is a left noetherian ring and ${}_SQ$ is finitely generated. Hence ${}_SQ$ is noetherian. And, since Q_R is finendo and quasi-injective, Q_T is injective by [3, Proposition 19.18], where $T = \mathrm{Biend}(Q_R)$. Hence Q_T is Δ-injective according to [4, Proposition 8.1]. Thus, Q_T is a finitely generated Δ-injective module with $S = \mathrm{End}(Q_T)$. Therefore S is a left artinian ring by Corollary 4.8.

THEOREM 5.3. *Let Q be a quasi-projective, quasi-injective, artinian right R-module. Then $S = \mathrm{End}(Q_R)$ is a left artinian ring.*

PROOF. Since Q_R is quasi-projective and artinian, S is a semi-primary ring by [2, Corollary 4.14]. On the other hand, since Q_R is quasi-injective and artinian, S is a left noetherian ring by [2, Corollary 4.4.] or [6, Corollary 4.4]. Hence S is a left artinian ring.

THEOREM 5.4. *Let Q be a quasi-projective, quasi-injective, noetherian right R-module. Then $S = \mathrm{End}(Q_R)$ is a right artinian ring.*

PROOF. Since Q_R is quasi-injective and noetherian, S is a semi-primary ring by [2, Corollary 4.5]. On the other hand, since Q_R is quasi-projective and noetherian, S is a right noetherian ring by [2, Corollary 4.12]. Hence S is a right artinian ring.

COROLLARY 5.5. *Let Q be a quasi-projective, quasi-injective, noetherian or artinian, right R-module such that R is Q-artinian. Then $S = \mathrm{End}(Q_R)$ is a left and right artinian ring.*

PROOF. First, suppose that Q_R is noetherian. Then S is a right artinian ring by Theorem 5.4, while S is a left artinian ring by Theorem 4.12. Next, consider the case where Q_R is artinian. As has been shown in the proof of Theorem 4.13, Q_R is necessarily noetherian. Hence the result is due to the first case.

Note. In connection with Theorems 4.12 and 4.13, it should be noticed that in general, if Q is a quasi-injective right R-module having the right artinian biendomorphism ring T, Q is necessarily injective as a right T-module. Indeed, since Q is a faithful right module over a right artinian ring T, Q_T is compactly faithful according to a result of Beachy [12, Proposition 1] (see also Vámos [13]). On the other hand, since any quasi-injective module Q_R remains quasi-injective as a module over $T = \mathrm{Biend}(Q_R)$, Q_T is compactly faithful and quasi-injective. Therefore Q_T is injective by [3, Proposition 19.15], as desired.

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Addendum

We are able to strengthen Corollary 4.5. Under the same assumption as in Corollary 4.5 we can conclude that $S = \mathrm{End}(Q_R)$ is a left artinian ring. For, since R is also Q-noetherian by [4, Theorem 7.1], it follows by Theorems 4.2 and 4.4 that S is both semi-primary and left noetherian. Hence S is left artinian. Thus, our Corollary 4.8 is needless.

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