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ON THE LONGEST BLOCK FUNCTION IN CONTINUED FRACTIONS

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Abstract

For an irrational number $x \in [0, 1)$, let $x = [a_1(x), a_2(x), \ldots]$ be its continued fraction expansion with partial quotients $\{a_n(x) : n \geq 1\}$. Given $\Theta \in \mathbb{N}$, for $n \geq 1$, the nth longest block function of x with respect to Θ is defined by $L_n(x, \Theta) = \max\{k \geq 1 : a_{j+1}(x) = \cdots = a_{j+k}(x) = \Theta$ for some j with $0 \leq j \leq n - k\}$, which represents the length of the longest consecutive sequence whose elements are all Θ from the first n partial quotients of x. We consider the growth rate of $L_n(x, \Theta)$ as $n \to \infty$ and calculate the Hausdorff dimensions of the level sets and exceptional sets arising from the longest block function.

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1. Introduction

For $x \in [0, 1)$ with dyadic expansion $x = \sum_{k=1}^{\infty} x_n/2^n$ ($x_n = 0$ or 1), we define the run-length function

$$Z_n(x) = \max\{l \geq 1 : x_{i+1} = \cdots = x_{i+l} = 0 \text{ for some } i \text{ with } 0 \leq i \leq n - l\},$$

which counts the longest run of 0's in the first n digits of the dyadic expansion of x. A classical result due to Erdős and Rényi [4] asserts that for almost all $x \in [0, 1)$,

$$\lim_{n\to\infty} \frac{Z_n(x)}{\log_2 n} = 1. \tag{1.1}$$

It is natural to study the exceptional set in the Erdős–Rényi limit theorem. Ma *et al.* [13] (see also [17]) proved that the set of all points $x \in [0, 1)$ for which (1.1) does not hold has Hausdorff dimension 1. Liu *et al.* [12] extended this result further by considering the sets

$$E^{\varphi}_{\alpha,\beta} = \left\{x \in [0, 1) : \liminf_{n\to\infty} \frac{Z_n(x)}{\varphi(n)} = \alpha, \limsup_{n\to\infty} \frac{Z_n(x)}{\varphi(n)} = \beta\right\}$$

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with $\varphi : \mathbb{N} \to (0, +\infty)$ an increasing function. They showed that if $\lim_{n\to\infty} \varphi(n) = \infty$ and $\lim_{n\to\infty} \varphi(n)/\varphi(n + \varphi(n)) = 1$, then the set $E^{\varphi}_{\alpha,\beta}$ is of full Hausdorff dimension for all α, β with $0 \le \alpha \le \beta \le \infty$. Tong *et al.* [19] generalised these results to the β-expansion $x = \sum_{k=1}^{\infty} x_n/\beta^n$ for $\beta \in (1, 2]$. (For more information on the β-expansion, see [2, 6, 22].)

The asymptotic behaviour of similar run-length functions arising in the continued fraction expansion was studied in [20]. With the help of the Gauss transformation $T : [0, 1) \to [0, 1)$ defined by

$$T(0) = 0, \quad T(x) = \frac{1}{x} \pmod 1 \quad \text{for } x \in (0, 1),$$

each irrational number $x \in [0, 1)$ can be uniquely expanded as a continued fraction

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots \, \cfrac{1}{a_n(x) + T^n x}}}} = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cfrac{1}{\ddots}}}} \tag{1.2}$$

with the $a_n(x) = \lfloor 1/T^{n-1}(x) \rfloor$, called the partial quotients of x. For simplicity of notation, we write (1.2) as

$$x = [a_1(x), a_2(x), \ldots, a_n(x) + T^n x] = [a_1(x), a_2(x), a_3(x), \ldots].$$

For any $n \ge 1$, we define the nth maximal run-length function of x by

$$R_n(x) = \max\{k \ge 1 : a_{j+1}(x) = \cdots = a_{j+k}(x) \text{ for some } j \text{ with } 0 \le j \le n - k\}.$$

Wang and Wu [20] considered the metrical properties of $R_n(x)$ and proved that

$$\lim_{n\to\infty} \frac{R_n(x)}{\log_{\frac{1}{2}(\sqrt{5}+1)} n} = \frac{1}{2}$$

for almost all $x \in [0, 1)$.

