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Decomposition of multicorrelation sequences and joint ergodicity

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Abstract. We show that, under finitely many ergodicity assumptions, any multicorrelation sequence defined by invertible measure-preserving $\mathbb{Z}^d$-actions with multivariable integer polynomial iterates is the sum of a nilsequence and a nullsequence, extending a recent result of the second author. To this end, we develop a new seminorm bound estimate for multiple averages by improving the results in a previous work of the first, third, and fourth authors. We also use this approach to obtain new criteria for joint ergodicity of multiple averages with multivariable polynomial iterates on $\mathbb{Z}^d$-systems.

Key words: multicorrelation sequences, nilsequences, nullsequence, joint ergodicity
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Contents

1. *Introduction*

1.1. *Decomposition of multicorrelation sequences.* The structure and limiting behavior of (averages of) *multicorrelation sequences,* that is, sequences of the form

$$(n_1, \ldots, n_k) \mapsto \int_X f_0 \cdot T_1^{n_1} f_1 \cdots T_k^{n_k} f_k \, d\mu,$$

where $k \in \mathbb{N}$, $T_1, \ldots, T_k \colon X \to X$ are invertible and commuting (that is, $T_i T_j = T_j T_i$ for all i, j) measure-preserving transformations on a probability space $(X, \mathcal{B}, \mu)$, $f_0, \ldots, f_k \in L^\infty(\mu)$ and $n_1, \ldots, n_k \in \mathbb{Z}$, is a central topic in ergodic theory. (We say that T *preserves* μ if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$. The tuple $(X, \mathcal{B}, \mu, T_1, \ldots, T_k)$ is a *(measure-preserving) system.*) For $k = 1$, Herglotz–Bochner's theorem implies that the sequence $\int_X f_0 \cdot T_1^n f_1 \, d\mu$ is given by the Fourier coefficients of some finite complex measure σ on $\mathbb{T}$ (see [22, 23]). More specifically, decomposing σ into the sum of its atomic part, σ_a, and continuous part, σ_c, we get

$$\int_X f_0 \cdot T_1^n f_1 \, d\mu = \int_{\mathbb{T}} e^{2\pi i n x} \, d\sigma(x) = \int_{\mathbb{T}} e^{2\pi i n x} \, d\sigma_a(x) + \int_{\mathbb{T}} e^{2\pi i n x} \, d\sigma_c(x) = \psi(n) + \nu(n),$$

where $(\psi(n))$ is *an almost periodic sequence* (that is, there exists a compact abelian group G, a continuous function $\phi : G \to \mathbb{C}$, and $a \in G$ such that $\psi(n) = \phi(a^n)$, $n \in \mathbb{N}$) and $(\nu(n))$ is a *nullsequence*, that is,

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\nu(n)|^2 = 0. \tag{1}$$

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More generally, after Furstenberg's celebrated ergodic theoretic proof of Szemerédi's theorem [15], for a single transformation T and iterates of the form in, $1 \leq i \leq k$, there has been a particular interest in the study of the corresponding multicorrelation sequences

$$\alpha(n) = \int_X f_0 \cdot T^n f_1 \cdots T^{kn} f_k \, d\mu. \tag{2}$$

For T ergodic (that is, every T-invariant set in $\mathcal{B}$ has trivial measure in $\{0, 1\}$), Bergelson, Host, and Kra [3] showed that the sequence $(\alpha(n))$ in equation (2) admits a decomposition of the form $a(n) = \phi(n) + \nu(n)$, where $(\phi(n))$ is a uniform limit of k-step nilsequences (see §3.1 for the definition) and $(\nu(n))$ satisfies equation (1). (Note that k is the number of linear iterates that appear in equation (2).) Leibman, in [28] for ergodic systems and [29] for general ones, extended the result of Bergelson, Host, and Kra to polynomial iterates, meaning that in equation (2), instead of $n, \ldots, kn$, we have $p_1(n), \ldots, p_k(n)$, for some $p_1, \ldots, p_k \in \mathbb{Z}[x]$.

For $d \in \mathbb{N}$, we say that a tuple $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ is a $\mathbb{Z}^d$*-measure-preserving system* (or a $\mathbb{Z}^d$*-system*) if $(X, \mathcal{B}, \mu)$ is a probability space and $T_n \colon X \to X, n \in \mathbb{Z}^d$, are measure-preserving transformations on X such that $T_{(0,\ldots,0)} = \mathrm{id}$ and $T_m \circ T_n = T_{m+n}$ for all $m, n \in \mathbb{Z}^d$. Notice here that we use the notation T_n to stress the fact that T is a $\mathbb{Z}^d$-action. If T is generated by the $\mathbb{Z}$-actions $T_1, \ldots, T_d$ and $p_i = (p_{i,1}, \ldots, p_{i,d})$, we have $T_{p_i(n)} = \prod_{j=1}^d T_j^{p_{i,j}(n)}$. It is natural to ask whether splitting results still hold for systems with commuting transformations.

Question 1.1. [27, Question 2] Let $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system, $k \in \mathbb{N}$, $p_1, \ldots, p_k \colon \mathbb{Z} \to \mathbb{Z}^d$ a family of polynomials, and $f_0, f_1, \ldots, f_k \in L^\infty(\mu)$. Under which conditions on the system can the multicorrelation sequence

$$\int_X f_0 \cdot T_{p_1(n)} f_1 \cdots T_{p_k(n)} f_k \, d\mu \tag{3}$$

be decomposed as the sum of a uniform limit of nilsequences and a nullsequence?

The extension of the aforementioned results from $\mathbb{Z}$ to $\mathbb{Z}^d$-actions is, to this day, a challenging open problem. The main issue is that the proofs of the splitting theorems crucially depend on the theory of characteristic factors via the structure theory developed by Host and Kra [18], a tool that is unavailable in the more general $\mathbb{Z}^d$-setting. By this, we mean that while nilfactors for $\mathbb{Z}^d$-analogs of Host–Kra uniformity norms are available (this can be found, for example, in [16]), it is in general not possible to relate averages such as equation (3) to those uniformity norms in the way one does for $d = 1$. As an aside, Frantzikinakis provided a partial answer to Question 1.1 (for $d = 1$) in [10] that avoided the use of characteristic factors. The answer was partial in the sense that the nullsequence part was allowed to have an $\ell^2(\mathbb{Z})$ error term. A similar decomposition result for general d was proven by Frantzikinakis and Host in [12]. (The third author showed in [25] the analog to this result for integer parts, or any combination of rounding functions, of real polynomial iterates. For a refinement of this result, with the average of the error term taken along primes, see [27].) From the point of view of applications, it is useful to have such splitting results for studying weighted averages, in particular for multiple commuting transformations. (It is worth mentioning that the splitting of equation (2), where the average in the null term is taken along primes, was used by Tao and Teräväinen to show the logarithmic Chowla conjecture for products of odd factors [32].)

It was demonstrated in [7] that under finitely many ergodicity assumptions (that is, we only have to assume that some iterates, coming from a finite set, of T are ergodic), the characteristic factors (defined in §2.3) for the corresponding averages

$$\frac{1}{N}\sum_{n=1}^{N} T_{p_1(n)} f_1 \cdots T_{p_k(n)} f_k \tag{4}$$

are, as in the case of $\mathbb{Z}$-actions, rotations on nilmanifolds. (A similar result was obtained in [20] under infinitely many ergodicity assumptions. Such multiple ergodic averages always have L^2-limits as $N \to \infty$ [34].) So, it is reasonable to expect that Question 1.1 holds after postulating finitely many ergodicity assumptions (this is an open problem even in the $k = 2$ case—see [12]).

A partial answer toward this direction was obtained in [9] by the second author. Namely, [9, Theorem 1.5] shows that for any system $(X, \mathcal{B}, \mu, T_1, \ldots, T_k)$ with T_i and $T_i T_j^{-1}$ ergodic (for all i and $j \neq i$) and $f_0, \ldots, f_k \in L^\infty(\mu)$, the sequence

$$\int_X f_0 \cdot T_1^n f_1 \cdots T_k^n f_k \, d\mu \tag{5}$$

can be decomposed as a sum of a uniform limit of k-step nilsequences plus a nullsequence.

For more general expressions (as in equation (3)), exploiting results from [20], it is also shown in [9] that if we further assume ergodicity in all directions, that is, $T_1^{a_1} \cdots T_d^{a_d}$ is ergodic for all $(a_1, \ldots, a_d) \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, then for any family of pairwise distinct polynomials $p_1, \ldots, p_k : \mathbb{Z} \to \mathbb{Z}^d$, the sequence

$$\int_X f_0 \cdot T_{p_1(n)} f_1 \cdots T_{p_k(n)} f_k \, d\mu \tag{6}$$

can be decomposed as a sum of a uniform limit of D-step nilsequences plus a nullsequence. (Here D depends on k, d and the maximum degree of the p_i terms. It also has a connection to the number of van der Corput operations we have to run in the induction (see Remark 5.14 for details).) The proof of this result makes essential use of a seminorm bound estimate obtained in [20], where the (infinitely many) ergodicity assumptions are reflected (see [9, Theorem 1.6]).

In [7], the first, third, and fourth authors improved the seminorm bound estimates of [20] by imposing only finitely many ergodic assumptions. Although the results in [7] are stronger than those in [20], one cannot apply them directly to [9] to improve the aforementioned results, due to the incompatibility of the methods between the two studies [7, 9] (see §2.3 for more details).

In this article, we extend results from [7] to obtain splitting theorems for multicorrelation sequences involving multiparameter polynomials, postulating ergodicity assumptions which are even weaker than those in [7] on the transformations that define the $\mathbb{Z}^d$-action in equation (6); for example, we will see that the sequence $\int_X f_0 \cdot T_1^{n^2} T_2^n f_1 \cdot T_3^{n^2} T_4^n f_2 \, d\mu$ admits the desired splitting if we assume that $T_1, T_3, T_1 T_3^{-1}$ are ergodic.

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1.2. *The joint ergodicity phenomenon.* In his ergodic theoretic proof of Szemerédi's theorem, Furstenberg [15] studied the averages of the multicorrelation sequence in equation (2). In particular, a stepping stone in the proof is the special case when the transformation T is weakly mixing (that is, $T \times T$ is ergodic for $\mu \times \mu$), in which he showed that the averages

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdots T^{kn} f_k \tag{7}$$

converge in $L^2(\mu)$ to $\prod_{i=1}^{k}\int_X f_i\, d\mu$ (which we will refer to as the 'expected limit') as $N \to \infty$. (Throughout this paper, unless otherwise stated, all limits of measurable functions on a measure-preserving system are taken in L^2.) It was Berend and Bergelson [1] who characterized when the average of the integrand of equation (5), that is, for multiple commuting transformations, converges to the expected limit (and this happens exactly when $T_1 \times \cdots \times T_k$ and $T_i T_j^{-1}$ for all $i \neq j$ are ergodic).

Generalizing Furstenberg's result, Bergelson showed (in [2]) that, for a weakly mixing transformation T and essentially distinct polynomials $p_1, \ldots, p_k$ (that is, p_i, $p_i - p_j$ are non-constant for all $1 \le i, j \le k, i \neq j$),

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T^{p_1(n)} f_1 \cdots T^{p_k(n)} f_k = \prod_{i=1}^{k}\int_X f_i\, d\mu.$$

(For T totally ergodic (that is, T^n is ergodic for all $n \in \mathbb{N}$) and $p_1, \ldots, p_k$ 'independent' integer polynomials, it is proved in [14] that we have the same conclusion. This fact remains true for an ergodic T and 'strongly independent' real-valued polynomials iterates, $[p_1(n)], \ldots, [p_k(n)]$ ($[\cdot]$ denotes the floor function), as well (see [21]). These last two results also follow by a recent work of Frantzikinakis, [11], in which, for single T, we have a plethora of joint ergodicity results for a number of classes of iterates (not just polynomial). Finally, for real variable polynomial iterates, one is referred to [26].) One can think of this last result as a strong independence property of the sequences $(T^{p_i(n)})_{n\in\mathbb{Z}}$, $1 \le i \le k$ in the weakly mixing case. It is reasonable to expect, under additional assumptions on the system and/or the polynomial iterates, convergence, of the averages appearing in the previous relation, to the expected limit, which naturally leads to a general notion of joint ergodicity (a sequence of finite subsets $(I_N)_{N\in\mathbb{N}}$ of $\mathbb{Z}^L$ with the property $\lim_{N\to\infty} |I_N|^{-1} \cdot |(g + I_N)\Delta I_N| = 0$ for all $g \in \mathbb{Z}^L$ is called a *Følner sequence* in $\mathbb{Z}^L$).

