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COMPACT DERIVATIONS ON VON NEUMANN ALGEBRAS

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ABSTRACT. If $\mathcal{R}$ is a von Neumann algebra that thas no nonzero finite discrete central projection, then there is no nontrivial compact derivation of $\mathcal{R}$ into itself.

1. Introduction. In [3] Ho proved a theorem that if an operator A on a Hilbert space $\mathcal{H}$ induces a compact derivation on $\mathcal{B}(\mathcal{H})$, the C^*-algebra of all bounded operators on $\mathcal{H}$, by

$$ad_A(T) = AT - TA, \ (T \in \mathcal{B}(\mathcal{H})),$$

then A must be of the form αI for some scalar α. Since all derivations of a von Neumann algebra are inner [4], [6], Ho's result can simply be rephrased as there is no non-trivial compact derivation of an infinite discrete factor (for terminology, see [7], 2.2.9, p. 86). It is natural to follow this line of investigation to ask whether there are any non-trivial compact derivations on a von Neumann algebra other than those obvious ones on its finite discrete direct summand. The answer provided in this note is "negative", and the method used to obtain the answer in the general case relies on some intrinsic properties of von Neumann algebra theory. Basically, the fact, from which the theorem is derived, is the abundence of partial isometries in a von Neumann algebra (see the proof of 2.1. Theorem).

2. The main theorem. We denote a von Neumann algebra by $\mathcal{R}$, and projections in $\mathcal{R}$ by E, F, P etc. Two projections E, F in $\mathcal{R}$ are said to be *equivalent*, if there exists a partial isometry U in $\mathcal{R}$ such that $U^*U = E$ and $UU^* = F$. We denote the equivalent relationship by $E \sim F$, and call E, the initial projection, and F, the final projection of U. Suppose that a projection E is equivalent to a subjection of F. We denote this relationship by $E \preccurlyeq F$, and $\preccurlyeq$ defines a partial ordering among projections in $\mathcal{R}$. We denote the central support ([7], 1.10.7, p. 25) of a projection E by $C(E)$. Let d be a derivation of

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a von Neumann algebra $\mathscr{R}$ into itself. Suppose P is a central projection in $\mathscr{R}$. We have $d(P)=d(P^2)=d(P)P+Pd(P)=2d(P)P$. Hence $d(P)P=2d(P)P$. So, $d(P)P=0$ and $d(P)=0$. For A in $\mathscr{R}$, consider $d(AP)=d(AP\cdot P)=d(AP)P+APd(P)=d(AP)P$. It follows that the restriction of d on $\mathscr{R}_P$ $(=\{AP \mid A\in\mathscr{R}\})$ is a derivation of $\mathscr{R}_P$ into itself.

Any von Neumann algebra $\mathscr{R}$ can be uniquely decomposed into a direct sum of a finite discrete von Neumann algebra $\mathscr{R}_1$ and another von Neumann algebra $\mathscr{R}_2$. In this notation we have the following:

2.1. THEOREM. *Let d be a compact derivation of $\mathscr{R}$ into itself Then, the restriction of d on $\mathscr{R}_2$ is trivial.*

Proof. Since all derivations of a von Neumann algebra into itself are inner, there exists an element A in $\mathscr{R}_2$ such that $d\,|_{\mathscr{R}_2}=ad_A$. From the decomposition $d=\frac{1}{2}(d-d^*)+i/2i(d+d^*)$, where $d^*(A)=d(A^*)^*$ for all A in $\mathscr{R}$, it suffices to prove the theorem for those compact derivations with $d=-d^*$. Then we may assume that $d\,|_{\mathscr{R}_2}=\mathrm{ad}_A$ for a self-adjoint element A in $\mathscr{R}_2$. Suppose that A is not central in $\mathscr{R}_2$ and ad_A is compact on $\mathscr{R}_2$. We denote the spectrum of A by $\sigma(A)$, and the spectral measure for A by $E(\lambda)$. $\mathscr{R}_2$ can be uniquely decomposed into a direct sum of a continuous von Neumann algebra $\mathscr{M}_1$ and a properly infinite discrete von Neumann algebra $\mathscr{M}_2$. Hence $A=A_1+A_2$, where $A_i\in\mathscr{M}_i$, $i=1,2$.

Case 1. Suppose that A_1 is not central.

