A UNIVERSAL SEMIGROUP

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1. Introduction

In [4] S. Ulam asks the following question.

'Does there exist a universal compact semigroup; i.e., a semigroup U such that every compact topological semigroup is continuously isomorphic to a subsemigroup of it?'

The author has not been able to answer this question. However, in this paper, a proof is given for the following related result.

Let Q denote the Hilbert cube of countably infinite dimension and $C(Q)$ the Banach space of continuous real-valued functions on Q with the usual norm. Let U denote the semigroup consisting of all bounded linear operators $T: C(Q) \to C(Q)$ with $||T|| \leqq 1$ and let U be endowed with the strong topology. Then, for every compact metric semigroup S with the property:

(1.1) for all $x, y \in S$, with $x \neq y$, there exists a $z \in S$, such that $xz \neq yz$ or $zx \neq zy$;

there exists a $1-1$ mapping φ of S into U such that φ is both a semigroup isomorphism and a homeomorphism.

U is metrizable, but is not compact; hence it does not provide an answer to the question of Ulam.

The proof of the above statement leans heavily on a result of S. Kakutani [1].

2. The space $C(Q)$ and the semigroup U

The Hilbert cube Q can be regarded as the set of all real sequences $a = \{a_n\}$, such that $0 \leqq a_n \leqq 1$ for $n = 1, 2, \ldots$, with a metric d defined by

$$d(a, b) = \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n|.$$

It is well known that Q is compact and that the Banach space $C(Q)$ of real-valued continuous functions on Q is separable.

As mentioned in **1**, U denotes the semigroup consisting of all bounded linear operators $T: C(Q) \to C(Q)$ with $||T|| \leqq 1$. We recall that a sequence $\{T_r\}$ in U

converges strongly to an element T of U if, and only if, for each $f \in C(Q)$,

$$\|T_r(f) - T(f)\| \to 0$$

as $r \to \infty$ (see, for example, [3] page 150).

Strong convergence in U can be characterized by the following metric. Let

$$\{f_1, f_2, \ldots\}$$

be a countable dense subset of the unit ball in $C(Q)$ and put

$$d(T_1, T_2) = \sum_{r=1}^{\infty} 2^{-r} \|T_1(f_r) - T_2(f_r)\|$$

for all $T_1, T_2 \in U$. The multiplication operation in U is continuous with respect to this metric.

3. The embedding theorem

We are now ready to prove that U has the universal property, mentioned in the introduction. We make use of the theorem of Urysohn [2, page 125]:

every separable metric space is homeomorphic to a subset of Q.

We also need the following special case of an extension theorem, proved by Kakutani in [1].

If F is a non-empty closed subset of a compact metric space X, then there exists a bounded linear operator $E : C(F) \to C(X)$ such that $\|E\| = 1$ and, for every $f \in C(F)$, $E(f)$ extends f.

The well-known extension theorem of Tietze provides an extension for each continuous function f on F, but the Kakutani theorem goes further. It provides an extension which is a linear operator with unit norm. We can now prove our theorem.

THEOREM. *Let S be a compact metric semigroup with the property* (1.1). *Then there exists a* $1-1$ *mapping φ of S into U such that φ is both a semigroup isomorphism and a homeomorphism.*

PROOF. (i) We suppose to begin with that S has the property

(A): for all $x, y \in S$, with $x \neq y$, there exists a $z \in S$ such that $zx \neq zy$.

Because of Urysohn's theorem we may suppose that S is a subset of Q. Then S is closed. By Kakutani's theorem, there exists a bounded linear operator

$$E : C(S) \to C(Q)$$

with $\|E\| = 1$ and such that for every $f \in C(S)$, $E(f)$ extends f. Let

$$R : C(Q) \to C(S)$$

be the bounded linear operator given by $R(g) = g|S$. Then $\|R\| \leqq 1$.

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For each $a \in S$, let $\psi(a)$ be the bounded linear operator of $C(S)$ into $C(S)$ given by

$$[\{\psi(a)\}(f)](x) = f(xa) \qquad x \in A.$$

Then $\|\psi(a)\| \leqq 1$ and

$$\psi(a)\psi(b) = \psi(ab) \tag{1}$$

for all $a, b \in S$.

For each $a \in S$, let $\varphi(a)$ be the bounded linear operator of $C(Q)$ into $C(Q)$, given by

$$\varphi(a) = E\psi(a)R.$$

Then $\|\varphi(a)\| \leqq 1$, hence $\varphi(a) \in U$.

