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STRONGLY q-ADDITIVE FUNCTIONS AND DISTRIBUTIONAL PROPERTIES OF THE LARGEST PRIME FACTOR

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Abstract

Let $P(n)$ denote the largest prime factor of an integer $n \geq 2$. In this paper, we study the distribution of the sequence $\{f(P(n)) : n \geq 1\}$ over the set of congruence classes modulo an integer $b \geq 2$, where f is a strongly q-additive integer-valued function (that is, $f(aq^j + b) = f(a) + f(b)$, with $(a, b, j) \in \mathbb{N}^3$, $0 \leq b < q^j$). We also show that the sequence $\{\alpha P(n) : n \geq 1, f(P(n)) \equiv a \pmod b\}$ is uniformly distributed modulo 1 if and only if $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

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1. Introduction

For a positive integer n, let $P(n)$ be the largest prime factor of n, with the usual convention that $P(1) = 1$. The distribution of the largest prime factor in congruence classes has been previously considered by Ivič [6] and Oon [13] for a fixed modulus k. Using a similar approach to that of Ivič [6], Banks *et al.* [1] obtained new bounds that are nontrivial for a wide range of values of the modulus k. In particular, if k is not too large relative to x, they derived the expected asymptotic formula

$$\sharp\{n \leq x : P(n) \equiv l \pmod k\} \sim \frac{x}{\varphi(k)}$$

with an explicit error term that is independent of l. Moreover, by bounding the exponential sum $\sum_{n\leq x} e(\alpha P(n))$ for a fixed irrational real number α, they deduced that the sequence $\{\alpha P(n) : n \geq 1\}$ is uniformly distributed modulo 1. This result is reminiscent of the classical theorem of Vinogradov [15] that, for a fixed irrational real number α, the sequence $\{\alpha p : p \text{ prime}\}$ is uniformly distributed modulo 1.

The main goal of this paper is to give asymptotic expansions for the cardinality of

$$\mathcal{A}(x, a, b) = \{n \leq x : f(P(n)) \equiv a \pmod b\},$$

where f is a strongly q-additive function, $b \geq 2$ and $a \in \mathbb{Z}$. In addition, we prove the uniform distribution modulo 1 of $\alpha P(n)$ when $f(P(n)) \equiv a \pmod{b}$. In Section 2, we define the basic notions which are standard in this area (see, for example, [1, 10]) and give some preliminary results. In Section 3, we give an asymptotic formula for the number of elements of $\mathcal{A}(x, a, b)$ and we prove that the sequence $\{\alpha P(n) : n \geq 1, f(P(n)) \equiv a \pmod{b}\}$ is uniformly distributed modulo 1.

Throughout this paper, p always denotes a prime number and φ denotes the Euler function. For any real x, we define $e(x) = e^{2\pi i x}$. The notations (a, b) and $[a, b]$ refer respectively to the greatest common divisor and the least common multiple of a and b. We denote by $|\mathcal{E}|$ the number of elements of a set $\mathcal{E}$. We recall that the notation $U \ll V$ is equivalent to the statement that $U = O(V)$ for positive functions U and V and the implied constants in the symbols 'O' and '$\ll$' are absolute. We also use the symbol 'o' with its usual meaning, that is, the statement $U = o(V)$ is equivalent to $U/V \to 0$.

2. Preliminaries

2.1. Digital functions and strongly q-additive functions. Let $q \geq 2$ be an integer. Then we can represent every positive integer n in a unique way as

$$n = \sum_{0 \leq j \leq \nu} n_j q^j \quad \text{and} \quad n_j \in \{0, \ldots, q-1\}.$$

This is the q-ary representation of n with q the base and $\{0, \ldots, q-1\}$ the set of digits.

