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REGULARITY OF BOUNDARY POINTS IN THE DIRICHLET PROBLEM FOR THE HEAT EQUATION

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Abstract

We show that the null limit hypothesis, in the definition of a barrier, can be relaxed for normal boundary points that satisfy a mild additional condition. We also give a simple necessary and sufficient condition for the regularity of semi-singular boundary points.

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1. Introduction

The regularity of boundary points in the Dirichlet problem for the heat equation has been studied by many authors, for example in [1–13]. Most have used the notion of a barrier, a positive supertemperature on a relative neighbourhood of the boundary point which has a null limit at that point. In this paper, we show that the null limit hypothesis can be relaxed for suitable boundary points. In fact, the null limit need not be taken in the usual topology; it suffices that there is a null limit through heat balls centred at the boundary point.

To achieve this, we first study the extension of the Green function to the whole space, which was given by Doob in [4, page 343]. Although his treatment of the corresponding extension in the context of Laplace's equation [4, page 90] included a formula for the value of the extension at any boundary point, he gave no such formula in the context of the heat equation. We give such a formula in Theorem 2.1 and use it in Theorem 2.2 to relax the null limit hypothesis in the Green function criterion for regularity, [13, Theorem 8.53], for those normal boundary points that satisfy a mild additional condition.

Motivated by this relaxation, in Section 3 we introduce the notion of a heat ball barrier, which differs from an ordinary barrier only in that the null limit is required only through heat balls centred at the boundary point in question. Our main result is that the existence of a heat ball barrier at the point is a sufficient condition for its regularity.

In our final section, we turn our attention to semi-singular boundary points, and give a simple necessary and sufficient condition for the regularity of such points.

Notation and terminology will follow [13], where further details can be found; but we briefly summarise it here. We work in $\mathbb{R}^{n+1} = \{(x,t) : x \in \mathbb{R}^n, t \in \mathbb{R}\}$, and denote a typical point by p or (x,t) as convenient. Let

$$W(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-\dfrac{|x|^2}{4t}\right) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

For any point $p_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and any positive number c, the set

$$\Omega(p_0; c) = \Omega(x_0, t_0; c) = \{(y,s) \in \mathbb{R}^{n+1} : W(x_0 - y, t_0 - s) > (4\pi c)^{-n/2}\}$$

is called the *heat ball* with *centre* (x_0, t_0) and *radius* c. Given a function u on the heat ball $\Omega(x_0, t_0; c)$ for which the integral exists, we define the *volume mean value* of u by

$$\mathcal{V}(u; x_0, t_0; c) = (4\pi c)^{-n/2} \iint_{\Omega(x_0,t_0;c)} \frac{|x_0 - x|^2}{4(t_0 - t)^2} u(x,t)\, dx\, dt.$$

Given any two points $p = (x,t)$ and $q = (y,s)$, we put $G(p;q) = W(x-y, t-s)$. For any open set E, the *Green function* G_E for E is the nonnegative, real-valued function defined on $E \times E$ by putting

$$G_E(p;q) = G(p;q) - h_E(p;q),$$

where for each $q \in E$ the function $h_E(\cdot\,;q)$ is the greatest thermic minorant of $G(\cdot\,;q)$ on E. We may refer to $G_E(\cdot\,;q)$ as the *Green function for E with pole at q*. The Green function for $\mathbb{R}^{n+1}$ is G. If $q = (y,s) \in E$, we denote by $\Lambda^*(q;E)$ the set of points $p \in E$ for which there is a polygonal path $\gamma \subseteq E$ joining q to p, along which the temporal variable is strictly *increasing*. The boundary of any set is taken relative to the one-point compactification of $\mathbb{R}^{n+1}$. Thus ∂E contains the point at infinity if and only if E is unbounded.