We give a more subtle characterisation of the function $R_n(x)$. More precisely, given $\Theta \in \mathbb{N}$, for $n \ge 1$, we define the longest block function of x as

$$L_n(x, \Theta) = \max\{k \ge 1 : a_{j+1}(x) = \cdots = a_{j+k}(x) = \Theta \text{ for some } j \text{ with } 0 \le j \le n - k\}.$$

It represents the length of the longest consecutive sequence whose elements are all Θ from the first n partial quotients of x. We obtain the following law of large numbers for $L_n(x, \Theta)$.

Theorem 1.1. *For almost all $x \in [0, 1)$,*

$$\lim_{n\to\infty} \frac{L_n(x, \Theta)}{\log_{\frac{1}{2}(\Theta + \sqrt{\Theta^2+4})} n} = \frac{1}{2}.$$

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Following Theorem 1.1, it is natural to study the metrical theory of the set

$$D_n(x,\Theta) = \max\{k \geq 1 : a_{j+1}(x)a_{j+2}(x) = \cdots = a_{j+k-1}(x)a_{j+k}(x) = \Theta \text{ for some } j \text{ with } 0 \leq j \leq n-k\}.$$

This is in turn related to the Dirichlet improvable sets discussed in [1, 8, 9]. This will be the subject of a forthcoming article.

The fractal structures of the level sets and exceptional sets with respect to the metrical result in Theorem 1.1 are also of interest. For $0 \leq \alpha \leq \infty$, we define the level set

$$E(\alpha) = \Big\{x \in [0,1) : \lim_{n\to\infty} \frac{L_n(x,\Theta)}{\log_{\frac{1}{2}(\Theta+\sqrt{\Theta^2+4})} n} = \alpha\Big\}.$$

For $\alpha,\beta \in [0,+\infty]$ with $\alpha \leq \beta$, we define exceptional sets $E(\alpha,\beta)$ and $\widehat{E}$ by

$$E(\alpha,\beta) = \Big\{x \in [0,1) : \liminf_{n\to\infty} \frac{L_n(x,\Theta)}{\log_{\frac{1}{2}(\Theta+\sqrt{\Theta^2+4})} n} = \alpha,\ \limsup_{n\to\infty} \frac{L_n(x,\Theta)}{\log_{\frac{1}{2}(\Theta+\sqrt{\Theta^2+4})} n} = \beta\Big\}$$

and

$$\widehat{E} = \Big\{x \in [0,1) : \liminf_{n\to\infty} \frac{L_n(x,\Theta)}{\log_{\frac{1}{2}(\Theta+\sqrt{\Theta^2+4})} n} < \limsup_{n\to\infty} \frac{L_n(x,\Theta)}{\log_{\frac{1}{2}(\Theta+\sqrt{\Theta^2+4})} n}\Big\}.$$

From a global measure theoretic point of view, they are zero sets. It is of interest to know whether the 'sizes' of the sets are also small from the perspective of dimension theory. We obtain the somewhat surprising result that all the exceptional sets have full dimension.

THEOREM 1.2. *For any α,β with $0 \leq \alpha \leq \beta \leq +\infty$, the Hausdorff dimension* $\dim_H E(\alpha,\beta)$ *of the exceptional set $E(\alpha,\beta)$ is equal to* 1.

By taking first $\alpha = \beta$ and second $\alpha = 0$, $\beta = \infty$ in Theorem 1.2 and noting that $E(\alpha,\beta) \subset \widehat{E}$, we obtain the following two corollaries.

COROLLARY 1.3. *For any α with $0 \leq \alpha \leq \infty$, we have* $\dim_H E(\alpha) = 1$.

COROLLARY 1.4. *We have* $\dim_H \widehat{E} = 1$.

The paper is organised as follows. Section 2 collects some basic results on continued fractions that will be used later. The proofs of Theorems 1.1 and 1.2 are given in Sections 3 and 4, respectively.

2. Preliminaries

In this section, we fix some notation and cite some elementary properties of continued fractions. For a wealth of classical results about the continued fraction expansion, see the book by Khintchine [11] and for more information see [14, 18, 21].

For any irrational number $x \in [0,1)$ with continued fraction expansion (1.2), we define the nth convergent of x by $p_n(x)/q_n(x) = [a_1(x),\ldots,a_n(x)]$ with the conventions

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$p_{-1}(x) = 1$, $q_{-1}(x) = 0$, $p_0(x) = 0$ and $q_0(x) = 1$. Then $p_n(x)$ and $q_n(x)$ can be given by the recursive relations

$$p_{n+1}(x) = a_{n+1}(x)p_n(x) + p_{n-1}(x), \quad q_{n+1}(x) = a_{n+1}(x)q_n(x) + q_{n-1}(x), \quad n \geq 0.$$

Clearly, $q_n(x)$ is determined by $a_1(x), \ldots, a_n(x)$, so we also write $q_n(a_1(x), \ldots, a_n(x))$ instead of $q_n(x)$. We write a_n and q_n in place of $a_n(x)$ and $q_n(x)$ for simplicity when no confusion can arise.