A function $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) = \sum_{0 \leq k < q} \alpha_k |n|_k$, with

$$|n|_k = |\{0 \leq j \leq \nu : n_j = k\}| \quad \text{and} \quad \alpha_0, \ldots, \alpha_{q-1} \in \mathbb{R},$$

is called a digital function. A function $f : \mathbb{N} \to \mathbb{R}$ is called strongly q-additive if $f(aq^i + b) = f(a) + f(b)$, where $(a, b, i) \in \mathbb{N}^3$ and $0 \leq b < q^i$. In particular, $f(0) = 0$ and

$$f(n) = \sum_{0 \leq j \leq \nu} f(n_j) = \sum_{1 \leq k < q} f(k)|n|_k.$$

A simple example of a strongly q-additive function is the sum of digits function,

$$s_q(n) = \sum_{0 \leq j \leq \nu} n_j = \sum_{1 \leq k < q} k|n|_k.$$

Strongly q-additive functions, particularly their asymptotic distribution, have been extensively discussed in the literature (see, for example, [2, 3, 10–12]).

Let $\mathcal{F}$ be the set of digital functions $f = \sum_{0 \leq k < q} a_k | \cdot |_k$ such that the real sequence $a_0, \ldots, a_{q-1}$ is not an arithmetic progression modulo 1 whose common difference r is an integer multiple of $1/(q-1)$ (that is, $r(q-1) \notin \mathbb{Z}$) and let $\mathcal{F}_0$ be the set of functions $f = \sum_{0 \leq k < q} a_k | \cdot |_k$ such that the sequence $a_0, \ldots, a_{q-1}$ is an arithmetic progression modulo 1. It is easily seen that $s_q(\cdot) \in \mathcal{F}_0$.

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For $f(n) = \sum_{0\le k<q} a_k|n|_k \in \mathcal{F}\cup\mathcal{F}_0$, we define real numbers $\lambda_q(f)$ by

$$\lambda_q(f) = \begin{cases} c_{1,q}\min_{t\in\mathbb{R}} \sum_{0\le j<i<q} \|a_i - a_j - (i-j)t\|^2 & \text{if } f\notin\mathcal{F}_0,\\ c_{2,q}\|(q-1)(a_1-a_0)\|^2 & \text{if } f\in\mathcal{F}_0\cap\mathcal{F}, \end{cases} \tag{2.1}$$

where $\|y\|$ denotes the distance from the real number y to the nearest integer, and $c_{1,q}$ and $c_{2,q}$ are constants depending only on q (defined in [10, page 27]). It was established in [10] that $\lambda_q(f) > 0$ and the theorems of Hadamard–de La Vallée Poussin and Vinogradov (see [4, 5, 15]) were extended to the case of prime numbers satisfying a digital constraint. The method is based on the following estimate of exponential sums.

Theorem 2.1 [10, Théorèmes 1 and 2]. *Suppose that $q\ge 2$ and $f\in\mathcal{F}\cup\mathcal{F}_0$. Then, for all $x\ge 2$ and $\beta\in\mathbb{R}$,*

$$\sum_{n\le x}\Lambda(n)e(f(n)+\beta n) \ll x^{1-\lambda_q(f)}(\log x)^4,$$

where $\lambda_q(f)$ is defined in (2.1) and the implied constant depends only on q.

We can see a generalised version of Theorem 2.1 in [12].

Let $\mathcal{F}_q^+$ be the set of strongly q-additive functions f such that

$$f = \sum_{1\le k<q} a_k|\cdot|_k \quad \text{with} \quad a_1,\dots,a_{q-1}\in\mathbb{Z} \quad \text{and} \quad \gcd(a_1,\dots,a_{q-1}) = 1.$$

Let $f\in\mathcal{F}_q^+$ and let $d = d_{f,b,q}\ge 1$ be the greatest divisor of $(b, q-1)$ such that $(f(1), d) = 1$ and, for all integers n,

$$f(n)\equiv f(1)s_q(n)\equiv f(1)n \bmod d. \tag{2.2}$$

By using the result of Martin *et al.* (see [10, Proposition 5]), we see that for all $j\in J_2 = \{0\le j<b : j \text{ is not a multiple of } b/d\}$,

$$\sum_{p\le N} e\Big(\frac{j}{b}f(p)+rp\Big) \ll N^{1-\sigma_{f,b,q}}(\log N)^3, \tag{2.3}$$

where the implied constant depends only on q.