Definition 1.2. Let $d, k, L \in \mathbb{N}$, $p_1, \ldots, p_k : \mathbb{Z}^L \to \mathbb{Z}^d$ be polynomials and $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system. We say that the sequence of tuples $(T_{p_1(n)}, \ldots, T_{p_k(n)})_{n\in\mathbb{Z}^L}$ is *jointly ergodic* for μ if for every $f_1, \ldots, f_k \in L^\infty(\mu)$ and every Følner sequence $(I_N)_{N\in\mathbb{N}}$ of $\mathbb{Z}^L$, we have that

$$\lim_{N\to\infty}\frac{1}{|I_N|}\sum_{n\in I_N} T_{p_1(n)} f_1 \cdots T_{p_k(n)} f_k = \int_X f_1\, d\mu \cdots \int_X f_k\, d\mu. \tag{8}$$

When $k = 1$, we also say that $(T_{p_1(n)})_{n\in\mathbb{Z}^L}$ is *ergodic* for μ.

The following conjecture was stated in [7].

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Conjecture 1.3. [7, Conjecture 1.5] Let $d, k, L \in \mathbb{N}$, $p_1, \ldots, p_k \colon \mathbb{Z}^L \to \mathbb{Z}^d$ be polynomials and $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system. Then the following are equivalent.

(C1) $(T_{p_1(n)}, \ldots, T_{p_k(n)})_{n\in\mathbb{Z}^L}$ is jointly ergodic for μ.

(C2) The following conditions are satisfied:

- (i) $(T_{p_i(n)-p_j(n)})_{n\in\mathbb{Z}^L}$ is ergodic for μ for all $1 \leq i, j \leq k$, $i \neq j$; and
- (ii) $(T_{p_1(n)} \times \cdots \times T_{p_k(n)})_{n\in\mathbb{Z}^L}$ is ergodic for the product measure $\mu^{\otimes k}$ on X^k.

Answering a question of Bergelson, it was shown in [7, Theorem 1.4] that for a polynomial $p : \mathbb{Z}^L \to \mathbb{Z}$, the sequence $(T_1^{p(n)}, \ldots, T_k^{p(n)})_{n\in\mathbb{Z}^L}$ is jointly ergodic for μ if and only if $((T_1 \times \cdots \times T_k)^{p(n)})_{n\in\mathbb{Z}^L}$ is ergodic for $\mu^{\otimes k}$ and $T_i T_j^{-1}$ is ergodic for μ for all $i \neq j$. In this paper, the strong decomposition results that we obtain allow us to deduce joint ergodicity results for a larger family of polynomials (see Theorems 2.5 and 2.9), thus addressing some additional cases in the aforementioned conjecture.

2. *Main results*

In this section, we state the main results of the paper and provide a number of examples to better illustrate them. We also comment on the approaches that we follow.

2.1. *Splitting results.*

Our first main concern is to resolve the incompatibility between [7] and [9], and improve the method in [7], to obtain an extension of the results in [9].

Before we state our first result, we need to introduce some notation.

For $d, L \in \mathbb{N}$, the polynomial $q = (q_1, \ldots, q_d) : \mathbb{Z}^L \to \mathbb{Z}^d$ is *non-constant* if some q_i is non-constant. Here we mean that each q_i is a member of $\mathbb{Q}[x_1, \ldots, x_L]$ with $q_i(\mathbb{Z}^L) \subseteq \mathbb{Z}$. The *degree* of q is defined as the maximum of the degrees of the q_i terms.

The polynomials $p_1, \ldots, p_k \colon \mathbb{Z}^L \to \mathbb{Z}^d$ are called *essentially distinct* if they are non-constant and $p_i - p_j$ is non-constant for all $i \neq j$. (In general, a polynomial $q \colon \mathbb{Z}^L \to \mathbb{Z}^d$ has rational coefficients (that is, vectors with rational coordinates).)

For a subset A of $\mathbb{Q}^d$, we denote $G(A) := \mathrm{span}_{\mathbb{Q}}\{a \in A\} \cap \mathbb{Z}^d$. The following subgroups of $\mathbb{Z}^d$ play an important role in this paper.

Definition 2.1. Let $\mathbf{p} = (p_1, \ldots, p_k)$, $p_1, \ldots, p_k \colon \mathbb{Z}^L \to \mathbb{Z}^d$ be a family of essentially distinct polynomials with $p_i(n) = \sum_{v\in\mathbb{N}_0^L} b_{i,v} n^v$ for some $b_{i,v} \in \mathbb{Q}^d$ with at most finitely many $b_{i,v}$, $v \in \mathbb{N}_0^L$ non-zero. (Here, we denote $n^v := n_1^{v_1} \ldots n_L^{v_L}$ for $n = (n_1, \ldots, n_L) \in \mathbb{Z}^L$ and $v = (v_1, \ldots, v_L) \in \mathbb{N}_0^L$, where $0^0 := 1$.) For convenience, we artificially denote p_0 as the constant zero polynomial and $b_{0,v} := 0$ for all $v \in \mathbb{N}_0^L$. For $0 \leq i, j \leq k$, set $d_{i,j} := \deg(p_i - p_j)$ and $G_{i,j}(\mathbf{p}) := G(\{b_{i,v} - b_{j,v} \colon |v| = d_{i,j}\})$, where, for $v = (v_1, \ldots, v_L) \in \mathbb{N}_0^L$, we write $|v| = v_1 + \cdots + v_L$.

Our main result provides an affirmative answer to Question 1.1 under finitely many ergodicity assumptions on the groups $G_{i,j}(\mathbf{p})$, which generalizes [9, Theorem 1.5]. We say that the group $G_{i,j}(\mathbf{p})$ is *ergodic for* μ if any function $f \in L^2(\mu)$ that is T_a-invariant for all $a \in G_{i,j}(\mathbf{p})$ is constant.

The definition of a D-step nilsequence will be given in §3.1. We say that $a \colon \mathbb{Z}^L \to \mathbb{C}$ is a *nullsequence* if for any Følner sequence $(I_N)_{N\in\mathbb{N}}$, $\lim_{N\to\infty} 1/|I_N| \sum_{n\in I_N} |a(n)|^2 = 0$.

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THEOREM 2.2. (Decomposition theorem under finitely many ergodicity assumptions) *For $d, k, K, L \in \mathbb{N}$, let $\mathbf{p} = (p_1, \ldots, p_k)$, where $p_1, \ldots, p_k : \mathbb{Z}^L \to \mathbb{Z}^d$ is a family of essentially distinct polynomials of degree at most K, and let $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system. If $G_{i,j}(\mathbf{p})$ is ergodic for μ for all $0 \le i, j \le k, i \ne j$, then for all $f_0, \ldots, f_k \in L^\infty(\mu)$, the multicorrelation sequence*

$$a(n) := \int_X f_0 \cdot T_{p_1(n)} f_1 \cdots T_{p_k(n)} f_k \, d\mu$$

can be decomposed as a sum of a uniform limit of D-step nilsequences and a nullsequence, where $D \in \mathbb{N}$ is a constant depending only on d, k, K, L.

We refer the reader to Remark 5.14 for a further discussion on the constant D. Also, note that Theorem 2.2 goes beyond Question 1.1 as it deals with multivariable polynomial iterates (that is, $L > 1$).

Example 2.3. It was proved in [9, Theorem 1.5] that for any probability space $(X, \mathcal{B}, \mu)$ and commuting transformations $T_1, \ldots, T_k$ acting on X, if T_i and $T_i T_j^{-1}$ are ergodic (for all i and all $j \ne i$, respectively), then for all $f_0, \ldots, f_k \in L^\infty(\mu)$, the multicorrelation sequence

$$a(n) := \int_X f_0 \cdot T_1^n f_1 \cdots T_k^n f_k \, d\mu$$

can be decomposed as a sum of a uniform limit of k-step nilsequences plus a nullsequence. While Theorem 2.2 does not specify the step D of the nilsequence, a quick argument shows that, in this case, one can indeed take $D = k$ (see Remark 6.1 for details).

The following example shows that Theorem 2.2 is stronger than [9, Theorem 1.6], which deals with single variable essentially distinct polynomial iterates.

Example 2.4. Let $(X, \mathcal{B}, \mu, T_1, \ldots, T_6)$ be a system with commuting transformations $T_1, \ldots, T_6$ and $f_0, f_1, \ldots, f_4 \in L^\infty(\mu)$. Using [9, Theorem 1.6], we have that the multicorrelation sequence

$$\alpha(n) = \int_X f_0 \cdot T_1^{n^2} T_2^n f_1 \cdot T_1^{n^2} T_3^n f_2 \cdot T_4^{n^3} f_3 \cdot T_5^{n^3} T_6^n f_4 \, d\mu \tag{9}$$

can be decomposed as the sum of a uniform limit of nilsequences and a nullsequence if $T_1^{a_1} \cdots T_6^{a_6}$ is ergodic for all $(a_1, \ldots, a_6) \in \mathbb{Z}^6 \backslash \{\mathbf{0}\}$. In contrast, via Theorem 2.2, one can get the same conclusion by only assuming that $T_1, T_2T_3^{-1}, T_4, T_5, T_4T_5^{-1}$ are ergodic. (Indeed, denoting $T_{(a_1,\ldots,a_6)} := T_1^{a_1} \cdots T_6^{a_6}$, and e_i the vector whose ith entry is 1 and all other entries are 0, since $\mathbf{p} = ((n^2, n, 0, 0, 0, 0), (n^2, 0, n, 0, 0, 0), (0, 0, 0, n^3, 0, 0), (0, 0, 0, 0, n^3, n))$, we have that $G_{1,0}(\mathbf{p}) = G_{2,0}(\mathbf{p}) = G(e_1)$, $G_{1,3}(\mathbf{p}) = G_{2,3}(\mathbf{p}) = G_{3,0}(\mathbf{p}) = G(e_4)$, $G_{1,4}(\mathbf{p}) = G_{2,4}(\mathbf{p}) = G_{4,0}(\mathbf{p}) = G(e_5)$, $G_{1,2}(\mathbf{p}) = G(e_2 - e_3)$, $G_{3,4}(\mathbf{p}) = G(e_4 - e_5)$.)

2.2. *Convergence to the expected limit.* In [7, Theorem 1.4], the first, third, and fourth authors proved the following case of Conjecture 1.3. If $T_1, \ldots, T_k$ are commuting transformations acting on a probability space $(X, \mathcal{B}, \mu)$, then $(T_1^{p(n)}, \ldots, T_k^{p(n)})_{n\in\mathbb{Z}^L}$ is

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jointly ergodic for μ if and only if $((T_1 \times \cdots \times T_k)^{p(n)})_{n\in\mathbb{Z}^L}$ is ergodic for $\mu^{\otimes k}$ and $T_i T_j^{-1}$ is ergodic for μ for all $i \neq j$. In this paper, we further extend this result.

THEOREM 2.5. *Let $k, d, L \in \mathbb{N}$ and $\mathbf{p} = (p_1 v_1, \ldots, p_k v_k)$, where $p_1, \ldots, p_k : \mathbb{Z}^L \to \mathbb{Z}$, $v_1, \ldots, v_k \in \mathbb{Z}^d$ be a family of essentially distinct polynomials. Suppose that for all $1 \leq i, j \leq k$, if $\deg(p_i) = \deg(p_j)$, then either v_i and v_j are linearly dependent over $\mathbb{Z}$, or p_i and p_j are linearly dependent over $\mathbb{Z}$ (that is, there is a non-trivial linear combination of them over $\mathbb{Z}$ which equals to a constant). Let $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system. Then the following are equivalent.*

(C1) $(T_{p_1(n)v_1}, \ldots, T_{p_k(n)v_k})_{n\in\mathbb{Z}^L}$ *is jointly ergodic for* μ.