Lemma 2.1 [11]. *Let $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$.*

(1) $q_n \geq 2^{(n-1)/2}$.

(2) *For $1 \leq k \leq n$,*

$$1 \leq \frac{q_n(a_1, \ldots, a_n)}{q_k(a_1, \ldots, a_k)q_{n-k}(a_{k+1}, \ldots, a_n)} \leq 2, \quad \prod_{k=1}^{n} a_k \leq q_n \leq \prod_{k=1}^{n}(a_k + 1).$$

(3) *If $a_1 = a_2 = \cdots = a_n = i$, then*

$$\tau^n(i) \leq q_n(i, \ldots, i) = \frac{\tau^{n+1}(i) - \zeta^{n+1}(i)}{\tau(i) - \zeta(i)} \leq 2\tau^n(i),$$

where $\tau(i) = \frac{1}{2}(i + \sqrt{i^2 + 4})$ and $\zeta(i) = \frac{1}{2}(i - \sqrt{i^2 + 4})$.

For $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, we write

$$I_n(a_1, \ldots, a_n) = \{x \in [0, 1) : a_k(x) = a_k, 1 \leq k \leq n\}$$

and call it a basic interval of order n; this interval is the collection of points whose continued fraction expansions begin with $(a_1, \ldots, a_n)$.

Lemma 2.2 [11]. *For any $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$,*

$$\frac{1}{2q_n^2} \leq |I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n+1})} \leq \frac{1}{q_n^2}.$$

The Gauss transformation T is ergodic with respect to the Gauss measure μ, defined by

$$d\mu = \frac{1}{\log 2}\frac{1}{x + 1}\, dx.$$

From the definition of μ, we see that μ is absolutely continuous with respect to Lebesgue measure. Philipp [15] showed that T is not only ergodic but also strongly mixing with respect to μ and this result is critical in the metrical theory of the longest block function $L_n(x, \Theta)$.

Lemma 2.3 [15]. *For any $k \geq 1$, let $\mathbb{B}_1^k = \sigma(a_1, \ldots, a_k)$ and $\mathbb{B}_k^\infty = \sigma(a_k, a_{k+1}, \ldots)$ denote the σ-algebras generated by the random variables $(a_1, \ldots, a_k)$ and $(a_k, a_{k+1}, \ldots)$, respectively. Then, for any $A \in \mathbb{B}_1^k$ and $B \in \mathbb{B}_{k+n}^\infty$,*

$$\mu(A \cap B) = \mu(A)\mu(B)(1 + \theta\rho^n),$$

where $|\theta| \leq K$, $\rho < 1$ and K, ρ are positive constants independent of A, B, n, k.

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We cite some dimensional results on continued fractions (see [5] for more information on estimation of the dimension). Let E_M be the set consisting of all points in [0,1) whose partial quotients are not greater than M, that is,

$$E_M = \{x \in [0,1) : 1 \le a_n(x) \le M \text{ for } n \ge 1\}.$$

LEMMA 2.4 [10]. *For $M \ge 8$,*

$$1 - \frac{1}{M \log 2} \le \dim_H E_M \le 1 - \frac{1}{8M \log M}.$$

In particular, the set $E = \{x \in [0,1) : \sup_{n\ge1} a_n(x) < +\infty\}$ has Hausdorff dimension 1.

Good [7] obtained the more accurate estimate $\dim_H E_M = \lim_{n\to\infty} \sigma_{M,n}$, where $\sigma_{M,n}$ is the real root of

$$\sum_{1\le a_1,\ldots,a_n\le M} \left(\frac{1}{q_n(a_1,\ldots,a_n)}\right)^{2\sigma_{M,n}} = 1.$$

Let $\mathbf{K} = \{k_n\}_{n\ge1}$ be a subsequence of $\mathbb{N}$ which is not cofinite. Let $x = [a_1, a_2, \ldots]$ be an irrational number in $[0,1)$. Eliminating all the terms a_{k_n} from the sequence $a_1, a_2, \ldots$, we obtain an infinite subsequence $c_1, c_2, \ldots$ and we put $\phi_{\mathbf{K}}(x) = [c_1, c_2, \ldots]$. In this way, we define a mapping $\phi_{\mathbf{K}} : [0,1) \cap \mathbb{Q}^c \to [0,1) \cap \mathbb{Q}^c$.