Let $\pi(x;l,m)$ denote the number of primes less than or equal to x which are congruent to $l \pmod m$ for some real $x>0$ and positive coprime integers l,m. Using elementary means and the above result, Martin *et al.* [10] proved the following theorem.

Theorem 2.2 [10, Théorème 4]. *Let $q,\ b\ge 2$, $f\in\mathcal{F}^+$ and $d = d_{f,b,q}$ be the integer defined in (2.2). Let $c = f^*(1)$ be the multiplicative inverse of $f(1)$ modulo d. Then, for every $a\in\mathbb{Z}$,*

$$|\{p\le x : f(p)\equiv a \pmod b\}| = \begin{cases} 0 \text{ or } 1 & \text{if } (a,d)>1,\\ \dfrac{d}{b}\pi(x;ac,d) + O((\log x)^3 x^{1-\sigma_{f,b,q}}) & \text{otherwise,} \end{cases}$$

where the implied constant depends only on q.

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2.2. Auxiliary estimates. As usual, we say that a positive integer n is y-smooth if $P(n) \le y$. Let

$$\psi(x, y) = |\{n \le x : n \text{ is } y\text{-smooth}\}|.$$

The following estimate is a simplified version of [14, Theorem 1 of Ch. III.5].

LEMMA 2.3. *Let $u = \log x/\log y$, where $x \ge y > 0$. If $u \ge 1$, then*

$$\psi(x, y) \ll x \exp(-u/2). \tag{2.4}$$

In what follows, we denote by $\mathcal{P}$ the set of all prime numbers and by $\mathcal{P}[w, x]$ the set of primes p such that $w \le p \le x$. Given $x \ge y > 0$ and $m \ge 1$, we put

$$L_m = \max\{y, P(m)\}, \quad \mathcal{P}_m = \mathcal{P}[L_m, x/m].$$

LEMMA 2.4 [1, Lemma 3]. *Let $x \ge y > 0$. For any arithmetical functions h and g satisfying $\max\{|h(k)|, |g(k)|\} \le 1$ for all positive integers k,*

$$\sum_{n\le x} h(P(n))g(n) = \sum_{m\le x/y} \sum_{p\in\mathcal{P}_m} h(p)g(mp) + O(\psi(x, y)).$$

3. Main results

THEOREM 3.1. *Let $q, b \ge 2$ be integers, x a real number, $f \in \mathcal{F}_q^+$ and $d = d_{f,b,q}$ the integer defined in (2.2). Then, for every $a \in \mathbb{Z}$, there exists a constant $K_0 > 0$ such that for any $K < K_0$,*

$$|\mathcal{A}(x, a, b)| = \begin{cases} \dfrac{dx}{b\varphi(d)} + O(x \exp(-K \log^{1/3} x)) & \text{if } (a, d) = 1, \\ O(x \exp(-K \log^{1/3} x)) & \text{otherwise}. \end{cases}$$

PROOF. For every positive integer k, we consider the functions $g(k) = 1$ and

$$h(k) = \begin{cases} 1 & \text{if } f(k) \equiv a \pmod b, \\ 0 & \text{otherwise}. \end{cases}$$

For any real parameters x, y to be chosen later, with $0 < y < x$, Lemma 2.4 gives

$$\begin{aligned} |\mathcal{A}(x, a, b)| = \sum_{n\le x} h(P(n))g(P(n)) &= \sum_{m\le x/y} \sum_{p\in\mathcal{P}_m} h(p)g(mp) + O(\psi(x, y)) \\ &= \sum_{m\le x/y} \mathcal{N}(m, a, b) + O(\psi(x, y)), \end{aligned} \tag{3.1}$$

where $\mathcal{N}(m, a, b) = |\{p \in \mathcal{P}_m : f(p) \equiv a \pmod b\}|$. In view of Theorem 2.2, if $(a, d) > 1$,

$$\sum_{m\le x/y} \mathcal{N}(m, a, b) = 0.$$

In the other case, for any m with $mL_m \le x$,

$$\mathcal{N}(m, a, b) = \pi_f(x/m) - \pi_f(L_m) + O(1),$$

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where $\pi_f(x) = \sum_{p\le x, f(p)\equiv a \pmod b} 1$, and the sum is empty otherwise. In this case, since $(a,d) = 1$, Theorem 2.2 shows that there exists a constant $\sigma_{f,q,b} > 0$ such that

$$\pi_f(x) = \frac{d}{b}\pi(x; ac, d) + O(x^{1-\sigma_{f,q,b}}(\log x)^3). \tag{3.2}$$

