BULL. AUSTRAL. MATH. SOC. 19C09
VOL. 58 (1998) [495–499]

ON THE SCHUR MULTIPLIER OF A QUOTIENT OF A DIRECT PRODUCT OF GROUPS

GRAHAM ELLIS

We use a nonabelian exterior product to strengthen two old and basic results on the Schur multiplier of a (central) quotient of a direct product of groups.

This is one of a series of papers (see also [**5, 6, 7, 8**]) advertising the relevance of a certain 'nonabelian exterior product' to the development and exposition of the basic theory of the Schur multiplier of a group. We shall use the exterior product to prove the following generalisation of a result of Eckmann, Hilton and Stammbach [**3**].

THEOREM 1. *Let $A = M \times N$ be a direct product of groups, let $\pi_M : A \to M$, $\pi_N : A \to N$ be the projections, and let U be a normal subgroup of A. Set $G = A/U$, $\overline{M} = M/\pi_M U$, $\overline{N} = N/\pi_N U$. The Schur multiplier $H_2(G)$ fits into a short exact sequence*

$$0 \to B \to H_2(G) \to \frac{U \cap [A, A]}{[U, A]} \to 0 \tag{1}$$

where B is an Abelian group that fits into exact sequences

$$[U, A]_{ab} \oplus H_2(M) \oplus H_2(N) \to B \to \overline{M}_{ab} \otimes \overline{N}_{ab} \to 0, \tag{2}$$

$$M_{ab} \otimes N_{ab} \to B \to \ker\left(H_2\left(\overline{M}\right) \to \frac{\pi_M U}{[M, \pi_M U]}\right) \oplus \ker\left(H_2\left(\overline{N}\right) \to \frac{\pi_N U}{[N, \pi_N U]}\right). \tag{3}$$

A special case of this theorem, in which U is assumed to be central in A, was proved in [**3**]. As illustrated in [**3**], the theorem can be viewed as a tool for determining some of the structure of the Schur multiplier $H_2(G)$ from a knowledge of $H_2(A)$.

Theorem 1 also implies a result of Wiegold [**9**] which states that if $U \cong \pi_M U \cong \pi_N U$, if U is central in A, and if G is finite, then $\overline{M}_{ab} \otimes \overline{N}_{ab}$ is isomorphic to a subgroup of $H_2(G)$. To deduce this result it in fact suffices to assume that G is finite, for then $H_2(G)$ is finite, and thus (2) provides a surjection $B \twoheadrightarrow \overline{M}_{ab} \otimes \overline{N}_{ab}$ of finite groups. So $\overline{M}_{ab} \otimes \overline{N}_{ab}$ must be isomorphic to a subgroup of B, and hence isomorphic to a subgroup of $H_2(G)$.

Received 13th May, 1998

We shall show how Wiegold's result can be reworked into the following slightly more general proposition.

PROPOSITION 2. *Let M, N be normal subgroups of a group K such that $[M,N] = 1$. Set $G = MN$ and suppose that the image of the canonical homomorphism $\phi: G \to K_{ab}$ is a direct summand of K_{ab}, that is $K_{ab} \cong \phi(G) \oplus (K/G)_{ab}$. Then:*

- *(i)* $\big(\phi(M)/\phi(M \cap N)\big) \otimes \big(\phi(N)/\phi(M \cap N)\big)$ *is isomorphic to a quotient of* $H_2(K)$;
- *(ii) if M_{ab} and N_{ab} are finite then* $\big(\phi(M)/\phi(M \cap N)\big) \otimes \big(\phi(N)/\phi(M \cap N)\big)$ *is isomorphic to a subgroup of* $H_2(K)$.

Note that if $K = MN$ then $\big(\phi(M)/\phi(M \cap N)\big) \cong (M/M \cap N)_{ab}$ and $\big(\phi(N)/\phi(M \cap N)\big) \cong (N/M \cap N)_{ab}$.

For the proof of Theorem 1 we recall from [**2, 4**] that any group $E = PQ$, which is a product of two normal subgroups $P, Q \trianglelefteq E$, gives rise to a natural exact sequence

$$(4) \quad \ker\big(P \wedge Q \xrightarrow{\lambda} [P,Q]\big) \to H_2(E) \to H_2(E/P) \oplus H_2(E/Q) \to \frac{P \cap Q \cap [E,E]}{[P,Q]} \to 0.$$

The derivation given in [**4**] is purely algebraic and uses only elementary arguments based on Hopf's formula for the Schur multiplier and on an isomorphism

$$(5) \qquad H_2(E) \cong \ker\big(E \wedge E \xrightarrow{\lambda} E\big).$$

The exterior product $P \wedge Q$ is the group generated by symbols $x \wedge y$ ($x \in P$, $y \in Q$) subject to the relations

$$xx' \wedge y = \big(xx'x^{-1} \wedge xyx^{-1}\big)(x \wedge y),$$

$$x \wedge yy' = (x \wedge y)\big(yxy^{-1} \wedge yy'y^{-1}\big),$$

$$z \wedge z = 1,$$

for $x, x' \in P$, $y, y' \in Q$, $z \in P \cap Q$. The homomorphism λ is defined on generators by $\lambda(x \wedge y) = xyx^{-1}y^{-1}$.