Given any point $p_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and $r > 0$, we denote by $H(p_0, r)$ the open lower half-ball $\{(x,t) : |x - x_0|^2 + (t - t_0)^2 < r^2, t < t_0\}$, and by $H^*(p_0, r)$ the open upper half-ball $\{(x,t) : |x - x_0|^2 + (t - t_0)^2 < r^2, t > t_0\}$. Let q be a boundary point of the open set E, including the point at infinity if E is unbounded. In our classification of boundary points, we always suppose that the boundary of E does not contain any polar set whose union with E would give another open set. We call q a *normal* boundary point if either q is the point at infinity, or $q \in \mathbb{R}^{n+1}$ and every lower half-ball centred at q meets the complement of E. Otherwise, we call q an *abnormal* boundary point. The abnormal boundary points are of two kinds, according to whether they can be approached from above by points in E. If there is some $r_0 > 0$ such that $H^*(q, r_0) \cap E = \emptyset$, then q is called a *singular* boundary point. Otherwise, if for every $r > 0$ we have $H^*(q,r) \cap E \neq \emptyset$, then q is called a *semi-singular* boundary point. The set of all normal boundary points of E is denoted by $\partial_n E$, that of all abnormal points

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by $\partial_a E$, that of all singular points by $\partial_s E$, and that of all semi-singular points by $\partial_{ss} E$. Thus $\partial E = \partial_n E \cup \partial_a E$ and $\partial_a E = \partial_s E \cup \partial_{ss} E$. The *essential boundary* $\partial_e E$ is defined by $\partial_e E = \partial_n E \cup \partial_{ss} E = \partial E \backslash \partial_s E$.

If E is an open set, and f is an extended-real valued function defined on $\partial_e E$, the *upper class* determined by f, denoted by $\mathfrak{U}_f^E$, consists of all lower bounded hypertemperatures on E that satisfy

$$\liminf_{(x,t)\to(y,s)} w(x,t) \geq f(y,s) \quad \text{for all } (y,s) \in \partial_n E,$$

and

$$\liminf_{(x,t)\to(y,s+)} w(x,t) \geq f(y,s) \quad \text{for all } (y,s) \in \partial_{ss} E.$$

The *lower class* determined by f, denoted by $\mathfrak{L}_f^E$, consists of all upper bounded hypotemperatures on E that satisfy

$$\limsup_{(x,t)\to(y,s)} w(x,t) \leq f(y,s) \quad \text{for all } (y,s) \in \partial_n E,$$

and

$$\limsup_{(x,t)\to(y,s+)} w(x,t) \leq f(y,s) \quad \text{for all } (y,s) \in \partial_{ss} E.$$

The function $U_f^E = \inf\{w : w \in \mathfrak{U}_f^E\}$ is called the *upper solution* for f on E, and $L_f^E = \sup\{w : w \in \mathfrak{L}_f^E\}$ is called the *lower solution* for f on E. We say that f is *resolutive* for E if $L_f^E = U_f^E$ and is a temperature on E, in which case the function $S_f^E = L_f^E = U_f^E$ is called the *PWB solution* for f on E. A point $q \in \partial_n E$ is called *regular* if $\lim_{p\to q} S_f^E(p) = f(q)$ for all $f \in C(\partial_e E)$. A point $q = (y,s) \in \partial_{ss} E$ is called *regular* if $\lim_{(x,t)\to(y,s+)} S_f^E(x,t) = f(y,s)$ for all $f \in C(\partial_e E)$.

A function w is called a *barrier* at a finite point $q \in \partial_e E$ if it is defined on $N \cap E$ for some open neighbourhood N of q, is a supertemperature on $N \cap E$, is (strictly) positive on $N \cap E$, and satisfies $\lim_{(x,t)\to(y,s)} w(x,t) = 0$ if $q \in \partial_n E$, or $\lim_{(x,t)\to(y,s+)} w(x,t) = 0$ if $q \in \partial_{ss} E$.