We observe that the error term in (3.2) is an increasing function of x. Thus,

$$\mathcal{N}(m,a,b) = \frac{d}{b}\Big(\pi\Big(\frac{x}{m}; ac, d\Big) - \pi(L_m; ac, d)\Big) + O\Big(\Big(\log\frac{x}{m}\Big)^3\Big(\frac{x}{m}\Big)^{1-\sigma_{f,q,b}}\Big). \tag{3.3}$$

For any integers u, v such that $(u,v) = 1$, the following estimate holds (see [8]):

$$\pi(x; u, v) = \frac{1}{\varphi(v)}\mathrm{Li}(x) + O(x\exp(-c_1\sqrt{\log x})), \tag{3.4}$$

where c_1 is a positive constant. We note that an improved version of (3.4) can be found in [9]. So, (3.3) becomes

$$\mathcal{N}(m,a,b) = \frac{d}{b\varphi(d)}\Big(\mathrm{Li}\Big(\frac{x}{m}\Big) - \mathrm{Li}(L_m)\Big) + O\Big(\Big(\log\frac{x}{m}\Big)^3\Big(\frac{x}{m}\Big)^{1-\sigma_{f,q,b}}\Big)$$
$$+ O\Big(\frac{x}{m}\exp\Big(-c_1\sqrt{\log\frac{x}{m}}\Big)\Big).$$

Then

$$|\mathcal{A}(x,a,b)| = \frac{d}{b\varphi(d)}\sum_{m\le x/y}\Big(\mathrm{Li}\Big(\frac{x}{m}\Big) - \mathrm{Li}(L_m)\Big) + O(\psi(x,y) + R_1 + R_2),$$

where

$$R_1 = \sum_{m\le x/y}\Big(\log\frac{x}{m}\Big)^3\Big(\frac{x}{m}\Big)^{1-\sigma_{f,q,b}}, \quad R_2 = \sum_{m\le x/y}\frac{x}{m}\exp\Big(-c_1\sqrt{\log\frac{x}{m}}\Big).$$

The same arguments as applied in (3.1) with $h(k) = 1$ lead to the identity

$$\lfloor x\rfloor = \sum_{n\le x} 1 = \sum_{m\le x/y}\Big(\mathrm{Li}\Big(\frac{x}{m}\Big) - \mathrm{Li}(L_m)\Big) + O(\psi(x,y) + R_2).$$

Hence,

$$|\mathcal{A}(x,a,b)| = \frac{dx}{b\varphi(d)} + O(\psi(x,y) + R_1 + R_2). \tag{3.5}$$

By elementary estimates,

$$R_1 = O(x(\log x)^3 y^{-\sigma_{f,q,b}}), \quad R_2 = O(x\log x\exp(-c_1\sqrt{\log y})).$$

From Lemma 2.3, we have $\psi(x,y) = O(x\exp(-\log x/(2\log y)))$. For positive real numbers x, y, we define the functions θ_i with $1 \le i \le 3$ as follows:

$$\begin{cases} \theta_1(x,y) = (\log x)^3 y^{-\sigma_{f,q,b}}, \\ \theta_2(x,y) = \log x\exp(-c_1\sqrt{\log y}), \\ \theta_3(x,y) = \exp(-\log x/(2\log y)). \end{cases}$$