(C2') *The following subconditions hold:*

- (i)' $(T_{p_i(n)v_i - p_j(n)v_j})_{n\in\mathbb{Z}^L}$ *is ergodic for μ for all $1 \leq i, j \leq k, i \neq j$ with* $\deg(p_i) = \deg(p_j)$*;*
- (ii) $(T_{p_1(n)v_1} \times \cdots \times T_{p_k(n)v_k})_{n\in\mathbb{Z}^L}$ *is ergodic for* $\mu^{\otimes k}$.

Moreover, condition (C2') is equivalent to

(C2) *The following subconditions hold:*

- (i) $(T_{p_i(n)v_i - p_j(n)v_j})_{n\in\mathbb{Z}^L}$ *is ergodic for μ for all $1 \leq i, j \leq k, i \neq j$;*
- (ii) $(T_{p_1(n)v_1} \times \cdots \times T_{p_k(n)v_k})_{n\in\mathbb{Z}^L}$ *is ergodic for* $\mu^{\otimes k}$.

Note that the subconditions in condition (C2) are consistent with those in Conjecture 1.3. However, the reason we provide an alternative set of equivalent subconditions in condition (C2') is that these subconditions are easier to check in practice.

We now give some examples to illustrate Theorem 2.5. The first one is for polynomials of distinct degrees.

Example 2.6. Let $(X, \mathcal{B}, \mu, T_1, \ldots, T_k)$ be a system. Using Theorem 2.5, we conclude that $(T_1^n, T_2^{n^2}, \ldots, T_k^{n^k})_{n\in\mathbb{Z}}$ is jointly ergodic if and only if $(T_1^n \times \cdots \times T_k^{n^k})_{n\in\mathbb{Z}}$ is ergodic for $\mu^{\otimes k}$, and all the T_i terms are ergodic for μ.

We remark that Example 2.6 can also be proved by using arguments from [**6**]. We next present two examples in which some polynomials have the same degree and so cannot be recovered by the methods of [**6**].

Example 2.7. Let $(X, \mathcal{B}, \mu, T_1, T_2, T_3, T_4)$ be a system. Theorem 2.5 implies that $(T_1^n, T_2^n, T_3^{n^2}, T_4^{n^2})_{n\in\mathbb{Z}}$ is jointly ergodic if and only if $(T_1^n \times T_2^n \times T_3^{n^2} \times T_4^{n^2})_{n\in\mathbb{Z}}$ is ergodic for $\mu^{\otimes 4}$, and both $T_1 T_2^{-1}$ and $((T_3 T_4^{-1})^{n^2})_{n\in\mathbb{N}}$ are ergodic for μ.

Example 2.8. Let $(X, \mathcal{B}, \mu, T_1, T_2, T_3)$ be a system. Theorem 2.5 implies that $(T_1^{n^4+n^2}, T_1^{2n^4+3n}, T_2^{2n^2+2n+1}, T_3^{3n^2+3n})_{n\in\mathbb{Z}}$ is jointly ergodic if and only if $(T_1^{n^4+n^2} \times T_1^{2n^4+3n} \times T_2^{2n^2+2n+1} \times T_3^{3n^2+3n})_{n\in\mathbb{Z}}$ is ergodic for $\mu^{\otimes 4}$, and both sequences $(T_1^{-n^4+n^2-3n})_{n\in\mathbb{Z}}$ and $((T_2^2 T_3^{-3})^{n^2+n})_{n\in\mathbb{Z}}$ are ergodic for μ.

Another direction for the joint ergodicity problem is verifying whether condition (C1) implies condition (C2) in Conjecture 1.3. Namely, assume that $(T_{p_1(n)} \times \cdots \times T_{p_k(n)})_{n\in\mathbb{Z}^L}$ is ergodic for $\mu^{\otimes k}$ to find a condition, say (P), of certain sequences of actions to be ergodic,

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under which we have that $(T_{p_1(n)}, \ldots, T_{p_k(n)})_{n\in\mathbb{Z}^L}$ is jointly ergodic for μ. By combining existing results from [18, 20] (see also [7, Proposition 1.2]), (P) can be taken to be 'T_g is ergodic for μ for all $g \in \mathbb{Z}^d\backslash\{\mathbf{0}\}$'. Denoting $p_i(n) = \sum_{v\in\mathbb{N}_0^L, 0\leq|v|\leq K} b_{i,v}n^v$ for some $b_{i,v} \in \mathbb{Q}^d$ and $K \in \mathbb{N}_0$, this result was extended in [7, Theorem 1.3], where the previous property is replaced by 'T_g is ergodic for μ for all g that belongs to the finite set R', where

$$R = \bigcup_{0<|v|\leq K} \{b_{i,v}, b_{i,v} - b_{j,v} : 1 \leq i, j \leq k\}\backslash\{\mathbf{0}\}.$$

In this paper, we replace the latter condition with an even weaker one.

THEOREM 2.9. *Let $d, k, L \in \mathbb{N}$, $\mathbf{p} = (p_1, \ldots, p_k)$, $p_1, \ldots, p_k : \mathbb{Z}^L \to \mathbb{Z}^d$ be a family of essentially distinct polynomials and $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ a $\mathbb{Z}^d$-system. Then, $(T_{p_1(n)}, \ldots, T_{p_k(n)})_{n\in\mathbb{Z}^L}$ is jointly ergodic for μ if both of the following conditions hold:*
(i) *$G_{i,j}(\mathbf{p})$ is ergodic for μ for all $0 \leq i, j \leq k$, $i \neq j$;*
(ii) *$(T_{p_1(n)} \times \cdots \times T_{p_k(n)})_{n\in\mathbb{Z}^L}$ is ergodic for $\mu^{\otimes k}$.*

The last example for this section reflects the stronger nature of the previous theorem compared to what was previously known.

Example 2.10. Let $(X, \mathcal{B}, \mu, T_1, T_2, T_3, T_4)$ be a system. Then, [7, Theorem 1.3] implies that $(T_1^{n^2}T_2^n, T_3^{n^2}T_4^n)_{n\in\mathbb{Z}}$ is jointly ergodic if $((T_1^{n^2}T_2^n) \times (T_3^{n^2}T_4^n))_{n\in\mathbb{Z}}$ is ergodic for $\mu^{\otimes 2}$, and all $T_1, T_2, T_3, T_4, T_1T_3^{-1}, T_2T_4^{-1}$ are ergodic for μ. Using Theorem 2.9, we conclude that $(T_1^{n^2}T_2^n, T_3^{n^2}T_4^n)_{n\in\mathbb{Z}}$ is jointly ergodic if we instead only assume that $((T_1^{n^2}T_2^n) \times (T_3^{n^2}T_4^n))_{n\in\mathbb{Z}}$ is ergodic for $\mu^{\otimes 2}$, and all $T_1, T_3, T_1T_3^{-1}$ are ergodic for μ.

Unfortunately, Theorem 2.9 does not imply Conjecture 1.3 for the pair $(T_1^{n^2}T_2^n, T_3^{n^2}T_4^n)_{n\in\mathbb{Z}}$. This is because $T_1, T_3, T_1T_3^{-1}$ being ergodic for μ is independent of $((T_1T_3^{-1})^{n^2}(T_2T_4^{-1})^n)_{n\in\mathbb{Z}}$ being ergodic for μ. For example, if $T_1 = T_3 = T_4 = \text{id}$ and T_2 is any ergodic transformation, then $((T_1T_3^{-1})^{n^2}(T_2T_4^{-1})^n)_{n\in\mathbb{Z}}$ is ergodic for μ but $T_1, T_3, T_1T_3^{-1}$ are not. However, if $X = \{0, \ldots, 6\}$ with $\mu(\{i\}) = 1/7$, $T_1x := x + 1 \mod 7$, $T_3 = T_1^2$, and $T_2 = T_4 = \text{id}$, then $T_1, T_3, T_1T_3^{-1}$ are ergodic for μ but $((T_1T_3^{-1})^{n^2}(T_2T_4^{-1})^n)_{n\in\mathbb{Z}}$ is not.

2.3. *Strategy of the paper.* The central ingredient in proving the main results of the paper (Theorems 2.2, 2.5, and 2.9) is to find proper *characteristic factors* for the limit of the average in equation (4), that is, sub-σ-algebras $\mathcal{D}_1, \ldots, \mathcal{D}_k$ of $\mathcal{B}$ such that the average in equation (4) remains invariant if we replace each f_i by its conditional expectation (see below for the definition) with respect to $\mathcal{D}_i$. An important type of characteristic factor, called the *Host–Kra characteristic factor*, was invented in [18] to study multiple averages for $\mathbb{Z}$-systems (see below for the definition of these factors). This concept was generalized to systems with commuting transformations in [17] (see also [31]).

To introduce the main tool used in our results (Theorem 2.11), special cases of which have been studied extensively in the past (see for example [6, 14, 17, 18, 20]), we need to introduce the machinery of Host–Kra seminorms and factors.

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Host–Kra seminorms and their associated factors are arguably the main tools used to analyze the behavior of multiple averages and correlation sequences. In what follows, we give general results about these seminorms and factors, following the notation used in [7].

We first recall the notions of a factor and of the conditional expectation with respect to a factor. We say that the $\mathbb{Z}^d$-system $(Y, \mathcal{D}, \nu, (S_g)_{g\in\mathbb{Z}^d})$ is a *factor* of $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ if there exists a measurable map $\pi : X \to Y$ such that $\mu(\pi^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{D}$, and $\pi \circ T_g = S_g \circ \pi$ for all $g \in \mathbb{Z}^d$.

A factor $(Y, \mathcal{D}, \nu, (S_g)_{g\in\mathbb{Z}^d})$ of $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ can be identified with an invariant sub-σ-algebra $\mathcal{B}'$ of $\mathcal{B}$ by setting $\mathcal{B}' := \pi^{-1}(\mathcal{D})$. Given two σ-algebras, $\mathcal{B}_1$ and $\mathcal{B}_2$, their *joining* $\mathcal{B}_1 \vee \mathcal{B}_2$ is the σ-algebra generated by $B_1 \cap B_2$ for all $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$, that is, the smallest σ-algebra containing both $\mathcal{B}_1$ and $\mathcal{B}_2$.

Given a factor $\pi : (X, \mathcal{B}, \mu) \to (Y, \mathcal{D}, \nu)$ and a function $f \in L^2(\mu)$, the *conditional expectation of f with respect to Y* is the function $g \in L^2(\nu)$, which we denote by $\mathbb{E}(f \mid Y)$, with the property

$$\int_A g \circ \pi \, d\mu = \int_A f \, d\mu \quad \text{for all } A \in \pi^{-1}(\mathcal{D}).$$

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $\mathcal{B}_1$ be a sub-σ-algebra of $\mathcal{B}$. The *relatively independent joining* of $(X, \mathcal{B}, \mu)$ with itself with respect to $\mathcal{B}_1$ is the probability space $(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times_{\mathcal{B}_1} \mu)$, where the measure $\mu \times_{\mathcal{B}_1} \mu$ is given by the formula:

$$\int_{X\times X} f_1 \otimes f_2 \, d(\mu \times_{\mathcal{B}_1} \mu) = \int_X \mathbb{E}(f_1|\mathcal{B}_1)\mathbb{E}(f_2|\mathcal{B}_1) \, d\mu,$$

for all $f_1, f_2 \in L^\infty(\mu)$.

For a G-system $\mathbf{X} = (X, \mathcal{B}, \mu, (T_g)_{g\in G})$, if H is a subgroup of G, we denote by $\mathcal{I}(H)(\mathbf{X})$ the set of $A \in \mathcal{B}$ such that $T_g A = A$ for all $g \in H$. When there is no confusion, we write $\mathcal{I}(H)$.