Let $\{M_n\}_{n\ge1}$ be a sequence with $M_n \in \mathbb{N}$, $n \ge 1$. Set

$$S(\{M_n\}) = \{x \in [0,1) \cap \mathbb{Q}^c : 1 \le a_n(x) \le M_n \text{ for all } n \ge 1\}.$$

LEMMA 2.5 [3]. *Assume that $\{M_n\}_{n\ge1}$ is bounded. If the sequence $\mathbf{K} = \{k_n\}_{n\ge1}$ is of density zero in $\mathbb{N}$, then*

$$\dim_H S(\{M_n\}) = \dim_H \phi_{\mathbf{K}} S(\{M_n\}).$$

COROLLARY 2.6. *Given a set of positive integers $\mathbf{K} = \{j_1 < j_2 < \cdots\}$ and an infinite bounded sequence $\{b_i\}_{i\ge1}$ with $2 \le b_i \le B$ for some $B \in \mathbb{N}$, let*

$$E(\mathbf{K}, \{b_i\}) = \{x \in [0,1) : a_i(x) = b_i \text{ for all } i \in \mathbf{K}\}.$$

If the density of $\mathbf{K}$ is zero, that is,

$$\lim_{n\to\infty} \frac{\#\{i \le n : i \in \mathbf{K}\}}{n} = 0,$$

then $\dim_H E(\mathbf{K}, \{b_i\}) = 1$. Here and hereafter # denotes the cardinality of a finite set.

PROOF. The main idea of the proof is to construct Cantor-like subsets with Hausdorff dimensions approaching 1. Fix $M \ge \max\{8, B\}$. Let $E_M(\mathbf{K}, \{b_i\})$ be the set of $x \in [0,1)$ whose partial quotients satisfy

$$a_i(x) \begin{cases} = b_i, & i \in \mathbf{K}, \\ \in [1, M], & i \notin \mathbf{K}. \end{cases}$$

It is easy to check that $E_M(\mathbf{K}, \{b_i\}) \subset E(\mathbf{K}, \{b_i\})$ and $\phi_{\mathbf{K}} E_M(\mathbf{K}, \{b_i\}) = E_M$. Thus,

$$\dim_H E(\mathbf{K}, \{b_i\}) \ge \dim_H E_M(\mathbf{K}, \{b_i\}) = \dim_H \phi_{\mathbf{K}} E_M(\mathbf{K}, \{b_i\}) \ge 1 - \frac{1}{M \log 2}$$

by Lemmas 2.4 and 2.5. We complete the proof by letting $M \to \infty$. □

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3. Metrical properties of $L_n(x,\Theta)$

In this section, we prove Theorem 1.1. The main idea of the proof is borrowed from Theorem 7.1 in [16].

Lemma 3.1. *For almost all $x \in [0,1)$ and any $\epsilon > 0$,*

$$\liminf_{n\to\infty} \frac{L_n(x,\Theta)}{\log_{\tau(\Theta)} n} \geq \frac{1-\epsilon}{2}.$$

Proof. It suffices to show that

$$\mu\Big\{x \in [0,1) : L_n(x,\Theta) < \Big\lfloor \frac{1-\epsilon}{2} \log_{\tau(\Theta)} n \Big\rfloor \text{ for infinitely many } n \in \mathbb{N}\Big\} = 0.$$

Let $\mu_n = \lfloor \frac{1}{2}(1-\epsilon)\log_{\tau(\Theta)} n\rfloor$ and $k_n = \lfloor n/\mu_n^2 \rfloor$. For $n > m \geq 0$, set

$$L_{[m,n]}(x,\Theta) = L_{n-m}(a_{m+1}(x),\ldots,a_n(x)),$$

which represents the longest run of the same symbol in the first $n-m$ partial quotients of $T^m(x)$. Note the covering of the set

$$\begin{aligned}
&\{x \in [0,1) : L_n(x,\Theta) < \mu_n \text{ for infinitely many } n \in \mathbb{N}\}\\
&\quad = \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{x \in [0,1) : L_n(x,\Theta) < \mu_n\}\\
&\quad \subseteq \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{x \in [0,1) : L_{[i\mu_n^2, i\mu_n^2+\mu_n]}(x,\Theta) < \mu_n, 0 \leq i < k_n\}.
\end{aligned}$$