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For a fixed real number x, sufficiently large, we obtain

$$\theta_1(x,y) = \theta_3(x,y) \quad \text{for } y = y_0 = \exp\Big(\frac{6\log\log x + \sqrt{(6\log\log x)^2 + 8\sigma_{f,q,b}\log x}}{4\sigma_{f,q,b}}\Big),$$

$$\theta_2(x,y) = \theta_3(x,y) \quad \text{for } y = y_1 = \exp(C\log^{2/3} x + O(\log^{1/3} x \log\log x)),$$

with $C = (4c_1)^{-2/3}$, where the constant c_1 is defined in (3.4). Since $\theta_3(x,y)$ is an increasing function on y,

$$\theta_1(x,y_0) = \theta_3(x,y_0) \le \theta_3(x,y_1) = \theta_2(x,y_1).$$

So, by choosing $y = y_1$, we have proved that the error term in (3.5) is

$$O(x\log x \exp(-K_0 \log^{1/3} x)),$$

where $K_0 = 1/(2C)$ is a positive constant. The proof is completed. □

Next, we will prove the uniform distribution modulo 1 of $\{\alpha P(n) : n \in \mathcal{A}\}$ with $\mathcal{A} = \mathcal{A}(a,b) = \{n \in \mathbb{N}\setminus\{0\}, f(P(n)) \equiv a \pmod b\}$. We note that it is shown in [10] that the sequence $\{\alpha p : p \text{ prime}, f(p) \equiv a \pmod b\}$ is uniformly distributed modulo 1 if and only if α is irrational.

Theorem 3.2. *Let $q, b \ge 2$ be integers, $f \in \mathcal{F}_q^+$, $d = d_{f,b,q}$ the integer defined in (2.2), $a \in \mathbb{Z}$ such that $\gcd(a,d) = 1$ and $\alpha \in \mathbb{R}$. Then the sequence $\{\alpha P(n) : n \in \mathcal{A}\}$ is uniformly distributed modulo 1 if and only if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.*

Proof. If α is rational, then the sequence $\{\alpha P(n) : n \in \mathcal{A}\}$ contains only a finite number of terms modulo 1 and consequently is not uniformly distributed modulo 1.

Now, let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. By Weyl's criterion (see [7, Theorem 5.6]), it suffices to prove that for every $h \in \mathbb{Z}^*$,

$$\frac{1}{|\mathcal{A}(x,a,b)|}\sum_{n\in\mathcal{A}(x,a,b)} e(\alpha h P(n)) = o(1) \quad \text{as } x \to \infty.$$

To estimate the sum, we apply Lemma 2.4 to the functions $g(k) = 1$ and

$$h(k) = \begin{cases} e(\alpha h k) & \text{if } f(k) \equiv a \pmod b,\\ 0 & \text{otherwise.}\end{cases}$$

For $0 < y < x$,

$$\begin{aligned}\sum_{n\in\mathcal{A}(x,a,b)} e(\alpha h P(n)) &= \sum_{m\le x/y}\sum_{p\in\mathcal{P}_m} h(p)g(mp) + O(\psi(x,y))\\ &= \sum_{m\le x/y}\sum_{\substack{p\in\mathcal{P}_m\\ f(p)\equiv a \,(\mathrm{mod}\, b)}} e(\alpha h p) + O(\psi(x,y)).\end{aligned} \tag{3.6}$$

By the orthogonality formula,

$$\sum_{\substack{p\in\mathcal{P}_m\\ f(p)\equiv a \,(\mathrm{mod}\, b)}} e(\alpha h p) = \frac{1}{b}\sum_{j=0}^{b-1}\sum_{p\in\mathcal{P}_m} e\Big(\frac{j}{b}(f(p)-a) + \alpha h p\Big). \tag{3.7}$$

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We split the summation (3.7) over j into two parts according as $j \in J_1$ and $j \in J_2$, where $J_1 = \{0 \le j < b : j \text{ is a multiple of } b/d\}$ and $J_2 = \{0, \dots, b-1\} \setminus J_1$. We write

$$S_i = \frac{1}{b} \sum_{m \le x/y} \sum_{j \in J_i} \sum_{p \in \mathcal{P}_m} e\Big(\frac{j}{b}(f(p) - a) + \alpha h p\Big).$$