For a $\mathbb{Z}^d$-system $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ and $H_1, \ldots, H_k$ subgroups of $\mathbb{Z}^d$, define

$$\mu_{H_1} = \mu \times_{\mathcal{I}(H_1)} \mu$$

and for $k > 1$, let

$$\mu_{H_1,\ldots,H_k} = \mu_{H_1,\ldots,H_{k-1}} \times_{\mathcal{I}(H_k^{[k-1]})} \mu_{H_1,\ldots,H_{k-1}},$$

where $H_k^{[k-1]}$ denotes the subgroup of $(\mathbb{Z}^d)^{2^{k-1}}$ consisting of all the elements of the form $(h_k, \ldots, h_k)$ (2^{k-1} copies of h_k) for some $h_k \in H_k$. The *characteristic factor* $Z_{H_1,\ldots,H_k}(\mathbf{X})$ is defined to be the sub-σ-algebra of $\mathcal{B}$ characterized by

$$\mathbb{E}(f|Z_{H_1,\ldots,H_k}(\mathbf{X})) = 0 \text{ if and only if } |||f|||_{H_1,\ldots,H_k}^{2^k} := \int_{X^{[k]}} \bigotimes_{\epsilon\in\{0,1\}^k} C^{|\epsilon|} f \, d\mu_{H_1,\ldots,H_k} = 0$$

for all $f \in L^\infty(\mu)$, where $X^{[k]} = X \times \cdots \times X$ (2^k copies of X), $|\epsilon| = \epsilon_1 + \cdots + \epsilon_k$ for $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{0, 1\}^k$, and $C^{2r+1} f = \overline{f}$, the complex conjugate of f, $C^{2r} f = f$ for all $r \in \mathbb{Z}$. The quantity $|||f|||_{H_1,\ldots,H_k}$ denotes the *Host–Kra seminorm* of f with respect to the subgroups $H_1, \ldots, H_k$. Similar to the proof of [17, Lemma 4] or [18, Lemma 4.3], one can show that $Z_{H_1,\ldots,H_k}(\mathbf{X})$ is well defined.

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THEOREM 2.11. *Let $d, k, K, L \in \mathbb{N}$, $\mathbf{p} = (p_1, \ldots, p_k)$, $p_1, \ldots, p_k \in \mathbb{Z}^L \to \mathbb{Z}^d$ be a family of essentially distinct polynomials of degrees at most K. There exists $D \in \mathbb{N}_0$ depending only on d, k, K, L such that for every $\mathbb{Z}^d$-system $\mathbf{X} = (X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$, every $f_1, \ldots, f_k \in L^\infty(\mu)$, and every Følner sequence $(I_N)_{N\in\mathbb{N}}$ of $\mathbb{Z}^L$, if f_i is orthogonal to the Host–Kra characteristic factor $Z_{\{G_{i,j}(\mathbf{p})\}^{\times D}_{0\leq j\leq k, j\neq i}}(\mathbf{X})$ for some $1 \leq i \leq k$ (that is, the conditional expectation of f_i under $Z_{\{G_{i,j}(\mathbf{p})\}^{\times D}_{0\leq j\leq k, j\neq i}}(\mathbf{X})$ is 0), then we have that*

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \prod_{i=1}^{k} T_{p_i(n)} f_i = 0. \tag{10}$$

In particular, if for some $1 \leq i \leq k$, $G_{i,j}(\mathbf{p})$ is ergodic for μ for all $0 \leq j \leq k$, $j \neq i$ and f_i is orthogonal to the Host–Kra characteristic factor $Z_{(\mathbb{Z}^d)^{\times kD}}(\mathbf{X})$, then equation (10) holds.

It is worth noting that the factor $Z_{\{G_{i,j}(\mathbf{p})\}^{\times D}_{0\leq j\leq k, j\neq i}}(\mathbf{X})$ we obtain in Theorem 2.11 is not optimal, but it is good enough for our purposes.

A special case of Theorem 2.11 was proved in [**7**, Theorem 5.1]. In particular, Theorem 2.11 generalizes [**7**, Theorem 5.1] in the following ways.

(I) The characteristic factor obtained in Theorem 2.11 is of finite step, whereas that in [**7**, Theorem 5.1] is of infinite step.

(II) The groups $G_{i,j}(\mathbf{p})$ involved in Theorem 2.11 are larger than those in [**7**, Theorem 5.1], which makes the characteristic factors in Theorem 2.11 smaller.

We remark that the aforementioned technical distinctions have significant influences on the applications of Theorem 2.11. First, the essential reason why one cannot directly use [**7**, Theorem 5.1] to improve [**9**, Theorem 1.5] is that the method used in [**9**] requires a characteristic factor of finite step. This problem is resolved by generalization (I), enabling us to extend [**9**, Theorem 1.5] in this paper. Second, [**7**, Theorem 5.1] does not provide a strong enough characteristic factor in certain circumstances. For example, in the case of Example 2.6, [**6**, Theorem 6.5] suggests that the Host–Kra seminorms controlling equation (10) depend only on the transformations $T_1, \ldots, T_k$, whereas the upper bound provided by [**7**, Theorem 5.1] depends not only on the transformations $T_1, \ldots, T_k$ but also on many compositions of them. With the help of generalizations (I) and (II), we are able to obtain (and generalize) the aforementioned upper bound of [**6**, Theorem 6.5].

Roughly speaking, the achievement of generalization (I) relies on a sophisticated development of a Bessel-type inequality first obtained by Tao and Ziegler in [**33**, Proposition 3.6]. The most technical part of this paper is the approach we use to get generalization (II). In [**7**], a method was introduced to keep track of the coefficients of the polynomials while running a variation of the polynomial exhaustion technique (PET) induction. However, the tracking provided there is not strong enough to imply Theorem 2.11. To overcome this difficulty, we introduce more sophisticated machinery to have a better control of the coefficients.

The paper is organized as follows. We provide some background material in §3. In §4, we present the variation of PET induction that we use. In §5, we address how generalizations (I) and (II) above can be achieved with Propositions 5.2 and 5.4, which

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improve Propositions 5.6 and 5.5 of [7], respectively. We conclude the section by proving Theorem 2.11. This is the bulk of the paper. In §6, we use Theorem 2.11 to deduce Theorems 2.2, 2.5, and 2.9, which are the main results of the paper. We conclude with some discussions on future directions in §7.

2.4. *Notation.* We denote by $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ the sets of positive integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers, respectively. If X is a set and $d \in \mathbb{N}$, X^d denotes the Cartesian product $X \times \cdots \times X$ of d copies of X.

We will denote by e_i the vector that has 1 as its ith coordinate and 0 elsewhere. We use in general lower-case letters to symbolize both numbers and vectors but bold letters to symbolize vectors of vectors to highlight this exact fact. The only exception to this convention is the vector $\mathbf{0}$ (that is, the vector with coordinates only 0) which we always symbolize in bold.

Throughout this article, we use the following notation for averages. Let $(a(n))_{n\in\mathbb{Z}^L}$ be a sequence of complex numbers, or a sequence of measurable functions on a probability space $(X, \mathcal{B}, \mu)$. We let:

$\mathbb{E}_{n\in A} a(n) := (1/|A|) \sum_{n\in A} a(n)$, where A is a finite subset of $\mathbb{Z}^L$;

$\overline{\mathbb{E}}^{\square}_{n\in\mathbb{Z}^L} a(n) := \overline{\lim}_{N\to\infty} \mathbb{E}_{n\in[-N,N]^L} a(n)$ (we use the symbol $\square$ to highlight the fact that the averages are taken along the boxes $[-N, N]^L$);

$\overline{\mathbb{E}}_{n\in\mathbb{Z}^L} a(n) := \sup_{\substack{(I_N)_{N\in\mathbb{N}} \\ \text{Følner seq.}}} \overline{\lim}_{N\to\infty} \mathbb{E}_{n\in I_N} a(n)$;

$\mathbb{E}^{\square}_{n\in\mathbb{Z}^L} a(n) := \lim_{N\to\infty} \mathbb{E}_{n\in[-N,N]^L} a(n)$ (provided that the limit exists); and

$\mathbb{E}_{n\in\mathbb{Z}^L} a(n) := \lim_{N\to\infty} \mathbb{E}_{n\in I_N} a(n)$ (provided the limit exists for all Følner sequence $(I_N)_{N\in\mathbb{N}}$). It is worth noticing that if the limit $\lim_{N\to\infty} \mathbb{E}_{n\in I_N} a(n)$ exists for all Følner sequences (in $\mathbb{Z}^L$), then this limit does not depend on the chosen Følner sequence.

We also consider *iterated* averages. Let $(a(h_1, \ldots, h_s))_{h_1,\ldots,h_s\in\mathbb{Z}^L}$ be a multiparameter sequence. We let

$$\overline{\mathbb{E}}_{h_1,\ldots,h_s\in\mathbb{Z}^L} a(h_1, \ldots, h_s) := \overline{\mathbb{E}}_{h_1\in\mathbb{Z}^L} \ldots \overline{\mathbb{E}}_{h_s\in\mathbb{Z}^L} a(h_1, \ldots, h_s)$$

and adopt similar conventions for $\mathbb{E}_{h_1,\ldots,h_s\in\mathbb{Z}^L}$, $\overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s\in\mathbb{Z}^L}$, and $\mathbb{E}^{\square}_{h_1,\ldots,h_s\in\mathbb{Z}^L}$.

We end this section by recalling the notion of a system indexed by a countable abelian group $(G, +)$. We say that a tuple $(X, \mathcal{B}, \mu, (T_g)_{g\in G})$ is a *G-measure-preserving system* (or a *G-system*) if $(X, \mathcal{B}, \mu)$ is a probability space and $T_g \colon X \to X$ are measurable, measure-preserving transformations on X such that $T_{e_G} = \text{id}$ (e_G is the identity element of G) and $T_g \circ T_h = T_{g+h}$ for all $g, h \in G$. A G-system will be called *ergodic* if for any $A \in \mathcal{B}$ such that $T_g A = A$ for all $g \in G$, we have that $\mu(A) \in \{0, 1\}$. In this paper, we are mostly concerned about $\mathbb{Z}^d$-systems and $L^2(\mu)$-norm limits of (multiple) ergodic averages. For the corresponding norm, when it is clear from the context, we will write $\|\cdot\|_2$ instead of $\|\cdot\|_{L^2(\mu)}$.

3. *Background material*

In this section, we recall some background material and prove some intermediate results that will be used later throughout the paper.

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We summarize some basic properties of the Host–Kra seminorms and their associated factors.

PROPOSITION 3.1. [7, Lemma 2.4] *Let* $\mathbf{X} = (X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ *be a* $\mathbb{Z}^d$*-system,* $H_1, \ldots, H_k, H'$ *be subgroups of* $\mathbb{Z}^d$ *and* $f \in L^\infty(\mu)$.

(i) *For every permutation* $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$, *we have that*

$$Z_{H_1,\ldots,H_k}(\mathbf{X}) = Z_{H_{\sigma(1)},\ldots,H_{\sigma(k)}}(\mathbf{X}),$$

and hence the corresponding seminorm does not depend on the particular order taken for the subgroups $H_1, \ldots, H_k$.

(ii) *If* $\mathcal{I}(H_j) = \mathcal{I}(H')$, *then* $Z_{H_1,\ldots,H_j,\ldots,H_k}(\mathbf{X}) = Z_{H_1,\ldots,H_{j-1},H',H_{j+1},\ldots,H_k}(\mathbf{X})$.

(iii) *For* $k \geq 2$, *we have that*

$$|||f|||_{H_1,\ldots,H_k}^{2^k} = \mathbb{E}_{g\in H_k} |||f \cdot T_g\overline{f}|||_{H_1,\ldots,H_{k-1}}^{2^{k-1}},$$

while for $k = 1$,

$$|||f|||_{H_1}^2 = \mathbb{E}_{g\in H_1} \int_X f \cdot T_g\overline{f}\, d\mu.$$

(iv) *Let* $k \geq 2$. *If* $H' \leq H_j$ *is of finite index, then*

$$Z_{H_1,\ldots,H_j,\ldots,H_k}(\mathbf{X}) = Z_{H_1,\ldots,H_{j-1},H',H_{j+1},\ldots,H_k}(\mathbf{X}).$$

(v) *If* $H' \leq H_j$, *then* $Z_{H_1,\ldots,H_j,\ldots,H_k}(\mathbf{X}) \subseteq Z_{H_1,\ldots,H_{j-1},H',H_{j+1},\ldots,H_k}(\mathbf{X})$.