Since A_1 is not central there must be a real number β in $\sigma(A_1)$ such that the central support for $E((-\infty,\beta])$ is bigger than $E((-\infty,\beta])$. Now

$$I_{\mathscr{M}_1}-E((-\infty,\beta])=\bigcup_{\alpha>\beta}E([\alpha,\infty))\equiv F.$$

Since $C(E((-\infty,\beta]))\cdot F\neq 0$, it follows that

$$C(E((-\infty,\beta]))\cdot C(F)\neq 0,$$

and there exists an $\alpha>\beta$ such that

$$C(E((-\infty,\beta])\cdot C(E([\alpha,\infty)))\neq 0.$$

Denote

$$C(E((-\infty,\beta])\cdot C(E([\alpha,\infty))) \text{ by } Q.$$

By the theorem of comparability ([7]), 2.1.3, p. 80) applied to $QE((-\infty,\beta])$ and $QE([\alpha,\infty))$, there exists partial isometry V in M, such that $V^*V\leq E((-\infty,\beta])$ and $VV^*\leq E([\alpha,\infty))$. Choose a sequence of nontrivial pairwise orthogonal projections $\{E_i\}$ with sum V^*V and define partial isometries $U_i=VE_i$. Represent $\mathscr{R}$ σ-continuously, faithfully on a Hilbert space $\mathscr{H}$. For any pair of i, j with $i \neq j$ and any unit vector x in the range space of E_i, we get

$$\begin{aligned}
\|(\mathrm{ad}_{A_1}(U_i) - \mathrm{ad}_{A_1}(U_j))x\| &= \|A_1 U_i x - (U_i - U_j)A_1 x\| \\
&\geqq \|A_1 U_i x\| - \|(U_i - U_j)A_1 x\| \\
&\geq \left\{\int_{\sigma(A_1)} \lambda^2 \, d\langle E(\lambda)U_i x, U_i x\rangle\right\}^{1/2} - \left\{\int_{\sigma(A_1)} \lambda^2 \, d\langle E(\lambda)x, x\rangle\right\}^{1/2} \\
&= \left\{\int_{\lambda \geq \alpha} \lambda^2 \, d\langle E(\lambda)U_i x, U_i x\rangle\right\}^{1/2} - \left\{\int_{\lambda \leq \beta} \lambda^2 \, d\langle E(\lambda)x, x\rangle\right\}^{1/2} \\
&\geq \alpha - \beta
\end{aligned}$$

Hence, $\|\mathrm{ad}_{A_1}(U_i) - \mathrm{ad}_{A_1}(U_j)\| \geq \alpha - \beta$, when $i \neq j$. This contradicts the fact that ad_A is compact on $\mathcal{M}_1$. Therefore A_1 is central.

Case 2. Suppose that A_2 is not central.

We are going to consider two different situations.

2.1: The center of $\mathcal{M}_2$ is purely atomic. In this case $\mathcal{M}_2$ is a direct sum of a family of von Neumann algebras $\{\mathcal{N}_i\}_{i \in I}$, and each $\mathcal{N}_i$ is isomorphic to $\mathcal{B}(\mathcal{H}_i)$ for some infinite dimensional Hilbert space $\mathcal{H}_i$. Then, it follows from Ho's paper [3] that A_2 must be central. Hence it is a contradiction.

2.2: The center of M_2 is purely diffuse. Then we may regard M_2 as $\sum_{j \in J} \oplus [\mathcal{B}(\mathcal{H}_j) \otimes \mathcal{C}(X_j)]$, where $\mathcal{H}_j$'s are infinite dimensional Hilbert spaces, and none of X_j's contains any atomic points. We may as well consider $\mathcal{M}_2 = \mathcal{B}(\mathcal{H}_j) \otimes \mathcal{C}(X_j = \mathcal{C}(X_j, \mathcal{B}(\mathcal{H}_j))$ for one j in J. Since A is not central there exists an operator T in R such that $\|[A, T]\| \geq 1$. $[A, T]$ can be regarded as a continuous function on X_j with values in $\mathcal{B}(\mathcal{H}_j)$. Since $\mathrm{Sup}_{w \in X_j} \|[A, T](w)\| \geq 1$, and X_j is Stonean, there exists a closed-open subset S in X_j such that $\|[A, T](w)\| \geq \frac{1}{2}$ for all w in S. Since X_j is purely diffuse, there exists a sequence of pairwise disjoint non-empty closed-open subset S_i of S. Let P_i be the central projection corresponding to the characteristic function on S_i. We have

$$\|[A, T]P_i\| = \|[A, TP_i]\| \geq \tfrac{1}{2}$$

for all i. But $\{TP_i\}$ converges ultraweakly to zero, and hence by a compactness assumption $\|[A, TP_i]\| \to 0$. A contradiction. Q.E.D.