Estimation of S_1. For all $j \in J_1$, we can write $j = ub/d$ with $0 \le u < d$. From (2.2),

$$\sum_{p \le x} e\Big(\frac{j}{b} f(p) + \alpha h p\Big) = \sum_{p \le x} e\Big(p\Big(\frac{uf(1)}{d} + \alpha h\Big)\Big).$$

Since α is irrational, so is $(u/d)f(1) + \alpha h$. Thanks to [15], $(((u/d)f(1) + \alpha h)p)_{p \in \mathcal{P}}$ is uniformly distributed modulo 1. We deduce from Weyl's criterion that

$$\sum_{p \le x} e\Big(p\Big(\frac{uf(1)}{d} + \alpha h\Big)\Big) = o(\pi(x)) \quad \text{as } x \to \infty,$$

which gives, as $x \to \infty$,

$$\frac{1}{b} \sum_{m \le x/y} \sum_{j \in J_1} \Big| \sum_{p \in \mathcal{P}_m} e\Big(\frac{j}{b} f(p) + \alpha h p\Big)\Big| = o\Big(\sum_{m \le x/y} \pi\Big(\frac{x}{m}\Big)\Big) = o\Big(\frac{x \log(x/y)}{\log y}\Big). \tag{3.8}$$

Estimation of S_2. For all $j \in J_2$, we have from (2.3) that

$$\sum_{p \le x} e\Big(\frac{j}{b} f(p) + \alpha h p\Big) \ll x^{1-\sigma_{f,q,b}} (\log x)^3.$$

The same arguments as in the proof of Theorem 3.1 give

$$S_2 = O(xy^{-\sigma_{f,q,b}} (\log x)^3). \tag{3.9}$$

Assembling (3.6)–(3.9) and (2.4) yields

$$\Big| \sum_{n \in \mathcal{A}(x,a,b)} e(\alpha h P(n)) \Big| \ll x\Big(y^{-\sigma_{f,q,b}} (\log x)^3 + \frac{\log(x/y)}{\log y} + \exp\Big(-\frac{\log x}{2 \log y}\Big)\Big).$$

Now, from Theorem 3.1,

$$|\mathcal{A}(x,a,b)| \sim \frac{dx}{b\varphi(d)} \quad \text{as } x \to \infty$$

and, by choosing $y = \exp((\log x)^{2/3})$, we complete the proof. □

COROLLARY 3.3. *For $f \in \mathcal{F}$, the sequence $(\alpha f(P(n)))_{n \in \mathbb{N}}$ is uniformly distributed modulo 1 if and only if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.*

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Proof. If α is rational, then the sequence $(\alpha f(P(n)))_{n\in\mathbb{N}}$ contains only a finite number of terms modulo 1 and is not uniformly distributed modulo 1. Conversely, by Weyl's criterion (see [7, Theorem 5.6]), it suffices to prove that for every $h \in \mathbb{Z}^*$,

$$\frac{1}{x}\sum_{n\le x} e(\alpha h f(P(n))) = o(1) \quad \text{as } x \to \infty.$$

By Lemma 2.4, as in (3.6), we write

$$\sum_{n\le x} e(\alpha h f(P(n))) = \sum_{m\le x/y}\Big(\sum_{p\in\mathcal{P}_m} e(\alpha h f(p))\Big) + O(\psi(x,y)).$$

Now, we use [10, Théorème 3], which asserts that for every irrational α and $f \in \mathcal{F}$, the sequence $(\alpha f(p))_{p\in\mathcal{P}}$ is uniformly distributed modulo 1. So,

$$\sum_{p\le x} e(\alpha h f(p)) = o(\pi(x)) \quad \text{as } x \to \infty. \tag{3.10}$$

Applying (3.10) in (3.11) and using (2.4),

$$\sum_{n\le x} e(\alpha h f(P(n))) \ll x\Big(\frac{\log(x/y)}{\log y} + \exp\Big(-\frac{\log x}{2\log y}\Big)\Big). \tag{3.11}$$

By choosing $y = \exp((\log x)^{2/3})$, we complete the proof. □

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