(vi) *For* $k \geq 2$, $|||f|||_{H_1,\ldots,H_{k-1}} \leq |||f|||_{H_1,\ldots,H_{k-1},H_k}$ *and thus*

$$Z_{H_1,\ldots,H_{k-1}}(\mathbf{X}) \subseteq Z_{H_1,\ldots,H_{k-1},H_k}(\mathbf{X}).$$

(vii) *For* $k \geq 1$, *if* $H_1', \ldots, H_k'$ *are subgroups of* $\mathbb{Z}^d$, *then*

$$Z_{H_1,\ldots,H_k}(\mathbf{X}) \vee Z_{H_1',\ldots,H_k'}(\mathbf{X}) \subseteq Z_{H_1',\ldots,H_k',H_1,\ldots,H_k}(\mathbf{X}).$$

As an immediate corollary of Proposition 3.1(iv), we have the following corollary.

COROLLARY 3.2. [7, Corollary 2.5] *Let* $H_1, \ldots, H_k$ *be subgroups of* $\mathbb{Z}^d$. *If the* H_i*-action* $(T_g)_{g\in H_i}$ *is ergodic on* $\mathbf{X}$ *for all* $1 \leq i \leq k$, *then* $Z_{H_1,\ldots,H_k}(\mathbf{X}) = Z_{\mathbb{Z}^d,\ldots,\mathbb{Z}^d}(\mathbf{X})$.

Convention 3.3. Thanks to Proposition 3.1, we may adopt a flexible and convenient notation while writing the Host–Kra characteristic factors. For example, if $A = \{H_1, H_2\}^{\times 3}$, then the notation $Z_{A,H_3,H_4^{\times 2},(H_i)_{i=5,6}}(\mathbf{X})$ refers to $Z_{H_1,H_1,H_1,H_2,H_2,H_2,H_3,H_4,H_4,H_5,H_6}(\mathbf{X})$ (note that thanks to Proposition 3.1(i), $Z_{A,H_3,H_4^{\times 2},(H_i)_{i=5,6}}(\mathbf{X})$ is well defined regardless of the ordering of A).

Recall that for a subgroup $H \subseteq \mathbb{Z}^d$, $H^{[1]}$ denotes the subgroup $\{(h, h) : h \in H\} \subseteq \mathbb{Z}^d \times \mathbb{Z}^d$.

LEMMA 3.4. *Let* $d \in \mathbb{N}$. *Let* $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ *be a* $\mathbb{Z}^d$*-system and* $H_1, \ldots, H_k, H$ *be subgroups of* $\mathbb{Z}^d$. *Let* $f \in L^\infty(\mu)$. *Then,*

$$|||f \otimes \bar{f}|||_{H_1^{[1]},\ldots,H_k^{[1]}} \leq |||f|||_{H_1,\ldots,H_k,H}^2,$$

where in the left-hand side, we consider the product space $(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu, (T_m \times T_n)_{(m,n)\in\mathbb{Z}^{2d}})$.

Proof. We proceed by induction on k. For $k = 1$, using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|||f \otimes \overline{f}|||^2_{H_1^{[1]}} &= \mathbb{E}_{g\in H_1} \int f \otimes \overline{f} \cdot (T_g \times T_g)\overline{f} \otimes f \, d(\mu \times \mu) \\
&= \mathbb{E}_{g\in H_1} \bigg| \int T_g f \cdot \overline{f} d\mu \bigg|^2 = \mathbb{E}_{g\in H_1} \bigg| \int \mathbb{E}(T_g f \cdot \overline{f} | \mathcal{I}(H)) \, d\mu \bigg|^2 \\
&\leq \mathbb{E}_{g\in H_1} \int |\mathbb{E}(T_g f \cdot \overline{f} | \mathcal{I}(H))|^2 \, d\mu \\
&= \mathbb{E}_{g\in H_1} |||T_g f \cdot \overline{f}|||^2_H = |||f|||^4_{H,H_1} = |||f|||^4_{H_1,H},
\end{aligned}$$

where we used in the last two equalities Proposition 3.1(iii) and (i), respectively, from where we conclude the required relation by taking square roots.

Suppose that the result holds for $k - 1$. By Proposition 3.1(i) and the induction hypothesis,

$$\begin{aligned}
|||f \otimes \overline{f}|||^{2^k}_{H_1^{[1]},\ldots,H_k^{[1]}} &= \mathbb{E}_{g\in H_k} |||(T_g \times T_g) f \otimes \overline{f} \cdot \overline{f} \otimes f|||^{2^{k-1}}_{H_1^{[1]},\ldots,H_{k-1}^{[1]}} \\
&= \mathbb{E}_{g\in H_k} |||T_g f \cdot \overline{f} \otimes T_g \overline{f} \cdot f|||^{2^{k-1}}_{H_1^{[1]},\ldots,H_{k-1}^{[1]}} \\
&\leq \mathbb{E}_{g\in H_k} |||T_g f \cdot \overline{f}|||^{2^k}_{H_1,\ldots,H_{k-1},H} \\
&= |||f|||_{H_1,\ldots,H_{k-1},H,H_k} = |||f|||_{H_1,\ldots,H_{k-1},H_k,H}
\end{aligned}$$

and the claim follows. □

3.1. *Nilsystems, nilsequences, and structure theorem.* Let $X = N/\Gamma$, where N is a (k-step) nilpotent Lie group and Γ is a discrete cocompact subgroup of N. Let $\mathcal{B}$ be the Borel σ-algebra of X, μ the normalized Haar measure on X, and for $n \in \mathbb{Z}^d$, let $T_n \colon X \to X$ with $T_n x = b_n \cdot x$ for some group homomorphism $n \mapsto b_n$ from $\mathbb{Z}^d$ to N. We say that $\mathbf{X} = (X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ is a (k-*step*) $\mathbb{Z}^d$-*nilsystem*. For $k \geq 1$, we say that $(a_n)_{n\in\mathbb{Z}^d} \subseteq \mathbb{C}$ is a (k-*step*) $\mathbb{Z}^d$-*nilsequence* if there exist a (k-step) $\mathbb{Z}^d$-nilsystem $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$, a function $F \in C(X)$ and $x \in X$ such that $a_n = F(T_n x)$ for all $n \in \mathbb{Z}^d$. For $k = 0$, a 0-*step nilsequence* is a constant sequence. An important reason which makes the Host–Kra characteristic factors powerful is their connection with nilsystems. The following is a slight generalization of [**36**, Theorem 3.7] (see [**16**, Lemma 4.4.3 and Theorem 4.10.1], or Proposition 3.1(ii) and [**31**, Theorem 3.7]), which is a higher dimensional version of the Host–Kra structure theorem [**18**].

THEOREM 3.5. *Let* $\mathbf{X}$ *be an ergodic* $\mathbb{Z}^d$*-system. Then* $Z_{(\mathbb{Z}^d)^{\times k}}(\mathbf{X})$ *is an inverse limit of* $(k-1)$*-step* $\mathbb{Z}^d$*-nilsystems.*

3.2. *Bessel's inequality.* An essential difference in the study of multiple ergodic averages between $\mathbb{Z}$-systems and $\mathbb{Z}^d$-systems is that in the former case, one can usually bound the average by some Host–Kra seminorm of a function f appearing in the average, whereas in the latter, one can only bound the averages by an average of a family of Host–Kra seminorms of f. To overcome this difficulty, inspired by the work of Tao and Ziegler [**33**], in

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this subsection, we derive an upper bound for expressions of the form $\overline{\mathbb{E}}_{i\in I}|||f|||_{H_{i,1},\ldots,H_{i,s}}$, where I is a finite set and $H_{i,j}$ are subgroups of $\mathbb{Z}^d$.

The proof of the following statement is similar to [33, Corollary 1.22].

PROPOSITION 3.6. (Bessel's inequality) *Let $t\in\mathbb{N}$, $(X,\mathcal{B},\mu,(T_g)_{g\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system, I be a finite set of indices, and $H_{i,j}$, $i\in I$, $1\le j\le t$ be subgroups of $\mathbb{Z}^d$. Then for all $f\in L^\infty(\mu)$,*

$$\mathbb{E}_{i\in I}\|\mathbb{E}(f|Z_{H_{i,1},\ldots,H_{i,t}})\|_2^2\le\|f\|_2\cdot(\mathbb{E}_{i,j\in I}\|\mathbb{E}(f|Z_{\{H_{i,i'}+H_{j,j'}\}_{1\le i',j'\le t}})\|_2^2)^{1/2}.$$

Proof. For convenience, let $f_i:=\mathbb{E}(f|Z_{H_{i,1},\ldots,H_{i,t}})$. Then,

$$\mathbb{E}_{i\in I}\|\mathbb{E}(f|Z_{H_{i,1},\ldots,H_{i,t}})\|_2^2=\langle f,\mathbb{E}_{i\in I}f_i\rangle$$

which, by the Cauchy–Schwarz inequality, is bounded by

$$\|f\|_2\cdot|\mathbb{E}_{i,j\in I}\langle f_i,f_j\rangle|^{1/2}.$$

By [33, Corollary 1.21], $L^\infty(Z_{H_{i,1},\ldots,H_{i,t}})$ and $L^\infty(Z_{H_{j,1},\ldots,H_{j,t}})$ are orthogonal on the orthogonal complement of $L^\infty(Z_{\{H_{i,i'}+H_{j,j'}\}_{1\le i',j'\le t}})$, and hence

$$\langle f_i,f_j\rangle=\|\mathbb{E}(f|Z_{\{H_{i,i'}+H_{j,j'}\}_{1\le i',j'\le t}})\|_2^2,$$

and we have the conclusion. □

By repeatedly using Proposition 3.6, we have the following inequality.

COROLLARY 3.7. *Let $t,s\in\mathbb{N}$, $(X,\mathcal{B},\mu,(T_g)_{g\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system, I be a finite set of indices, and $H_{i,j}$, $i\in I$, $1\le j\le t$, be subgroups of $\mathbb{Z}^d$. Then for all $f\in L^\infty(\mu)$, we have*

$$(\mathbb{E}_{i\in I}\|\mathbb{E}(f|Z_{H_{i,1},\ldots,H_{i,t}})\|_2^2)^{2^s}\le\|f\|_2^{2\cdot 2^s-2}\cdot\mathbb{E}_{i_1,\ldots,i_{2^s}\in I}\Big\|\mathbb{E}\Big(f\Big|Z_{\{\sum_{j=1}^{2^s}H_{i_j,i'_j}\}_{1\le i'_1,\ldots,i'_{2^s}\le t}}\Big)\Big\|_2^2.$$

The next proposition provides an upper bound for $\mathbb{E}_{i\in I}|||f|||_{H_{i,1},\ldots,H_{i,t}}$ which can be combined with the previous two statements.

PROPOSITION 3.8. *Let $t\in\mathbb{N}$, $(X,\mathcal{B},\mu,(T_g)_{g\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system, I be a finite set of indices, and $H_{i,j}$, $i\in I$, $1\le j\le t$ be subgroups of $\mathbb{Z}^d$. Then, for all $f\in L^\infty(\mu)$, with $\|f\|_{L^\infty(\mu)}\le 1$,*

$$\mathbb{E}_{i\in I}|||f|||_{H_{i,1},\ldots,H_{i,t}}\le(\mathbb{E}_{i\in I}\|\mathbb{E}(f|Z_{H_{i,1},\ldots,H_{i,t}})\|_2^2)^{1/2^t}.$$

Proof. Note that

$$|||f|||_{H_{i,1},\ldots,H_{i,t}}\le\|f\|_{L^{2^t}(\mu)}\le\|f\|_2^{1/2^{t-1}}. \tag{11}$$

Also, for all i, we have

$$\begin{aligned}|||f|||_{H_{i,1},\ldots,H_{i,t}}&\le|||f-\mathbb{E}(f|Z_{H_{i,1},\ldots,H_{i,t}})|||_{H_{i,1},\ldots,H_{i,t}}+|||\mathbb{E}(f|Z_{H_{i,1},\ldots,H_{i,t}})|||_{H_{i,1},\ldots,H_{i,t}}\\&=|||\mathbb{E}(f|Z_{H_{i,1},\ldots,H_{i,t}})|||_{H_{i,1},\ldots,H_{i,t}},\end{aligned}$$

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so

$$\mathbb{E}_{i\in I}|||f|||_{H_{i,1},\ldots,H_{i,t}} \leq \mathbb{E}_{i\in I}|||\mathbb{E}(f|Z_{H_{i,1},\ldots,H_{i,t}})|||_{H_{i,1},\ldots,H_{i,t}}$$
$$\leq \mathbb{E}_{i\in I}\|\mathbb{E}(f|Z_{H_{i,1},\ldots,H_{i,t}})\|_2^{1/2^{t-1}} \leq (\mathbb{E}_{i\in I}\|\mathbb{E}(f|Z_{H_{i,1},\ldots,H_{i,t}})\|_2^2)^{1/2^t}, \tag{12}$$

as was to be shown. □

3.3. *General properties of subgroups of $\mathbb{Z}^d$ and properties of polynomials.* Recall that for a subset A of $\mathbb{Q}^d$, we denote $G(A) := \text{span}_{\mathbb{Q}}\{a \in A\} \cap \mathbb{Z}^d$. Next, we summarize some properties of these sets.