On taking $E = A, P = U$ and $Q = A$, sequence (4) reduces to an exact sequence

$$\ker\big(U \wedge A \xrightarrow{\lambda} [U,A]\big) \xrightarrow{\beta} H_2(A) \xrightarrow{\alpha} H_2(G) \to \frac{U \cap [A,A]}{[U,A]} \to 0.$$

We set $B = \operatorname{coker}(\beta) = \operatorname{im}(\alpha)$ and note that this definition of B leads to the exact sequence (1).

If $P, Q \trianglelefteq E$ are such that $[P,Q] = 1$ then it is readily shown (see [**2**] for details) that

$$(6) \qquad \ker\big(P \wedge Q \xrightarrow{\lambda} [P,Q]\big) \cong P_{ab} \otimes Q_{ab}/\Delta$$

where Δ is the subgroup of $P_{ab} \otimes Q_{ab}$ generated by the tensors $z[P,P] \otimes z[Q,Q]$ for $z \in P \cap Q$. We set

$$P_{ab} \wedge Q_{ab} = P_{ab} \otimes Q_{ab}/\Delta.$$

The naturality of sequence (4) and the isomorphisms $G/M \cong \overline{N}$, $G/N \cong \overline{M}$ lead to the following commutative diagram in which the rows and columns are exact.

$$\begin{array}{ccccccc}
0 & \to & M_{ab} \otimes N_{ab} & \to & H_2(A) & \to & H_2(M) \oplus H_2(N) & \to & 0 \\
 & & \downarrow & & \downarrow \alpha & & \downarrow & & \\
 & & (MU/U)_{ab} \wedge (NU/U)_{ab} & \to & H_2(G) & \to & H_2\left(\overline{M}\right) \oplus H_2\left(\overline{N}\right) & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & \dfrac{\pi_M(U)}{[M,\pi_M U]} \oplus \dfrac{\pi_N(U)}{[N,\pi_N U]} & &
\end{array}$$

The exact sequence (3) follows immediately from this diagram.

In order to derive sequence (2) note that the composition of the inclusion $H_2(M) \oplus H_2(N) \overset{\iota}{\hookrightarrow} H_2(A)$ with the surjection $H_2(A) \overset{\alpha}{\twoheadrightarrow} B$ yields a map with cokernel

$$\begin{aligned}
&\operatorname{coker}\left(H_2(M) \oplus H_2(N) \overset{\alpha\iota}{\twoheadrightarrow} B\right) \\
&\qquad = \operatorname{coker}\left(H_2(M) \oplus H_2(N) \oplus \ker\left(U \wedge A \overset{\lambda}{\twoheadrightarrow} [U,A]\right) \overset{\gamma}{\twoheadrightarrow} H_2(A)\right).
\end{aligned}$$

The natural isomorphism (5) leads to a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H_2(M) \oplus H_2(N) \oplus \ker(\lambda) & \xrightarrow{\gamma} & H_2(A) \\
\downarrow & & \downarrow \\
(M \wedge M) \oplus (N \wedge N) \oplus (U \wedge A) & \xrightarrow{\delta} & A \wedge A \\
\downarrow & & \downarrow \\
[M,M] \oplus [N,N] \oplus [U,A] & \xrightarrow{\nu} & [A,A] \\
\downarrow & & \downarrow \\
1 & & 1
\end{array} \tag{7}$$

in which the columns are exact. Note that $\operatorname{coker}(\nu) = 0$ and $\operatorname{coker}(\delta) = \overline{M}_{ab} \otimes \overline{N}_{ab}$. (To see the latter equality, recall [1] that $A \wedge A \cong (M \wedge M) \oplus (N \wedge N) \oplus (M_{ab} \otimes N_{ab})$, and note that if $(x,y) \in U \trianglelefteq M \times N$ and $(a,b) \in M \times N$ then working in $A \wedge A$ we have