The *thermal fine topology* is the coarsest topology on $\mathbb{R}^{n+1}$ that makes every supertemperature continuous. Concepts relative to the thermal fine topology will be prefixed with '$\Theta - f$'; for example, $\Theta - f$ lim. Concepts with no prefix will refer to the Euclidean topology. A set $L \subseteq \mathbb{R}^{n+1}$ is called a *semipolar subset of* $\mathbb{R}^{n+1}$ if it can be written in the form $L = \bigcup_{i=1}^{\infty} L_i$, where each set L_i has no thermal fine limit point in $\mathbb{R}^{n+1}$.

2. The extension of the Green function of an arbitrary open set

In this section, we supplement Doob's treatment of the extension of $G_E(\cdot\,; q)$ to $\mathbb{R}^{n+1}$ by giving a formula for the value of the extension at any finite boundary point of E. We also use this formula to show that the null limit hypothesis in the Green function criterion for regularity can be relaxed.

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THEOREM 2.1. *Let E be an open set, and let $q \in E$. Then $G_E(\cdot;q)$ can be uniquely extended to a function $G_E^{=}(\cdot;q)$ on $\mathbb{R}^{n+1}$ with the following properties:*

(a) $G_E^{=}(\cdot;q)$ *is a nonnegative subtemperature on* $\mathbb{R}^{n+1}\backslash\{q\}$.

(b) $G_E^{=}(\cdot;q) = 0$ *on* $\mathbb{R}^{n+1}\backslash\overline{E}$ *and at every finite regular point of* $\partial_n E$.

(c) *For each finite point $q_0 \in \partial E$, either there is $c > 0$ such that $\Omega(q_0;c) \cap E = \emptyset$, in which case $G_E^{=}(q_0;q) = 0$, or there is no such c, in which case*

$$G_E^{=}(q_0;q) = \lim_{c\to 0+}\Big(\sup_{\Omega(q_0;c)\cap E} G_E(\cdot;q)\Big). \tag{2.1}$$

PROOF. All except part (c) is given in [4, page 343], but we include a proof for the convenience of the reader.

We first show that such an extension is unique. Two extensions that satisfy (a) would both be thermal fine continuous on $\mathbb{R}^{n+1}$ by [13, Lemma 9.3]. Two extensions that satisfy (b) would be equal outside the union Z of $\partial_a E$ with the set of irregular points of $\partial_n E$, which is a semipolar subset of $\mathbb{R}^{n+1}$ by [13, Corollary 9.47], and hence thermal fine nowhere dense in $\mathbb{R}^{n+1}$ by [13, Lemma 9.24]. Two extensions that satisfy both (a) and (b) would therefore be equal everywhere on $\mathbb{R}^{n+1}$.

Let $\{D_k\}$ be a sequence of open circular cylinders such that $\overline{D}_k \subseteq E$ for all k, $\bigcup_{k=1}^{\infty} D_k = E$, and each cylinder in the collection $\{D_k : k \in \mathbb{N}\}$ occurs infinitely often in the sequence. Using the notation of [13, Theorem 3.21], for each k we put $w_k = \pi_{D_k}\cdots\pi_{D_1}G(\cdot;q)$ on $\mathbb{R}^{n+1}$. It follows from [13, Theorem 3.21] that $\{w_k\}$ is a decreasing sequence of supertemperatures on $\mathbb{R}^{n+1}$. Moreover, by [13, Theorem 6.8], the restriction to E of the limit of this sequence is the greatest thermic minorant of $G(\cdot;q)$ on E. We put $u(\cdot;q) = \lim_{k\to\infty} w_k$ on $\mathbb{R}^{n+1}$, and note that $u(\cdot;q) = G(\cdot;q)$ on $\mathbb{R}^{n+1}\backslash E$. By the fundamental convergence theorem ([4, page 314], [13, Theorem 9.30]), the lower semicontinuous smoothing $\widehat{u}(\cdot;q)$ is a supertemperature on $\mathbb{R}^{n+1}$, and