LEMMA 3.9. *The following properties hold.*

(i) *For any set $A \subseteq \mathbb{Z}^d$, $G(A)$ is a subgroup of $\mathbb{Z}^d$.*

(ii) *Let $A \subseteq \mathbb{Q}^d$ be a finite set and $M(A)$ the matrix whose columns are the elements of A. Then $G(A) = (M(A) \cdot \mathbb{Q}^{|A|}) \cap \mathbb{Z}^d$.*

(iii) *If $H \subseteq \mathbb{Z}^d$ is the subgroup generated by $h_1, \ldots, h_k \in \mathbb{Z}^d$, then $G(H) = G(\{h_1, \ldots, h_k\})$. In particular, letting $M(h_1, \ldots, h_k)$ be the matrix whose columns are $h_1, \ldots, h_k$, we have that $G(\langle h_1, \ldots, h_k\rangle) = (M(h_1, \ldots, h_k) \cdot \mathbb{Q}^k) \cap \mathbb{Z}^d$.*

(iv) *For any subgroup $H \subseteq \mathbb{Z}^d$, H has finite index in $G(H)$. Moreover, $G(H)$ is the largest subgroup of $\mathbb{Z}^d$ which is a finite index extension of H.*

(v) *If not all of $a_1, \ldots, a_k$ belong to a common proper subspace of $\mathbb{Q}^d$, then $G(\{a_1, \ldots, a_k\}) = \mathbb{Z}^d$.*

Proof. Properties (i), (ii), and (iii) follow directly from the definitions.

To prove property (iv), let $\{g_1, \ldots, g_k\}$ be a set such that $\langle g_1, \ldots, g_k\rangle = G(H)$. For each $i = 1, \ldots, k$, there exist m_i and $h_i \in H$ such that $g_i = h_i/m_i$. The group $\langle m_1 g_1, \ldots, m_k g_k\rangle$ is of finite index in $\langle g_1, \ldots, g_k\rangle = G(H)$ and is contained in H. Therefore, H is of finite index in $G(H)$.

To see that $G(H)$ is the largest finite index extension of H, take H' to be any finite index extension of H and take $h' \in H'$. Since H' is a finite index extension of H, we have that there exists $n \in \mathbb{N}$ such that $nh' \in H$. This implies that $h' \in G(H)$.

To show property (v), reordering $a_1, \ldots, a_k$ if needed, we may assume that $a_1, \ldots, a_d$ are linearly independent vectors over $\mathbb{Q}$. It follows that $\text{span}_{\mathbb{Q}}(\{a_1, \ldots, a_d\}) = \mathbb{Q}^d$ and then $G(\{a_1, \ldots, a_k\}) \supseteq G(\{a_1, \ldots, a_d\}) = \mathbb{Z}^d$. □

Remark 3.10. If H_1 and H_2 are subgroups of $\mathbb{Z}^d$, then $G(H_1) + G(H_2) \subseteq G(H_1 + H_2)$, with the inclusion possibly being strict. For instance, for $H_1 = \langle(1, 2)\rangle$, $H_2 = \langle(2, 1)\rangle$, we have that $G(H_1) = H_1$, $G(H_2) = H_2$, and $H_1 + H_2 \subsetneq G(H_1 + H_2) = \mathbb{Z}^2$. Nevertheless, Lemma 3.9 implies that that $G(H_1) + G(H_2)$ has finite index in $G(H_1 + H_2)$.

In the remainder of the section, we provide some algebraic lemmas that will be used later in the paper. For a set $E \subseteq \mathbb{Z}^d$, we define its *upper Banach density* (or just *upper density* when there is no confusion) by $d^*(E) := \overline{\lim}_{N\to\infty} \max_{t\in\mathbb{Z}^d} |(E - t) \cap \{1, \ldots, N\}^d|/N^d$. If the limit exists, we say that its value is the *Banach density* (or just *density*) of E. The proof of the following lemma is routine (see also [7, Lemma 2.11] for a more general version).

LEMMA 3.11. [7, Lemma 2.11] *Let* $\mathbf{c}\colon (\mathbb{Z}^L)^s \to \mathbb{R}$ *be a polynomial. Then either* $\mathbf{c} \equiv 0$ *or the set of* $\mathbf{h} \in (\mathbb{Z}^L)^s$ *such that* $\mathbf{c}(\mathbf{h}) = 0$ *is of (upper) Banach density* 0.

LEMMA 3.12. *Let* $v_i \in \mathbb{Z}^L$, $1 \leq i \leq k$ *and* U *be a subset of* $\mathbb{Z}^k$ *of positive density. Then,*

$$G\bigg(\bigg\{ \sum_{1\leq i\leq k} h_i v_i : \mathbf{h} = (h_1, \ldots, h_k) \in U \bigg\}\bigg) = G(\{v_i : 1 \leq i \leq k\}). \tag{13}$$

Proof. Note that in equation (13), the right-hand side clearly includes the left-hand side. To prove the converse inclusion, it suffices to show that

$$\mathrm{span}_{\mathbb{Q}}\{\mathbf{h} \colon \mathbf{h} \in U\} = \mathbb{Q}^k. \tag{14}$$

Since U has positive density, it cannot be contained in any hyperplane of $\mathbb{Q}^k$, so it must have at least k elements that are linearly independent over $\mathbb{Q}$. Thus, equation (14) follows immediately. □

Definition 3.13. Let $P\colon (\mathbb{Z}^L)^D \to \mathbb{R}$ be a polynomial. Denote by $\Delta P\colon (\mathbb{Z}^L)^{D+1} \to \mathbb{R}$ the polynomial given by $\Delta P(n, h_1, \ldots, h_D) := P(n + h_D, h_1, \ldots, h_{D-1}) - P(n, h_1, \ldots, h_{D-1})$ for all $n, h_1, \ldots, h_D \in \mathbb{Z}^L$. For a polynomial $P\colon \mathbb{Z}^L \to \mathbb{R}$, let $\Delta^0 P = P$, and for $K > 1$, $\Delta^K P\colon (\mathbb{Z}^L)^{D+K} \to \mathbb{R}$ is $\Delta^K P := \Delta \cdots \Delta P$ (where Δ acts K times).

LEMMA 3.14. *Let* $K \in \mathbb{N}$ *and* $Q\colon \mathbb{Z}^L \to \mathbb{R}$ *be a homogeneous polynomial with* $\deg(Q) > K$. *If* $Q(n) \notin \mathbb{Q}[n]$, *then the set of* $(h_1, \ldots, h_K) \in (\mathbb{Z}^L)^K$ *such that* $\Delta^K Q(n, h_1, \ldots, h_K) \notin \mathbb{Q}[n]$ *is of density* 1 *in* $(\mathbb{Z}^L)^K$.

Proof. We may write $Q(n) = \sum_{i=1}^M a_i Q_i(n)$ for some $M \in \mathbb{N}$, homogeneous polynomials $Q_1, \ldots, Q_M$ in $\mathbb{Q}[n]$ of degrees $\deg(Q)$, and real numbers $a_1, \ldots, a_M \in \mathbb{R}$ which are linearly independent over $\mathbb{Q}$ (this can be done by taking $a_1 \ldots, a_M$ to be a basis of the $\mathbb{Q}$-span of the coefficients of Q). Since $Q(n) \notin \mathbb{Q}[n]$, there exists some $1 \leq i \leq M$ such that $a_i \notin \mathbb{Q}$ and $Q_i \not\equiv 0$. Without loss of generality, assume that $i = 1$. Since $\deg(Q_1) > K$, we have that $\Delta^K Q_1 \not\equiv 0$.

Suppose that $\Delta^K Q(n, h_1, \ldots, h_K) \in \mathbb{Q}[n]$ for some $(h_1, \ldots, h_K) \in (\mathbb{Z}^L)^K$. Note that $\Delta^K Q(n, h_1, \ldots, h_K) = \sum_{i=1}^M a_i \Delta^K Q_i(n, h_1, \ldots, h_K)$. Since each $\Delta^K Q_i(n, h_1, \ldots, h_K)$ is a rational polynomial in terms of n of degree $\deg(Q) - K$ and $a_1, \ldots, a_M \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, we must have that $\Delta^K Q_1(\cdot, h_1, \ldots, h_K) \equiv 0$. So if the set of $(h_1, \ldots, h_K) \in (\mathbb{Z}^L)^K$ such that $\Delta^K Q(n, h_1, \ldots, h_K) \in \mathbb{Q}[n]$ has positive density, then the set of $(n, h_1, \ldots, h_K) \in (\mathbb{Z}^L)^{K+1}$ such that $\Delta^K Q_1(n, h_1, \ldots, h_K) = 0$ has positive density too. By [7, Lemma 2.11], $\Delta^K Q_1 \equiv 0$, which is a contradiction. This finishes the proof. □

4. *PET induction*

In this section, we present the method we use to reduce the complexity of the polynomial iterates, that is, PET induction (PET is an abbreviation for 'Polynomial Exhaustion Technique'), which was first introduced in [2]. To this end, we start by recalling a variation of van der Corput's lemma from [7] that is convenient for our study. We then continue by presenting the inductive scheme via the use of van der Corput operations.

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4.1. *The van der Corput lemma.* The standard tool used in reducing the complexity of polynomial families of iterates is van der Corput's lemma (also known as 'van der Corput's trick'). We will use the following variation of it, the proof of which can be found in [7, Lemma 2.2].

LEMMA 4.1. (van der Corput lemma) *Let* $(a(n; h_1, \ldots, h_s))_{(n;h_1,\ldots,h_s)\in(\mathbb{Z}^L)^{s+1}}$, $s \in \mathbb{N}_0$, *be a sequence bounded by* 1 *in a Hilbert space* $\mathcal{H}$. *Then, for all* $\tau \in \mathbb{N}_0$,

$$\overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s\in\mathbb{Z}^L} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} a(n; h_1, \ldots, h_s)\|^{2\tau}$$

$$\leq 4^{\tau}\overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s,h_{s+1}\in\mathbb{Z}^L} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} |\mathbb{E}_{n\in I_N}\langle a(n + h_{s+1}; h_1, \ldots, h_s), a(n; h_1, \ldots, h_s)\rangle|^{\tau}.$$

Remark 4.2. We use this unorthodox notation to separate the variable n from the h_i terms. The variable n plays a different role in our study than the h_i terms.

We also provide two applications of Lemma 4.1 for later use. The first one is to get an upper bound for single averages with polynomial iterates and a polynomial exponential weight. Let $\exp(x) := e^{2\pi i x}$ and recall Definition 3.13 for the polynomial $\Delta^K P$.