$$xy \wedge ab = (y \wedge {}^{x}a)(y \wedge b)(x \wedge a)({}^{a}x \wedge b).$$

Thus

$$\operatorname{coker}(\delta) \cong M_{ab} \otimes N_{ab}/\Gamma$$

where Γ is the subgroup of $M_{ab} \otimes N_{ab}$ generated by the elements $u[M,M] \otimes b[N,N]$ and $a[M,M] \otimes v[N,N]$ for $a \in M$, $b \in N$, $u \in \pi_M U$, $v \in \pi_N U$. It follows that $\text{coker}(\delta) = \overline{M}_{ab} \otimes \overline{N}_{ab}$.) Diagram (7) yields an exact sequence

$$\to \ker(\nu) \to \text{coker}(\gamma) \to \text{coker}(\delta) \to \text{coker}(\nu)$$

which we recognise as

$$\to [U, A] \to \text{coker}(\alpha\iota) \to \overline{M}_{ab} \otimes \overline{N}_{ab} \to 0.$$

The exact sequence (2) follows from this sequence and the fact that $\text{coker}(\alpha\iota)$ is Abelian.

Let us now turn to the proof of Proposition 2. The quotient homomorphism

$$K \twoheadrightarrow (K/M \cap N)_{ab} \cong \phi(M)/\phi(M \cap N) \oplus \phi(N)/\phi(M \cap N) \oplus (K/G)_{ab}$$

induces a homology homomorphism

$$H_2(K) \twoheadrightarrow H_2\Big(\phi(M)/\phi(M \cap N) \oplus \phi(N)/\phi(M \cap N) \oplus (K/G)_{ab} \Big) \cong$$

$$H_2\big(\phi(M)/\phi(M \cap N)\big) \quad \oplus \quad H_2\big(\phi(N)/\phi(M \cap N) \oplus (K/G)_{ab}\big) \quad \oplus$$

$$\big(\phi(M)/\phi(M \cap N) \otimes \phi(N)/\phi(M \cap N)\big) \quad \oplus \quad \big(\phi(M)/\phi(M \cap N) \otimes (K/G)_{ab}\big).$$

By projecting onto the penultimate summand we obtain a homomorphism

$$\rho \colon H_2(K) \to \phi(M)/\phi(M \cap N) \otimes \phi(N)/\phi(M \cap N).$$

The homomorphism ρ is surjective because the condition $[M,N] = 1$ implies there is a surjective composite homomorphism

$$M_{ab} \wedge N_{ab} \cong M \wedge N \xrightarrow{\mu} H_2(K) \xrightarrow{\rho} \phi(M)/\phi(M \cap N) \otimes \phi(N)/\phi(M \cap N).$$

(The isomorphism follows from (6), and the homomorphism μ is derived from (4).) This proves part (i) of Proposition 2. If M_{ab} and N_{ab} are finite then so too is $M_{ab} \wedge N_{ab}$; hence $\text{im}(\mu)$ is finite and thus contains a subgroup isomorphic to its quotient $\phi(M)/\phi(M \cap N) \otimes \phi(N)/\phi(M \cap N)$. This proves part (ii) of Proposition 2.

References

[1] R. Brown, D.L. Johnson and E.F. Robertson, 'Some computations of nonabelian tensor products of groups', *J. Algebra* **111** (1987), 177–202.

[2] R. Brown and J.-L. Loday, 'Van Kampen theorems for diagrams of spaces', *Topology* **26** (1987), 311–335.

[3] B. Eckmann, P.J. Hilton and U. Stammbach, 'On the Schur multiplicator of a central quotient of a direct product of groups', *J. Pure Appl. Algebra* **3** (1973), 73–82.

https://doi.org/10.1017/S0004972700032470 Published online by Cambridge University Press

[4] G. Ellis, 'Nonabelian exterior products of groups and an exact sequence in the homology of groups', *Glasgow Math. J.* **29** (1987), 13–19.

[5] G. Ellis, 'Capability, homology, and central series of a pair of groups', *J. Algebra* **179** (1995), 31–46.

[6] G. Ellis, 'The Schur multiplier of a pair of groups', *Appl. Categ. Structures* (to appear).

[7] G. Ellis, 'On the capability of groups', *Proc. Edinburgh Math. Soc.* (to appear).

[8] G. Ellis, 'A bound for the derived and Frattini subgroups of a prime-power group', *Proc. Amer. Math. Soc.* **126** (1998), 2513–2523.

[9] J. Wiegold, 'Some groups with nontrivial multiplicators', *Math. Z.* **120** (1971), 307–308.

Department of Mathematics
National University of Ireland
Galway
Ireland
e-mail: graham.ellis@ucg.ie

Current address:
Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn
Germany