$$\widehat{u}(r;q) = \Theta - f\lim_{p\to r} u(p;q)$$

for all $r \in \mathbb{R}^{n+1}$. If r is a finite regular point of $\partial_n E$, then r is a thermal fine limit point of $\mathbb{R}^{n+1}\backslash E$, by [13, Theorem 9.40], so that

$$\widehat{u}(r;q) = \Theta - f\lim_{p\to r} G(p;q) = G(r;q)$$

because $u(\cdot;q) = G(\cdot;q)$ on $\mathbb{R}^{n+1}\backslash E$. Defining

$$G_E^{=}(\cdot;q) = G(\cdot;q) - \widehat{u}(\cdot;q)$$

on $\mathbb{R}^{n+1}$, we obtain a function which possesses properties (a) and (b).

We know from (b) that Z contains the set of points outside E where $G_E^{=}(\cdot;q) > 0$. We have already noted that Z is a semipolar subset of $\mathbb{R}^{n+1}$, and so Z has Lebesgue measure zero by [13, Theorem 9.27]. Thus $G_E^{=}(\cdot;q) = 0$ almost everywhere on $\mathbb{R}^{n+1}\backslash E$. Let q_0 be a finite point of ∂E. If there is $c > 0$ such that $\Omega(q_0;c) \cap E = \emptyset$, then $q_0 \in \partial_n E$ and is a regular point by any one of several tests, the most elementary being given by Pini [10] (for $n = 1$) and Watson [12, Theorem 36], [13, Theorem 8.49]. Hence $G_E^{=}(q_0;q) = 0$ by (b). On the other hand, if there is no such c, we put l equal to the

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upper limit in (2.1). Then

$$l \leq \limsup_{p\to q_0,\, p\in E} G_E(p;q) \leq \limsup_{p\to q_0} G_E^{=}(p;q) \leq G_E^{=}(q_0;q) < +\infty, \tag{2.2}$$

because $G_E^{=}(\cdot;q)$ is upper semicontinuous and upper finite on $\mathbb{R}^{n+1}\backslash\{q\}$, by part (a). Given any number $L > l$, we can find a heat ball $\Omega(q_0;c_0)$ such that $q \notin \overline{\Omega}(q_0;c_0)$ and $G_E(p;q) \leq L$ for all points $p \in \Omega(q_0;c_0) \cap E$. Therefore, by part (a) and [13, Theorem 3.51],

$$G_E^{=}(q_0;q) \leq \mathcal{V}(G_E^{=}(\cdot;q);q_0;c_0) = \mathcal{V}(G_E(\cdot;q)\chi_E;q_0;c_0) \leq \mathcal{V}(L\chi_E;q_0;c_0) \leq L.$$

Thus $G_E^{=}(q_0;q) \leq L$ whenever $l < L$, so that $G_E^{=}(q_0;q) \leq l$. This, together with (2.2), shows that $G_E^{=}(q_0;q) = l$. □

Theorem 2.1(c) leads to the following sharpening of the regularity criterion in [13, Theorem 8.53(c)] for normal boundary points of E. The improvement lies in the replacement of the condition $\lim_{p\to q_0} G_E(p;p_k) = 0$ by condition (2.3) below.

Theorem 2.2. *Let E be an open set and let q_0 be a finite point of $\partial_n E$. If there is a countable set $\{p_k : k \in I \subseteq \mathbb{N}\}$ of points in E such that*

$$\bigcup_{k\in I} \Lambda^*(p_k;E) \supseteq E \cap N$$

for some neighbourhood N of q_0, and if for each $k \in I$ such that $\Omega(q_0;c)$ meets $\Lambda^*(p_k;E)$ for every $c > 0$ we have

$$\lim_{c\to 0+}\Big(\sup_{\Omega(q_0;c)\cap E} G_E(\cdot;p_k)\Big) = 0, \tag{2.3}$$

then q_0 is regular for E.