LEMMA 4.3. *Let* $P\colon \mathbb{Z}^L \to \mathbb{R}$ *and* $p\colon \mathbb{Z}^L \to \mathbb{Z}^d$ *be polynomials. For all* $K \in \mathbb{N}_0$ *and* $\tau \in \mathbb{N}$, *there exists* $C_{K,\tau} > 0$ *such that for every* $\mathbb{Z}^d$*-system,* $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$, *and* $f \in L^\infty(\mu)$ *bounded by 1, we have*

$$\sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} \exp(P(n))T_{p(n)} f\|_2^{2\tau}$$

$$\leq C_{K,\tau}\overline{\mathbb{E}}^{\square}_{\mathbf{h}=(h_1,\ldots,h_K)\in(\mathbb{Z}^L)^K} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} \exp(\Delta^K P(n, \mathbf{h}))T_{\Delta^K p(n,\mathbf{h})} f\|_2^{\tau}.$$

Proof. When $K = 0$, there is nothing to prove. We now assume that the relation holds for some $K \in \mathbb{N}_0$ and we show it for $K + 1$. Using Lemma 4.1 and the T-invariance of μ, we get

$$\overline{\mathbb{E}}^{\square}_{\mathbf{h}=(h_1,\ldots,h_K)\in(\mathbb{Z}^L)^K} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} \exp(\Delta^K P(n, \mathbf{h}))T_{\Delta^K p(n,\mathbf{h})} f\|_2^{2\tau}$$

$$\leq 4^{\tau}\overline{\mathbb{E}}^{\square}_{\mathbf{h}=(h_1,\ldots,h_{K+1})\in(\mathbb{Z}^L)^{K+1}} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \Big|\mathbb{E}_{n\in I_N}\int_X \exp(\Delta^{K+1}P(n, \mathbf{h}))T_{\Delta^{K+1}p(n,\mathbf{h})} f\cdot\overline{f}\, d\mu\Big|^{\tau}$$

$$\leq 4^{\tau}\overline{\mathbb{E}}^{\square}_{\mathbf{h}=(h_1,\ldots,h_{K+1})\in(\mathbb{Z}^L)^{K+1}} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} \exp(\Delta^{K+1} P(n, \mathbf{h}))T_{\Delta^{K+1} p(n,\mathbf{h})} f\|_2^{\tau},$$

and hence the result (the constant that appears depends only on τ and K). □

The second application of Lemma 4.1 provides an upper bound for single averages, with linear iterates and an exponential weight evaluated at a linear polynomial, on a product system. The proof is inspired by [7, Lemma 5.2] and [19, Proposition 2.9].

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LEMMA 4.4. *Let $(X, \mathcal{B}, \mu)$ be a probability space, $k, L \in \mathbb{N}$ and $T_{i,j}$, $1 \le i \le k$, $1 \le j \le L$ be commuting measure-preserving transformations on X. Denote $S_j = T_{1,j} \times \cdots \times T_{k,j}$ for $1 \le j \le L$. Let G_i be the group generated by $T_{i,1}, \ldots, T_{i,L}$. Then, for any polynomial $P\colon \mathbb{Z}^L \to \mathbb{R}$ of degree 1 and $f_1, \ldots, f_k \in L^\infty(\mu)$ bounded by 1, we have that*

$$\sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} \exp(P(n)) R_n f\|_{L^2(\mu^{\otimes k})} \le 2 \min_{1\le i\le k} |||f_i|||_{G_i^{\times 2}}, \tag{15}$$

where $f = f_1 \otimes \cdots \otimes f_k$ and for $n = (n_1, \ldots, n_L)$, $R_n := S_1^{n_1} \cdots S_L^{n_L}$.

Proof. Fix $1 \le i \le k$ and let $P(n) = a \cdot n + b$ for some $a \in \mathbb{R}^L, b \in \mathbb{R}$. Then, by Lemma 4.1 for $\tau = 2$ and $s = 0$, the fourth power of the left-hand side of equation (15) is bounded by

$$\begin{aligned}
&16 \cdot \mathbb{E}^{\square}_{h\in\mathbb{Z}^L} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \bigg| \int_{X^k} \mathbb{E}_{n\in I_N} \exp(P(n+h) - P(n)) R_{n+h} f \cdot R_n \overline{f}\, d\mu^{\otimes k} \bigg|^2 \\
&= 16 \cdot \mathbb{E}^{\square}_{h\in\mathbb{Z}^L} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \bigg| \int_{X^k} \mathbb{E}_{n\in I_N} \exp(a\cdot h) R_h f \cdot \overline{f}\, d\mu^{\otimes k} \bigg|^2 \\
&= 16 \cdot \mathbb{E}^{\square}_{h\in\mathbb{Z}^L} \bigg| \int_{X^k} R_h f \cdot \overline{f}\, d\mu^{\otimes k} \bigg|^2 \\
&= 16 \cdot \mathbb{E}^{\square}_{h=(h_1,\ldots,h_L)\in\mathbb{Z}^L} \bigg| \int_{X^k} \bigotimes_{i=1}^{k} \bigg(\Big(\prod_{j=1}^{L} T_{i,j}^{h_j}\Big) f_i \cdot \overline{f}_i \bigg)\, d\mu^{\otimes k} \bigg|^2 \\
&\le 16 \cdot \mathbb{E}^{\square}_{h=(h_1,\ldots,h_L)\in\mathbb{Z}^L} \bigg| \int_{X} \bigg(\prod_{j=1}^{L} T_{i,j}^{h_j} \bigg) f_i \cdot \overline{f}_i\, d\mu \bigg|^2 \\
&\le 16 \cdot \mathbb{E}^{\square}_{h=(h_1,\ldots,h_L)\in\mathbb{Z}^L} \bigg| \int_{X} \mathbb{E}\bigg(\Big(\prod_{j=1}^{L} T_{i,j}^{h_j}\Big) f_i \cdot \overline{f}_i | \mathcal{I}(G_i) \bigg)\, d\mu \bigg|^2 \\
&= 16 |||f_i|||^4_{G_i^{\times 2}},
\end{aligned}$$

from where the result follows. □

4.2. *The van der Corput operation.* To review the PET induction scheme, we will follow, and slightly modify, the approach from [7]. To this end, we extend the definitions that we have already given on the polynomial families of interest (see the beginning of §2.1), taking into account that we treat the first L-tuple of variables of the polynomials differently.

Before we list the steps of the van der Corput operation, we will present the manipulations of the inner product in Lemma 4.1 in a simple example where we have three essentially distinct polynomial iterates $(p_1(n), p_2(n), p_3(n)) = (n^2, 2n, n)$, to show how, by repeatedly running the van der Corput trick, we get an expression of linear iterates. This will be extended to general expressions in Theorem 4.9. Here, we want to study, for

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bounded by 1 functions f_1, f_2, f_3, the average of the sequence $a(n) = T_1^{n^2} f_1 \cdot T_2^{2n} f_2 \cdot T_2^n f_3$. Notice that we can write this sequence as a $\mathbb{Z}^2$-action, $a(n) = T_{(n^2,0)} f_1 \cdot T_{(0,2n)} f_2 \cdot T_{(0,n)} f_3$ for the triple of polynomials $((n^2, 0), (0, 2n), (0, n))$. Using Lemma 4.1, we have

$$\sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} a(n)\|^2$$

$$\leq 4\overline{\mathbb{E}}^{\square}_{h_1\in\mathbb{Z}} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} |\mathbb{E}_{n\in I_N} \langle a(n+h_1), a(n)\rangle|$$

$$= 4\overline{\mathbb{E}}^{\square}_{h_1\in\mathbb{Z}} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \Big|\mathbb{E}_{n\in I_N} \int_X T_1^{(n+h_1)^2} f_1 \cdot T_2^{2(n+h_1)} f_2 \cdot T_2^{n+h_1} f_3 \cdot T_1^{n^2} \bar{f}_1$$

$$\times\, T_2^{2n} \bar{f}_2 \cdot T_2^n \bar{f}_3 \, d\mu\Big|.$$

Using the fact that T_2 is measure-preserving, we compose by T_2^{-n} (notice that n is the polynomial of the minimum degree in the expression) to get

$$4\overline{\mathbb{E}}^{\square}_{h_1\in\mathbb{Z}} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \bigg|\mathbb{E}_{n\in I_N} \int_X \bar{f}_3 \cdot T_2^{h_1} f_3 \cdot T_1^{n^2+2h_1 n} T_2^{-n} (T_1^{h_1^2} f_1)$$

$$\times\, T_1^{n^2} T_2^{-n} \bar{f}_1 \cdot T_2^n (T_2^{2h_1} f_2 \cdot \bar{f}_2) \, d\mu\bigg|,$$

where we grouped the functions with the same linear terms.

Using the Cauchy–Schwarz inequality (to discard the terms that have iterates independent of n), the previous relation is bounded by

$$\overline{\mathbb{E}}^{\square}_{h_1\in\mathbb{Z}} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} T_1^{n^2+2h_1 n} T_2^{-n} (T_1^{h_1^2} f_1) \cdot T_1^{n^2} T_2^{-n} \bar{f}_1 \cdot T_2^n (T_2^{2h_1} f_2 \cdot \bar{f}_2)\|.$$

Exactly because of the grouping of the terms of the same linear iterates, the resulting polynomial iterates, $(n^2 + 2h_1 n, -n)$, $(n^2, -n)$, $(0, n)$, have the property that they are non-constant and that their pairwise differences are non-constant (this will lead us below to the notion of the 'essentially distinct' vector-valued polynomials).

Similarly, skipping the details, using Lemma 4.1, composing with T_2^{-n} (the polynomial $(0, n)$ is of minimum 'degree'—see below for the definition of the degree of a vector-valued polynomial), the square of the previous quantity can be bounded by

$$\overline{\mathbb{E}}^{\square}_{(h_1,h_2)\in\mathbb{Z}^2} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} T_1^{n^2+2(h_1+h_2)n} T_2^{-2n} (T_1^{(h_1+h_2)^2} T_2^{-h_2} f_1) \cdot T_1^{n^2+2h_1 n} T_2^{-2n} (T_1^{h_1^2} \bar{f}_1)$$

$$\times\, T_1^{n^2+2h_2 n} T_2^{-2n} (T_1^{h_2^2} T_2^{-h_2} \bar{f}_1) \cdot T_1^{n^2} T_2^{-2n} f_1\|.$$

Note that the iterates in the previous relation are 'essentially distinct' for 'almost all' tuples $(h_1, h_2) \in \mathbb{Z}^2$.

Analogously, using Lemma 4.1 once more, noticing that all the resulting terms in the expression will have the factor $T_1^{n^2} T_2^{-2n}$, where $(n^2, -2n)$ is the polynomial of minimum 'degree', we can bound, composing with the term $T_1^{-n^2}T_2^{2n}$, the square of the previous relation by

$$\overline{\mathbb{E}}^{\square}_{(h_1,h_2,h_3)\in\mathbb{Z}^3} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} T_1^{2(h_1+h_2+h_3)n}(T_1^{(h_1+h_2+h_3)^2}T_2^{-h_2-2h_3}f_1)$$
$$\times T_1^{2(h_2+h_3)n}(T_1^{(h_2+h_3)^2}T_2^{-h_2-2h_3}\bar{f}_1)\cdot T_1^{2(h_1+h_3)n}(T_1^{(h_1+h_3)^2}T_2^{-2h_3}\bar{f}_1)\cdot T_1^{2h_3n}(T_1^{h_3^2}T_2^{-2h_3}f_1)$$
$$\times T_1^{2(h_1+h_2)n}(T_1^{(h_1+h_2)^2}T_2^{-h_2}\bar{f}_1)\cdot T_1^{2h_2n}(T_1^{h_2^2}T_2^{-h_2}f_1)\cdot T_1^{2h_1n}(T_1^{h_1^2}f_1)\,d\mu\|.$$

The iterates in this last relation are linear with distinct coefficients for 'almost all' tuples $(h_1, h_2, h_3) \in \mathbb{Z}^3$. So, the eighth power of the initial expression is bounded by the previous relation.

The previous example leads naturally to the following notions.

Definition 4.5. For a polynomial $p(n; h_1, \ldots, h_s) \colon (\mathbb{Z}^L)^{s+1} \to \mathbb{Z}$, we denote by $\deg(p)$ *the degree of p with respect to n* (for example, for $s = 1$, $L = 2$, the degree of $p(n_1, n_2; h_{1,1}, h_{1,2}) = h_{1,1}h_{1,2}n_1^2 + h_{1,1}^5 n_2$ is 2).