Now φ is $1-1$, because if $a \neq b$, there exists a $z \in S$, with $za \neq zb$ and there exists an $f \in C(S)$ with $f(za) \neq f(zb)$, hence

$$[\{\psi(a)\}(f)](z) \neq [\{\psi(b)\}(f)](z)$$

so that $\{\psi(a)\}(f) \neq \{\psi(b)\}(f)$; therefore, since $R\{E(f)\} = f$, we have

$$\begin{aligned}[\varphi(a)]\{E(f)\} &= E[\{\psi(a)\}(f)] \neq E[\{\psi(b)\}(f)] \\ &= [\varphi(b)]\{E(f)\};\end{aligned}$$

hence $\varphi(a) \neq \varphi(b)$.

φ is an isomorphism, because

$$\begin{aligned}\varphi(a)\varphi(b) &= E\psi(a)RE\psi(b)R, \\ &= E\psi(a)\psi(b)R,\end{aligned}$$

and by (1)

$$\begin{aligned}&= E\psi(ab)R \\ &= \varphi(ab).\end{aligned}$$

Finally, we show that φ is continuous. (Since S is compact, the continuity of φ^{-1} will follow from the continuity of φ). Let $\{a_r\}$ be a sequence in S converging to a point a of S. Consider an arbitrary function f of $C(Q)$. Put $g = R(f) \in C(S)$. Then

$$\begin{aligned}\|\{\varphi(a_r)\}(f) - \{\varphi(a)\}(f)\| &= \|\{E\psi(a_r)\}(g) - \{E\psi(a)\}(g)\| \\ &\leqq \|\{\psi(a_r)\}(g) - \{\psi(a)\}(g)\| \\ &= \sup_{x \in S} |g(xa_r) - g(xa)| \\ &\to 0 \qquad \text{as } r \to \infty;\end{aligned}$$

i.e., $\{\varphi(a_r)\}$ converges strongly to $\varphi(a)$. Thus we have shown that φ is continuous

(ii) We now consider the general case.

Let d be a metric for S. We introduce a special metric ρ by defining

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$$\rho_1(x, y) = \sup_{u \in S} d(ux, uy),$$

$$\rho_2(x, y) = \sup_{v \in S} d(xv, yv),$$

$$\rho_3(x, y) = \sup_{u, v \in S} d(uxv, uyv)$$

and

$$\rho(x, y) = \sup \{d(x, y), \rho_1(x, y), \rho_2(x, y), \rho_3(x, y)\}.$$

ρ is a metric for S, it is topologically equivalent to d and it has the property:

$$\rho(xz, yz) \leqq \rho(x, y)$$

and

$$\rho(zx, zy) \leqq \rho(x, y),$$

for all $x, y, z \in S$.

Let $\mathscr{I}$ denote the semigroup consisting of all transformations f of S into S such that

$$\rho[f(x), f(x')] \leqq \rho(x, x')$$

for all $x, x' \in S$. Define a metric D for $\mathscr{I}$, by

$$D(f, g) = \sup_{x \in S} \rho[f(x), g(x)].$$

Then $\mathscr{I}$ is compact. Let $\mathscr{U}$ be the semigroup, whose underlying space is $\mathscr{I} \times \mathscr{I}$, with the product topology and with multiplication defined by

$$(f_1, f_2)(g_1, g_2) = (f_1 g_1, g_2 f_2).$$

Since $\mathscr{I}$ contains the identity transformation, $\mathscr{U}$ clearly has the property (A). Also $\mathscr{U}$ is compact and metrizable. Hence by (i), there exists a $1-1$ mapping η of $\mathscr{U}$ into U, such that η is both a semigroup isomorphism and a homeomorphism.

We now define a mapping $\psi : S \to \mathscr{U}$ by putting

$$[\psi_1(a)](x) = ax, \qquad [\psi_2(a)](x) = xa$$

and

$$\psi = (\psi_1, \psi_2).$$

Routine computations show that ψ is a $1-1$ mapping,

$$\psi(ab) = \psi(a)\psi(b)$$

for all $a, b \in S$ and ψ is continuous. Since S is compact, ψ is a homeomorphism. By putting

$$\varphi = \eta \circ \psi,$$

we obtain the required mapping φ of S into U.

References

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