Proof. For each $k \in I$, we choose a point $r_k \in E$ such that $p_k \in \Lambda^*(r_k;E)$. We denote by G_k the Green function of $\Lambda^*(r_k;E)$, which is the restriction of G_E to $\Lambda^*(r_k;E) \times \Lambda^*(r_k;E)$ by [13, Theorem 6.7(b)]. Applying Theorem 2.1 to $\Lambda^*(r_k;E)$, we obtain an extension $G_k^{=}(\cdot;p_k)$ of $G_k(\cdot;p_k)$ such that (i) $G_k^{=}(\cdot;p_k)$ is a nonnegative subtemperature on $\mathbb{R}^{n+1}\backslash\{p_k\}$, (ii) $G_k^{=}(\cdot;p_k) = 0$ outside the closure of $\Lambda^*(r_k;E)$, and (iii) if $q_0 \in \partial\Lambda^*(r_k;E)$ then either there is $c > 0$ such that $\Omega(q_0;c) \cap \Lambda^*(r_k;E) = \emptyset$, in which case $G_k^{=}(q_0;p_k) = 0$, or there is no such c, in which case

$$G_k^{=}(q_0;p_k) = \lim_{c\to 0+}\Big(\sup_{\Omega(q_0;c)\cap\Lambda^*(r_k;E)} G_k(\cdot;p_k)\Big).$$

In the latter case, our hypothesis (2.3) gives

$$G_k^{=}(q_0;p_k) \leq \lim_{c\to 0+}\Big(\sup_{\Omega(q_0;c)\cap E} G_E(\cdot;p_k)\Big) = 0.$$

Hence $G_k^{=}(q_0;p_k) = 0$ whenever $q_0 \in \partial\Lambda^*(r_k;E)$. Since $G_k^{=}(\cdot;p_k)$ is a nonnegative subtemperature on $\mathbb{R}^{n+1}\backslash\{p_k\}$, we obtain

$$0 \leq \limsup_{p\to q_0,\, p\in\Lambda^*(r_k;E)} G_k(p;p_k) \leq \limsup_{p\to q_0} G_k^{=}(p;p_k) \leq G_k^{=}(q_0;p_k) = 0.$$

Since $G_k = G_E$ on $\Lambda^*(r_k; E) \times \Lambda^*(r_k; E)$, we deduce that

$$\lim_{p\to q_0, p\in\Lambda^*(r_k;E)} G_E(p; p_k) = 0 \tag{2.4}$$

for all $k \in I$ such that $q_0 \in \partial\Lambda^*(r_k; E)$.

We now define a function u on E by putting

$$u = \sum_{k\in I} (G_E(\cdot; p_k) \wedge 2^{-k-1}),$$

and show that u is a barrier for E at q_0. Since $G_E(\cdot; p_k) > 0$ on $\Lambda^*(p_k; E)$ for every k, we have $u > 0$ on $\bigcup_{k\in I} \Lambda^*(p_k; E) \supseteq E \cap N$. Since a finite sum of supertemperatures is itself a supertemperature, u is clearly a supertemperature if I is finite. On the other hand, if I is infinite, [13, Theorem 3.60] and the uniform convergence of the defining series imply that u is a supertemperature on E. We put $J = \{k \in I : q_0 \in \partial\Lambda^*(p_k; E)\}$, and define

$$v = \sum_{k\in J} (G_E(\cdot; p_k) \wedge 2^{-k-1}), \quad w = \sum_{k\in I\backslash J} (G_E(\cdot; p_k) \wedge 2^{-k-1})$$

on E, so that $u = v + w$. If $I\backslash J \neq \emptyset$, we arrange its elements as a finite or infinite sequence $\{k_i\}_{1\le i<m}$, where $1 \le m \le \infty$. For each i there is a neighbourhood N_i of q_0 such that $G_E(\cdot; p_{k_i}) = 0$ on $N_i \cap E$, because $G_E(\cdot; p_{k_i}) = 0$ on $E\backslash\Lambda^*(p_{k_i}; E)$. Given any positive integer j, we put $V_j = \bigcap_{k_i\in I\backslash J, i\le j} N_i$. Then V_j is a neighbourhood of q_0 such that