Based on this covering,

$$\begin{aligned}
&\mu\{x \in [0,1) : L_n(x,\Theta) < \mu_n \text{ for infinitely many } n \in \mathbb{N}\}\\
&\quad \leq \liminf_{m\to\infty}\sum_{n=m}^{\infty}(\mu\{x \in [0,1) : L_{[1,\mu_n]}(x,\Theta) < \mu_n\})^{k_n}(1+\theta\rho^{\mu_n^2-\mu_n})^{k_n}\\
&\quad \leq \liminf_{m\to\infty}\sum_{n=m}^{\infty}(1-\mu(I_{\mu_n}(\Theta,\ldots,\Theta)))^{k_n}(1+\theta\rho^{\mu_n^2-\mu_n})^{k_n}\\
&\quad \leq \liminf_{m\to\infty}\sum_{n=m}^{\infty} e^{-k_n\mu(I_{\mu_n}(\Theta,\ldots,\Theta))}e^{k_n\theta\rho^{\mu_n^2-\mu_n}}\\
&\quad \leq M\liminf_{m\to\infty}\sum_{n=m}^{\infty} e^{-(n/16\mu_n^2)(1/\tau(\Theta))^{2\mu_n}}\\
&\quad \leq M\liminf_{m\to\infty}\sum_{n=m}^{\infty} e^{-n^{\epsilon}/16\mu_n^2} \leq M\liminf_{m\to\infty}\sum_{n=m}^{\infty}\frac{1}{n^{1+\epsilon}} = 0.
\end{aligned}$$

Here the first inequality is obtained by Lemma 2.3 and the fourth inequality follows from Lemmas 2.1 and 2.2 as well as the fact that $\lim_{n\to\infty} e^{k_n\theta\rho^{\mu_n^2-\mu_n}} = 1$, so that there exists $M \in (0,\infty)$ with $e^{k_n\theta\rho^{\mu_n^2-\mu_n}} \leq M$ for n large enough. The estimate in the lemma therefore follows from the remark made at the beginning of the proof. □

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LEMMA 3.2. *For almost all $x \in [0,1)$, for any $\epsilon > 0$,*

$$\limsup_{n\to\infty} \frac{L_n(x,\Theta)}{\log_{\tau(\Theta)} n} \le \frac{1+\epsilon}{2}.$$

PROOF. It suffices to show that

$$\mu\Big\{x \in [0,1) : L_n(x,\Theta) \ge \Big\lfloor \frac{1+\epsilon}{2} \log_{\tau(\Theta)} n \Big\rfloor + 1 \text{ for infinitely many } n\Big\} = 0.$$

Let $\mu_n = \lfloor \frac{1}{2}(1+\epsilon)\log_{\tau(\Theta)} n\rfloor$. Note the covering of the set

$$\begin{aligned}&\{x \in [0,1) : L_n(x,\Theta) \ge \mu_n \text{ for infinitely many } n\}\\ &\quad= \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{x \in [0,1) : L_n(x,\Theta) \ge \mu_n\}\\ &\quad\subseteq \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\bigcup_{k=\mu_n}^{\infty}\bigcup_{j=0}^{n-k}\{x \in [0,1) : a_{j+1}(x) = \cdots = a_{j+k}(x) = \Theta\}.\end{aligned}$$

Based on this covering,

$$\begin{aligned}&\mu\{x \in [0,1) : L_n(x,\Theta) \ge \mu_n \text{ for infinitely many } n\}\\ &\quad\le \frac{1}{\log 2}\liminf_{m\to\infty}\sum_{n=m}^{\infty}\sum_{k=\mu_n}^{\infty} n|I_k(\Theta,\ldots,\Theta)|\\ &\quad\le \frac{1}{\log 2}\liminf_{m\to\infty}\sum_{n=m}^{\infty}\sum_{k=\mu_n}^{\infty} n\Big(\frac{1}{\tau(\Theta)}\Big)^{2k}\\ &\quad\le \frac{1}{\log 2}\liminf_{m\to\infty}\sum_{n=m}^{\infty} n\cdot n^{-(1+\epsilon)} = \frac{1}{\log 2}\liminf_{m\to\infty}\sum_{n=m}^{\infty}\frac{1}{n^{\epsilon}}.\end{aligned}$$

Choose a sequence $\{n_k\}_{k\ge1}$, where $n_k = k^{\tau}$ and $\tau\epsilon > 1$. Then $\liminf_{k\to\infty}\sum_{n=n_k}^{\infty} n^{-\epsilon} = 0$. It follows from the remark at the beginning of the proof that for almost all $x \in [0,1)$,

$$\limsup_{k\to\infty}\frac{L_{n_k}(x,\Theta)}{\log_{\tau(\Theta)} n_k} \le \frac{1+\epsilon}{2}.$$