Not all finite type I von Neumann algebras admit a nontrivial compact derivation (into itself).

2.2. Theorem. *A finite type I von Neumann algebra $\mathcal{R}$ admits a nontrivial compact derivation d (of $\mathcal{R}$ into itself) if and only if $\mathcal{R}$ has a direct summand isomorphic to $\sum_{i=1}^{k} \oplus M_{n_i}$, where M_{n_i} is the $n_i \times n_i$ matrix algebra, and k may be countably infinite. Furthermore, for a given compact derivation d we have*

$$\left\| d - d \Big|_{\sum_{i=1}^{s} \oplus M_{n_i}} \right\| \to 0$$

as $s \to k$.

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Proof. The sufficient condition is obvious. We show the condition is also necessary. Suppose that d is a nontrivial compact derivation of $\mathcal{R}$ into $\mathcal{R}$. Consider $Q = V\{Q_\alpha \mid Q_\alpha$: central projection in $\mathcal{R}$ and $\|d\|_{\mathcal{R}Q_\alpha}\| \leq \frac{1}{2}\|d\|\}$, and let $Q^c = I - Q$. We claim that there must exist an "atomic" central projection P in Q^c, i.e., $\mathcal{R}P$ has no nonzero central subprojections. In fact, otherwise, there exists a sequence of nonzero central projections $\{P_i \mid i = 1, 2, \ldots\}$ such that $Q^c = \sum_{i=1}^{\infty} P_i$. We note that $\|d\,|_{\mathcal{R}P_i}\| > \frac{1}{2}\|d\|$ for all $i = 1, 2, \ldots$. We may find a sequence of unit elements A_i in $\mathcal{R}P_i$ such that $\|d(A_i)\| \geq \frac{1}{2}\|d\|$ for all $i = 1, 2, \ldots$. It follows that $\|d(A_i) - d(A_j)\| \geq \frac{1}{2}\|d\|$ for all $i \neq j$, which contradicts to the compactness assumption of d. Let P be such an atomic central projection. Hence $\mathcal{R}P$ must be isomorphic to M_n for some $n > 1$. Let $P = \bigvee_{\alpha \in J}\{P_\alpha \mid P_\alpha \cdot P_\beta = 0,\ \alpha \neq \beta,\ P_\alpha$ atomic central projections and $d\,|_{\mathcal{R}P_\alpha} \neq 0\}$. The restriction of d to $\mathcal{R}(I-P)$ is trivial. Since $d\,|_{\mathcal{R}P}$ is compact,

$$J_n = \left\{\alpha \in J \mid \|d_{\mathcal{R}P_\alpha}\| \geq \frac{1}{n}\right\}$$

must be finite. It follows that J must be countable, and

$$\lim_{i \to \infty} \|d\,|_{\mathcal{R}P_i}\| = 0,$$

when J is infinite. Q.E.D.

2.3. Question and Remark. The next investigation is on the compact derivations of a von Neumann aubalgebras $\mathcal{R}_1$ of $\mathcal{R}$ into $\mathcal{R}$. The innerness of derivations of some kinds of subalgebras $\mathcal{R}_1$ of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$ has been extensively studied in [1]. However, it is not possible to use the innerness of the derivations, whenever it is true, to prove a theorem like Theorem 2.1. The time when this paper is being prepared the author learns that a C^*-version of Theorem 2.2. is being worked out by Charles Akemann and Steve Wright.

References

1. Christensen, Eric, "Extensions of derivations," *J. Functional Analysis*, Vol. **27** (1978) p. 234–247.

2. Dixmier, Jacques, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1969.

3. Ho, Yang, "A note on derivations," *Bulletin of the Institute of Mathematics*, Academic Sinica, Vol. **5**, No. 1, June 1977, p. 1–5.

4. Kadison, Richard *Derivations of operator algebras*, Ann. of Math., Vol. **83** (1969), p. 280–293.

5. Rudin, Walter, *Functional Analysis*, McGraw-Hill, 1973.

6. Sakai, Shôichirô, *Derivations of W*-algebras*, Ann. of Math., Vol. **83** (1966), p. 273–279.

7. Sakai, Shôichirô, *C*-algebras and W*-algebras*, Springer-Verlag, 1971.

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