$$\sum_{k_i\in I\backslash J, i\le j} (G_E(\cdot; p_{k_i}) \wedge 2^{-k_i-1}) = 0$$

on $V_j \cap E$. Therefore

$$w = \sum_{k_i\in I\backslash J, i>j} (G_E(\cdot; p_{k_i}) \wedge 2^{-k_i-1}) \le \sum_{k_i\in I\backslash J, i>j} 2^{-k_i-1} \le \sum_{i=j+1}^{\infty} 2^{-i-1} = 2^{-j-1}$$

on $V_j \cap E$. Thus $\lim_{p\to q_0} w(p) = 0$. To consider v, we arrange the elements of J as a finite or infinite sequence $\{l_i\}_{1\le i<h}$, where $1 \le h \le \infty$. Let ϵ be a given positive number. Since (2.4) holds for each $k \in J$, for each i we can find a neighbourhood B_i of q_0 such that $G_E(\cdot; p_{l_i}) < 2^{-l_i-1}\epsilon$ on $B_i \cap \Lambda^*(r_{l_i}; E)$, and therefore on $B_i \cap E$ because $G_E(\cdot; p_{l_i}) = 0$ on $E\backslash\Lambda^*(p_{l_i}; E) \supseteq E\backslash\Lambda^*(r_{l_i}; E)$. Given any positive integer j such that $2^{-j} < \epsilon$, we put $U_j = \bigcap_{i=1}^{j} B_i$. Then on $U_j \cap E$ we have

$$\begin{aligned} v &\le \sum_{l_i\in J,\, i\le j} 2^{-l_i-1}\epsilon + \sum_{l_i\in J,\, i>j} 2^{-l_i-1} \\ &\le \sum_{i=1}^{j} 2^{-i-1}\epsilon + \sum_{i=j+1}^{\infty} 2^{-i-1} \\ &\le \frac{\epsilon}{2} + 2^{-j-1} < \epsilon. \end{aligned}$$

Thus $\lim_{p\to q_0} v(p) = 0$, and hence $\lim_{p\to q_0} u(p) = 0$. Therefore u is a barrier for E at q_0, and [13, Theorem 8.46] shows that q_0 is regular for E. □

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3. The regularity of normal boundary points

The relaxation of the null limit criterion, given in Theorem 2.2, suggests that a similar relaxation should be possible using barriers other than those derived from the Green function. In this section we show that this is the case.

DEFINITION 3.1. Let E be an open set, and let q_0 be a finite point of $\partial_n E$ such that, for all $c > 0$, $\Omega(q_0; c) \cap E \neq \emptyset$. A function w is called a *heat ball barrier* at q_0 if it is defined on $N \cap E$ for some open neighbourhood N of q, and possesses the following properties:

(a) w is a supertemperature on $N \cap E$;
(b) $w > 0$ on $N \cap E$;
(c) w has a null limit through heat balls centred at q_0, that is

$$\lim_{c \to 0+} \Big(\sup_{\Omega(q_0;c) \cap E} w \Big) = 0.$$

The condition that, for all $c > 0$, $\Omega(q_0; c) \cap E \neq \emptyset$, is not an important restriction, because if it is not satisfied then there is $c_0 > 0$ such that $\Omega(q_0; c_0) \subseteq \mathbb{R}^{n+1} \backslash E$, and so q_0 is regular.