For $n \ge 1$, there exists $k \in \mathbb{N}$ such that $n_k \le n < n_{k+1}$. As a consequence, we have $L_{n_k}(x,\Theta) \le L_n(x,\Theta) \le L_{n_{k+1}}(x,\Theta)$, so, for almost all $x \in [0,1)$,

$$\begin{aligned}\limsup_{n\to\infty}\frac{L_n(x,\Theta)}{\log_{\tau(\Theta)} n} &\le \limsup_{k\to\infty}\frac{L_{n_{k+1}}(x,\Theta)}{\log_{\tau(\Theta)} n_k}\\ &\le \limsup_{k\to\infty}\frac{L_{n_{k+1}}(x,\Theta)}{\log_{\tau(\Theta)} n_{k+1}}\cdot\lim_{k\to\infty}\frac{\log_{\tau(\Theta)} n_{k+1}}{\log_{\tau(\Theta)} n_k} \le \frac{1+\epsilon}{2}.\end{aligned}$$ □

Lemmas 3.1 and 3.2 together establish Theorem 1.1.

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4. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. The proof relies on the application of Corollary 2.6 by constructing a Cantor-like subset with Hausdorff dimension 1. We divide the whole proof into six parts, giving a detailed proof for the two cases with $0 < \alpha \le \beta < +\infty$ and sketches of proofs for the remaining cases.

Proof of theorem 1.2. Fix $d = \Theta + 1$.

Case 1. $0 < \alpha = \beta < +\infty$.

Choose two sequences $\{m_k\}_{k\ge1}$ and $\{n_k\}_{k\ge1}$ satisfying, for each $k \ge 1$,

$$n_k = \lfloor \tau(\Theta)^k \rfloor, \ m_k = n_k + \lfloor k\alpha \rfloor.$$

Then it is clear that $\{m_k\}_{k\ge1}$ increases exponentially and there exists $N \ge 1$ such that $n_k < m_k < n_{k+1}$ for $k \ge N$. For $k \ge N$, let

$$t_k = \max\{t : m_k + t(m_k - n_k) < n_{k+1}\}.$$

Define a marked set $\mathbf{K}$ of positive integers by

$$\begin{aligned}\mathbf{K} = \mathbf{K}(\{m_k\},\{n_k\}) = \{&1, 2, \ldots, n_K - 1, \text{ and } n_k, n_k + 1, \ldots, m_k,\\ &m_k + (m_k - n_k), \ldots, m_k + t_k(m_k - n_k), \text{ for } k \ge N\}.\end{aligned}$$

Define a sequence $\{a_i\}_{i\ge1}$ as follows. For $1 \le i < n_N$, set

$$a_i = d.$$

For $k \ge N$, set

$$a_{n_k} = d, \ a_{n_k+1} = \cdots = a_{m_k-1} = \Theta, \ a_{m_k} = d,$$
$$a_{m_k+(m_k-n_k)} = a_{m_k+2(m_k-n_k)} = \cdots = a_{m_k+t_k(m_k-n_k)} = d.$$

Now we consider the set $E(\mathbf{K}, \{a_i\})$ of real numbers $x \in [0, 1)$ whose continued fraction expansion $x = [a_1(x), a_2(x), \ldots]$ satisfies $a_i(x) = a_i$ for all $i \in \mathbf{K}$, that is,

$$E(\mathbf{K}, a_i) = \{x \in [0, 1) : a_i(x) = a_i \text{ for } i \in \mathbf{K}\}.$$

We claim that $E(\mathbf{K}, \{a_i\}) \subset E(\alpha, \alpha)$.