THEOREM 3.2. *Let E be an open set, and let q_0 be a finite point of $\partial_n E$ such that, for all $c > 0$, $\Omega(q_0; c) \cap E \neq \emptyset$. If there is a heat ball barrier u at q_0, then there is also a heat ball barrier v at q_0 such that v is a supertemperature on the whole of E and $\inf_{E \backslash N} v > 0$ for each neighbourhood N of q_0.*

PROOF. The proof is an easy modification of the proof of the corresponding result for standard barriers given, for example, in [13, Ch. 8]. □

THEOREM 3.3. *Let E be an open set, and let q_0 be a finite point of $\partial_n E$ such that, for all $c > 0$, $\Omega(q_0; c) \cap E \neq \emptyset$. If f is an upper bounded function on $\partial_e E$, and there is a heat ball barrier at q_0, then*

$$\lim_{c \to 0+} \Big(\sup_{\Omega(q_0;c) \cap E} U_f^E \Big) \leq \limsup_{q \to q_0} f(q).$$

PROOF. By Theorem 3.2, there is a heat ball barrier v at q_0 such that v is a supertemperature on the whole of E and $\inf_{E \backslash N} v > 0$ for each neighbourhood N of q_0. We put $L = \limsup_{q \to q_0} f(q)$, and note that $L < +\infty$ because f is upper bounded. Given any number $M > L$, we can find a closed neighbourhood V of q_0 such that $f(q) < M$ for all points $q \in (V \cap \partial_e E) \backslash \{q_0\}$. Since $\inf_{E \backslash V} v > 0$, we can choose a positive number κ such that

$$M + \kappa \inf_{E \backslash V} v > \sup_{\partial_e E} f.$$

By [13, Theorem 7.53], there is a nonnegative supertemperature w on $\mathbb{R}^{n+1}$ such that $w(q_0) = +\infty$ and $w(p) < +\infty$ for all $p \neq q_0$. The lower semicontinuity of w implies that $w(p) \to +\infty$ as $p \to q_0$. Let ϵ be any positive number. We now put $u = M + \kappa v + \epsilon w$

on E, and note that u is a lower bounded supertemperature on E. For all points $q \in (\partial_e E)\backslash V$, we have

$$\liminf_{p \to q} u(p) \geq M + \kappa \inf_{E\backslash V} v > \sup_{\partial_e E} f \geq f(q).$$

For all points $q \in (V \cap \partial_e E)\backslash\{q_0\}$ we have $f(q) < M$, so that

$$\liminf_{p \to q} u(p) \geq M > f(q).$$

Moreover,

$$\liminf_{p \to q_0} u(p) \geq M + \epsilon \lim_{p \to q_0} w(p) = +\infty \geq f(q_0).$$

It follows that $u \in \mathfrak{U}_f^E$, so that $u \geq U_f^E$ on E. Since ϵ is arbitrary and $w < +\infty$ on E, we deduce that $M + \kappa v \geq U_f^E$. Hence

$$\lim_{c \to 0+}\Big(\sup_{\Omega(q_0;c)\cap E} U_f^E\Big) \leq M + \kappa \lim_{c \to 0+}\Big(\sup_{\Omega(q_0;c)\cap E} v\Big) = M.$$

Since M is arbitrary, the result follows. □

Corollary 3.4. *Let E be an open set, and let q_0 be a finite point of $\partial_n E$ such that, for all $c > 0$, $\Omega(q_0; c) \cap E \neq \emptyset$. If f is a bounded function on $\partial_e E$ which is continuous at q_0, and there is a heat ball barrier at q_0, then*

$$\lim_{c \to 0+}\Big(\sup_{\Omega(q_0;c)\cap E} U_f^E\Big) = \lim_{c \to 0+}\Big(\inf_{\Omega(q_0;c)\cap E} U_f^E\Big) = f(q_0).$$

Proof. Theorem 3.3 shows that

$$\lim_{c \to 0+}\Big(\sup_{\Omega(q_0;c)\cap E} U_f^E\Big) \leq f(q_0)$$

and that

$$\lim_{c \to 0+}\Big(\sup_{\Omega(q_0;c)\cap E} U_{-f}^E\Big) \leq -f(q_0),$$

which implies that

$$\lim_{c \to 0+}\Big(\inf_{\Omega(q_0;c)\cap E} L_f^E\Big) \geq f(q_0).$$

Therefore, because $L_f^E \leq U_f^E$, we have

$$f(q_0) \leq \lim_{c \to 0+}\Big(\inf_{\Omega(q_0;c)\cap E} L_f^E\Big) \leq \lim_{c \to 0+}\Big(\sup_{\Omega(q_0;c)\cap E} U_f^E\Big) \leq f(q_0).$$ □

Theorem 3.5. *Let E be an open set, and let q_0 be a finite point of $\partial_n E$ such that, for all $c > 0$, $\Omega(q_0; c) \cap E \neq \emptyset$. If there is a heat ball barrier at q_0, then q_0 is regular for E.*

Proof. By [13, Theorem 8.53(a)], for each point $q \in E$ the Green function for E has the representation $G_E(\cdot; q) = G(\cdot; q) - S^E_{G(\cdot;q)}$. Therefore, if there is a heat ball barrier at q_0, then the above corollary shows that

$$\lim_{c \to 0+}\Big(\sup_{\Omega(q_0;c)\cap E} G_E(\cdot; q)\Big) \leq G(q_0; q) - \lim_{c \to 0+}\Big(\inf_{\Omega(q_0;c)\cap E} S^E_{G(\cdot;q)}\Big) = 0.$$

It now follows from Theorem 2.2 and the Lindelöf theorem that q_0 is regular for E. □

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4. The regularity of semi-singular boundary points

For semi-singular boundary points, we have the following criterion for regularity.

THEOREM 4.1. *Let E be an open set, and let $q_0 = (y_0, s_0) \in \partial_{ss}E$. Then q_0 is regular for E if and only if there is a positive number r_1 such that $H(q_0, r_1)$ is a component of $E \cap B(q_0, r_1)$.*

PROOF. If there is such a number r_1, then the restriction of the function w, defined by

$$w(x,t) = \begin{cases} \int_{\mathbb{R}^n} W(x-z, t-s_0)|z-y_0|\,dz & \text{if } t > s_0, \\ 1 & \text{if } t \le s_0, \end{cases}$$

to $E \cap B(q_0, r_1)$ is a positive temperature which satisfies $\lim_{(x,t)\to(y_0,s_0+)} w(x,t) = 0$, in view of [13, Theorem 4.8]. It is therefore a barrier for E at q_0, and hence q_0 is regular for E, by [13, Theorem 8.46(a)].

We now suppose, conversely, that there is no such number r_1. We choose a number $r_0 \in]0,1[$ such that $H(q_0, r_0) \subseteq E$, and define a function f on $\partial_e E$ by putting $f(p) = |p - q_0| \wedge 1$. Then f is resolutive for E by [13, Theorem 8.26]. If $D = \{(x,t) \in E : t < s_0\}$, then the restriction of f to $\partial_e D$ is resolutive for D, and [13, Lemma 8.10] shows that $S_f^E = S_f^D$ on D. Since $H(q_0, r_0) \subseteq E$, we have $f(x,t) \ge r_0$ whenever $t < s_0$. Therefore $f \ge r_0$ on $\partial_e D$, and so $S_f^D \ge r_0$ on D by [13, Lemma 8.14(c)]. Hence $S_f^E(x,t) \ge r_0$ whenever $t \le s_0$. Our hypothesis implies that, for any positive integer j, we can find a point $(x_j, s_0) \in E$ such that $|x_j - y_0| < 2^{-j}$. Since S_f^E is continuous at (x_j, s_0) and $S_f^E(x_j, s_0) \ge r_0$, we can find a point $(x_j, s_j) \in E$ such that $2^{-j} > s_j - s_0 > 0$ and $S_f^E(x_j, s_j) > r_0/2$. Making $j \to \infty$, we have $(x_j, s_j) \to (y, s_0+)$ but $S_f^E(x_j, s_j) \nrightarrow 0 = f(y_0, s_0)$. Hence q_0 is irregular for E. □

COROLLARY 4.2. *Let E be an open set, let C be a component of E, and let q_0 be a point in $\partial_{ss}C \cap \partial_{ss}E$. Then q_0 is regular for C if and only if q_0 is regular for E.*

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