Suppose that $x \in E(\mathbf{K}, \{a_i\})$ and $n_k \le n < n_{k+1}$ with some $k \ge N$. From the construction of the set $E(\mathbf{K}, \{a_i\})$,

$$L_n(x, \Theta) = \begin{cases} m_{k-1} - n_{k-1} - 1 = \lfloor (k-1)\alpha \rfloor - 1 & \text{if } n_k \le n \le n_k + m_{k-1} - n_{k-1} - 1,\\ n - n_k & \text{if } n_k + m_{k-1} - n_{k-1} \le n \le m_k - 1,\\ m_k - n_k - 1 = \lfloor k\alpha \rfloor - 1 & \text{if } m_k \le n < n_{k+1}.\end{cases}$$

Thus,

$$\begin{aligned}\liminf_{n\to\infty} \frac{L_n(x,\Theta)}{\log_{\tau(\Theta)} n} &= \liminf_{k\to\infty} \min\left\{\frac{L_{n_k+m_{k-1}-n_{k-1}-1}(x,\Theta)}{\log_{\tau(\Theta)}(n_k + m_{k-1} - n_{k-1} - 1)}, \frac{L_{n_{k+1}-1}(x,\Theta)}{\log_{\tau(\Theta)}(n_{k+1} - 1)}\right\}\\ &= \liminf_{k\to\infty} \min\left\{\frac{\lfloor (k-1)\alpha \rfloor - 1}{\log_{\tau(\Theta)}(n_k + \lfloor (k-1)\alpha \rfloor - 1)}, \frac{\lfloor k\alpha \rfloor - 1}{\log_{\tau(\Theta)}(n_{k+1} - 1)}\right\}\\ &= \alpha\end{aligned}$$

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and

$$\limsup_{n\to\infty}\frac{L_n(x,\Theta)}{\log_{\tau(\Theta)} n} = \limsup_{k\to\infty}\max\left\{\frac{L_{n_k}(x,\Theta)}{\log_{\tau(\Theta)} n_k}, \frac{L_{m_k-1}(x,\Theta)}{\log_{\tau(\Theta)}(m_k-1)}\right\}$$
$$= \limsup_{k\to\infty}\max\left\{\frac{\lfloor (k-1)\alpha\rfloor - 1}{\log_{\tau(\Theta)} n_k}, \frac{\lfloor k\alpha\rfloor - 1}{\log_{\tau(\Theta)}(m_k-1)}\right\}$$
$$= \alpha.$$

Hence, $x \in E(\alpha,\alpha)$.

It remains to prove that the density of $\mathbf{K} \subset \mathbb{N}$ is zero. For $n_k \le n < n_{k+1}$ with some $k \ge N$,

$$\#\{i \le n : i \in \mathbf{K}\} = \begin{cases} n_N + \sum_{j=N}^{k-1}[(m_j - n_j + 1) + t_j] + n - n_k, & n_k \le n \le m_k, \\ n_N + \sum_{j=N}^{k-1}[(m_j - n_j + 1) + t_j] + m_k - n_k + \left\lfloor \frac{n - m_k}{m_k - n_k}\right\rfloor, & m_k < n \le n_{k+1}. \end{cases}$$

Consequently,

$$\limsup_{n\to\infty}\frac{\#\{i \le n : i \in \mathbf{K}\}}{n} \le \limsup_{k\to\infty}\max\left\{\frac{1}{n_k}\sum_{j=K}^{k-1}[(m_j - n_j + 1) + t_j] + \frac{m_k - n_k}{m_k},\right.$$
$$\left.\frac{1}{m_k}\sum_{j=K}^{k-1}[(m_j - n_j + 1) + t_j] + \frac{m_k - n_k}{m_k} + \frac{1}{m_k - n_k}\right\}$$
$$\le \limsup_{k\to\infty}\max\left\{\frac{(m_k - n_k + 1) + t_{k-1}}{n_k - n_{k-1}}, \frac{(m_k - n_k + 1) + t_{k-1}}{m_k - m_{k-1}}\right\}$$
$$\le \limsup_{k\to\infty}\left(\frac{m_{k-1} - n_{k-1} + 1}{n_k - n_{k-1}} + \frac{n_k - m_{k-1}}{(n_k - n_{k-1})(m_{k-1} - n_{k-1})}\right)$$
$$=0.$$

So, $\dim_H E(\mathbf{K}, \{a_i\}) = 1$ by Corollary 2.6.

Case 2. $0 < \alpha < \beta < +\infty$.

Take $n_k = \lfloor \tau(\Theta)^{\beta/\alpha^k}\rfloor$ and $m_k = n_k + \lfloor \beta \log_{\tau(\Theta)} n_k\rfloor$ for $k \ge 1$. Clearly, $\{n_k\}_{k\ge1}$ and $\{m_k\}_{k\ge1}$ increase super-exponentially, so there exists $K \ge 1$ such that $n_k < m_k < n_{k+1}$ for any $k \ge N$.

Define $\mathbf{K} = \mathbf{K}(\{n_k\}\{m_k\})$ and $E(\mathbf{K}, \{a_i\})$ as in Case 1 and consider $x \in E(\mathbf{K}, \{a_i\})$ and $n_k \le n < n_{k+1}$ with some $k \ge N$. From the construction of the set $E(\mathbf{K}, \{a_i\})$,

$$L_n(x,\Theta) = \begin{cases} m_{k-1} - n_{k-1} - 1 = \lfloor \beta \log_{\tau(\Theta)} n_{k-1}\rfloor - 1 & \text{if } n_k \le n \le n_k + m_{k-1} + n_{k-1} - 1, \\ n - n_k & \text{if } n_k + m_{k-1} + n_{k-1} \le n \le m_k - 1, \\ m_k - n_k - 1 = \lfloor \beta \log_{\tau(\Theta)} n_k\rfloor - 1 & \text{if } m_k \le n < n_{k+1}. \end{cases}$$

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Thus,

$$\liminf_{n\to\infty}\frac{L_n(x,\Theta)}{\log_{\tau(\Theta)} n} = \liminf_{k\to\infty}\min\Big\{\frac{L_{n_k+m_{k-1}-n_{k-1}-1}(x,\Theta)}{\log_{\tau(\Theta)}(n_k+m_{k-1}-n_{k-1}-1)}, \frac{L_{n_{k+1}-1}(x,\Theta)}{\log_{\tau(\Theta)}(n_{k+1}-1)}\Big\}$$
$$= \liminf_{k\to\infty}\min\Big\{\frac{\lfloor\beta\log_{\tau(\Theta)} n_{k-1}\rfloor-1}{\log_{\tau(\Theta)}(n_k+\lfloor\beta\log_{\tau(\Theta)} n_k\rfloor-1)}, \frac{\lfloor\beta\log_{\tau(\Theta)} n_k\rfloor-1}{\log_{\tau(\Theta)}(n_{k+1}-1)}\Big\}$$
$$=\alpha$$

and

$$\limsup_{n\to\infty}\frac{L_n(x,\Theta)}{\log_{\tau(\Theta)} n} = \limsup_{k\to\infty}\max\Big\{\frac{L_{n_k}(x,\Theta)}{\log_{\tau(\Theta)} n_k}, \frac{L_{m_k-1}(x,\Theta)}{\log_{\tau(\Theta)}(m_k-1)}\Big\}$$
$$= \limsup_{k\to\infty}\max\Big\{\frac{\lfloor\beta\log_{\tau(\Theta)} n_{k-1}\rfloor-1}{\log_{\tau(\Theta)} n_k}, \frac{\lfloor\beta\log_{\tau(\Theta)} n_k\rfloor-1}{\log_{\tau(\Theta)}(m_k-1)}\Big\}$$
$$= \limsup_{k\to\infty}\max\Big\{\alpha, \frac{\lfloor\beta\log_{\tau(\Theta)} n_k\rfloor-1}{\log_{\tau(\Theta)}(n_k+\lfloor\beta\log_{\tau(\Theta)} n_k\rfloor-1)}\Big\} = \beta.$$

Hence, $x \in E(\alpha,\beta)$. It is readily seen that the density of $\mathbf{K} \subset \mathbb{N}$ is zero from the definitions of the sequences $\{n_k\}_{k\geq 1}$ and $\{m_k\}_{k\geq 1}$.

Similar arguments apply to the remaining cases. Here, we only give the constructions for the proper sequences $\{n_k\}_{k\geq 1}$ and $\{m_k\}_{k\geq 1}$. It is easy to check that the corresponding $\mathbf{K}$ is of density zero and $E(\mathbf{K},\{a_i\})$ with full Hausdorff dimension is a subset of $E(\alpha,\beta)$ in each case.

Case 3. $\alpha=\beta=0$. Take $n_k=\lfloor\tau(\Theta)^k\rfloor, m_k=n_k+\lfloor\sqrt{k}\rfloor$ for each $k\geq 1$.

Case 4. $\alpha=0<\beta<\infty$. Take $n_1=2, n_{k+1}=n_k^k, m_k=n_k+\lfloor\beta\log_{\tau(\Theta)} n_k\rfloor$ for each $k\geq 1$.

Case 5. $\alpha=0,\beta=+\infty$. Take $n_1=2, n_{k+1}=n_k^k, m_k=n_k+\lfloor k\log_{\tau(\Theta)} n_k\rfloor$ for each $k\geq 1$.

Case 6. $\alpha=\beta=+\infty$. Take $n_k=k^2, m_k=n_k+\lfloor\sqrt{n_k}\rfloor$ for each $k\geq 1$. □

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