For a polynomial $p(n; h_1, \ldots, h_s) = (p_1(n; h_1, \ldots, h_s), \ldots, p_d(n; h_1, \ldots, h_s)) \colon (\mathbb{Z}^L)^{s+1} \to \mathbb{Z}^d$, we let $\deg(p) = \max_{1\le i\le d} \deg(p_i)$ and we say that p is *non-constant* if $\deg(p) > 0$ (that is, some p_i is a non-constant function of n), otherwise, we say that p is *constant*. The polynomials $q_1, \ldots, q_k \colon (\mathbb{Z}^L)^{s+1} \to \mathbb{Z}^d$ are called *essentially distinct* if they are non-constant and $q_i - q_j$ is non-constant for all $i \neq j$. Finally, for a tuple $\mathbf{q} = (q_1, \ldots, q_k)$, we let $\deg(\mathbf{q}) = \max_{1\le i\le k} \deg(q_i)$. (For clarity, we use non-bold letters for vectors (of polynomials) and bold letters for vectors of vectors (of polynomials).)

Let $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system, $q_1, \ldots, q_k \colon (\mathbb{Z}^L)^{s+1} \to \mathbb{Z}^d$ be polynomials, and $g_1, \ldots, g_k \colon X \times (\mathbb{Z}^L)^s \to \mathbb{R}$ be functions such that $g_m(\cdot; h_1, \ldots, h_s)$ is an $L^\infty(\mu)$ function bounded by 1 for all $h_1, \ldots, h_s \in \mathbb{Z}^L$, $1 \le m \le k$. If $\mathbf{q} = (q_1, \ldots, q_k)$ and $\mathbf{g} = (g_1, \ldots, g_k)$, we say that $A = (L, s, k, \mathbf{g}, \mathbf{q})$ is a *PET-tuple*, and for $\tau \in \mathbb{N}_0$, we set

$$S(A,\tau) := \overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s\in\mathbb{Z}^L} \sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner sequence}}} \overline{\lim_{N\to\infty}} \left\| \mathbb{E}_{n\in I_N} \prod_{m=1}^{k} T_{q_m(n;h_1,\ldots,h_s)} g_m(x; h_1, \ldots, h_s) \right\|_2^\tau. \tag{16}$$

We define $\deg(A) = \deg(\mathbf{q})$, and say that A is *non-degenerate* if $\mathbf{q}$ is a family of essentially distinct polynomials (for convenience, $\mathbf{q}$ will be called *non-degenerate* as well). For $1 \le m \le k$, the tuple A is *m-standard* for $f \in L^\infty(\mu)$ if $\deg(A) = \deg(q_m)$ and $g_m(x; h_1, \ldots, h_s) = f(x)$ for every $x, h_1, \ldots, h_s$. That is, f is the mth function in $\mathbf{g}$, only depending on the first variable, and the polynomial q_m that acts on f is of the highest degree. (Here, we say m-standard for f to highlight the function of interest as, after running the vdC-operation, the position of the functions in the expression we deal with changes.) The tuple A will be called *semi-standard* for f if there exists $1 \le m \le k$ such that $g_m(x; h_1, \ldots, h_s) = f(x)$ for every $x, h_1, \ldots, h_s$. In this case, we do not require the function f to have a specific position in $\mathbf{g}$ nor that the polynomial acting on f be of the highest degree.

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As an example, for a $\mathbb{Z}$-system $(X, \mathcal{B}, \mu, T)$, take $L = s = 1, k = 3, q_1(n, h) = n^3$, $q_2(n, h) = 3n^2h, q_3(n, h) = 3nh^2$, and, for $f, g \in L^\infty(\mu)$, let $g_1(x, h) = f(x)$, $g_2(x, h) = g(x)$, and $g_3(x, h) = T^{h^3} f(x)$.

Then, we have that A is 1-standard for f, semi-standard for f and g, and, for $\kappa \in \mathbb{N}_0$,

$$S(A, \kappa) := \overline{\mathbb{E}}^{\square}_{h\in\mathbb{Z}} \sup_{\substack{(I_N)_{N\in\mathbb{N}} \\ \text{Følner sequence}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} T^{n^3} f(x) \cdot T^{3n^2h} g(x) \cdot T^{3nh^2}(T^{h^3} f(x))\|_2^\kappa.$$

For each non-degenerate PET-tuple $A = (L, s, k, \mathbf{g}, \mathbf{q})$ and polynomial $q : (\mathbb{Z}^L)^{s+1} \to \mathbb{Z}^d$, we define the *vdC-operation*, $\partial_q A$, according to the following three steps. (Actually, the vdC-operation can be defined for any PET-tuple, not just for non-degenerate ones. Similarly, being a procedure that reduces complexity, PET induction can be applied to any family of polynomials. As the expressions of interest in this paper correspond to non-degenerate tuples, we consider only this case.)

Step 1: For all $1 \le m \le k$, let $g^*_m = g^*_{m+k} = g_m$, and $q^*_1, \ldots, q^*_{2k} : (\mathbb{Z}^L)^{s+2} \to \mathbb{Z}^d$ be the polynomials defined as

$$q^*_m(n; h_1, \ldots, h_{s+1}) = \begin{cases} q_m(n + h_{s+1}; h_1, \ldots, h_s) - q(n; h_1, \ldots, h_s), & 1 \le m \le k, \\ q_{m-k}(n; h_1, \ldots, h_s) - q(n; h_1, \ldots, h_s), & k+1 \le m \le 2k, \end{cases}$$

that is, we subtract the polynomial q from the first k polynomials after we have shifted by h_{s+1} the first L variables, and for the second k ones, we subtract q. (In practice, this q will be one of the q_i terms of minimum degree.) Denote $\mathbf{q}^* = (q^*_1, \ldots, q^*_{2k})$.

Step 2: We remove from $q^*_1, \ldots, q^*_{2k}$ the polynomials which are constant and the associated functions g^*_i in the expression (we group all these terms together and we see the resulting term as a single constant one, in terms of n). As we already saw in the example at the beginning of this subsection, this is justified via the use of the Cauchy–Schwarz inequality and the fact that the functions g_m are bounded. Then we put the non-constant ones in groups $J_i = \{\tilde{q}_{i,1}, \ldots, \tilde{q}_{i,t_i}\}$, $1 \le i \le k'$ for some $k', t_i \in \mathbb{N}$ such that any two polynomials are essentially distinct if and only if they belong to different groups. Next, we write $\tilde{q}_{i,j}(n; h_1, \ldots, h_{s+1}) = \tilde{q}_{i,1}(n; h_1, \ldots, h_{s+1}) + \tilde{p}_{i,j}(h_1, \ldots, h_{s+1})$ for some polynomial $\tilde{p}_{i,j}$ for all $1 \le j \le t_i$, $1 \le i \le k'$. For convenience, we also relabel what remains, as some of the initial terms may have been removed because of the grouping of the polynomials, of the $g^*_1, \ldots, g^*_{2k}$ accordingly as $\tilde{g}_{i,j}$ for all $1 \le j \le t_i$, $1 \le i \le k'$.

Step 3: For all $1 \le i \le k'$, let $q'_i = \tilde{q}_{i,1}$ and

$$g'_i(x; h_1, \ldots, h_{s+1}) = \tilde{g}_{i,1}(x; h_1, \ldots, h_{s+1}) \prod_{j=2}^{t_i} T_{\tilde{p}_{i,j}(h_1,\ldots,h_{s+1})} \tilde{g}_{i,j}(x; h_1, \ldots, h_{s+1}).$$

We set $\mathbf{q}' = (q'_1, \ldots, q'_{k'})$, $\mathbf{g}' = (g'_1, \ldots, g'_{k'})$ and we denote the new PET-tuple by $\partial_q A := (L, s + 1, k', \mathbf{g}', \mathbf{q}')$. (Here we abuse the notation by writing $\partial_q A$ to denote any such tuple, obtained from Steps 1 to 3. Strictly speaking, $\partial_q A$ is not uniquely defined as the order of the grouping of $q'_1, \ldots, q'_{2k}$ in Step 2 is ambiguous. However, this is done without loss of generality, since the order does not affect the value of $S(\partial_q A, \cdot)$.) It follows from the construction that $\partial_q A$ is non-degenerate because each q'_i comes from a distinct grouping J_i.

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If $q = q_t$ for some $1 \leq t \leq k$, we write $\partial_t \mathbf{q}$ instead of $\mathbf{q}'$ to highlight the fact that we have subtracted the polynomial q_t; we also write $\partial_t A$ instead of $\partial_{q_t} A$ to lighten the notation.

Definition 4.6. We say that the operation $A \to \partial_t A$ is *1-inherited* if we did not drop q_1^* or group it with any other q_i^* in Step 2. Note that his implies that $q_1' = q_1^*$ and $g_1' = g_1$.

Example 4.7. Let $\mathbf{p} = (p_1, p_2)$ with $p_1, p_2 \colon \mathbb{Z} \to \mathbb{Z}^d$ the polynomials given by $p_i(n) = b_{i,2}n^2 + b_{i,1}n$ for some $b_{i,1}, b_{i,2} \in \mathbb{Z}^d$ for $1 \leq i \leq 2$ with $b_{1,2}, b_{2,2}, b_{1,2} - b_{2,2} \neq \mathbf{0}$ (hence, we have that $L = 1$, $s = 0$, and $k = 2$). We will calculate $\partial_2 \mathbf{p}$. Subtracting p_2 in the Step 1 of the vdC operation, we have that $\partial_2 \mathbf{p} = (q_1, q_2, q_3)$ is a tuple of three polynomials, $q_1, q_2, q_3 \colon \mathbb{Z}^2 \to \mathbb{Z}^d$, given by

$$\begin{aligned} q_1(n, h_1) &= p_1(n + h_1) - p_2(n) \\ &= (b_{1,2} - b_{2,2})n^2 + 2b_{1,2}nh_1 + (b_{1,1} - b_{2,1})n + b_{1,1}h_1 + b_{1,2}h_1^2, \\ q_2(n, h_1) &= p_2(n + h_1) - p_2(n) = 2b_{2,2}nh_1 + b_{2,1}h_1 + b_{2,2}h_1^2, \\ q_3(n, h_1) &= p_1(n) - p_2(n) = (b_{1,2} - b_{2,2})n^2 + (b_{1,1} - b_{2,1})n, \end{aligned}$$

where we removed one essentially constant polynomial in Step 2 of the vdC operation. (Here we use q_i terms instead of p_i' terms in the first step to ease the notation of Example 5.8 that is given in the next section.) Notice that here we have $L = 1$, $s' = 1$, and $k' = 3$.

Actually, as in the example at the beginning of this subsection, after using a series of vdC operations, one can convert $\mathbf{p}$ into a PET-tuple of linear polynomials. Indeed, if we run the vdC operation once more by subtracting q_2 in Step 1 of the vdC operation, we have that $\partial_2\partial_2 \mathbf{p} = (q_1', \ldots, q_4')$ is a tuple of four polynomials, $q_1', \ldots, q_4' \colon \mathbb{Z}^3 \to \mathbb{Z}^d$, given by

$$\begin{aligned} q_1'(n, h_1, h_2) &= (b_{1,2} - b_{2,2})n^2 + 2(b_{1,2} - b_{2,2})nh_1 + 2(b_{1,2} - b_{2,2})nh_2 \\ &\quad + (b_{1,1} - b_{2,1})n + r_1'(h_1, h_2), \\ q_2'(n, h_1, h_2) &= (b_{1,2} - b_{2,2})n^2 - 2b_{2,2}nh_1 + 2(b_{1,2} - b_{2,2})nh_2 \\ &\quad + (b_{1,1} - b_{2,1})n + r_2'(h_1, h_2), \\ q_3'(n, h_1, h_2) &= (b_{1,2} - b_{2,2})n^2 + 2(b_{1,2} - b_{2,2})nh_1 + (b_{1,1} - b_{2,1})n + r_3'(h_1, h_2), \\ q_4'(n, h_1, h_2) &= (b_{1,2} - b_{2,2})n^2 - 2b_{2,2}nh_1 + (b_{1,1} - b_{2,1})n + r_4'(h_1, h_2), \end{aligned}$$

where $r_i' \colon \mathbb{Z}^2 \to \mathbb{Z}^d$, $1 \leq i \leq 4$, are polynomials in h_1, h_2, and we removed two essentially constant polynomials (that is, $q_2(n, h_1) - q_2(n, h_1)$ and $q_2(n + h_2, h_1) - q_2(n, h_1)$) in Step 2 of the vdC operation.

Finally, if we apply the vdC operation again by subtracting q_4' in Step 1 of the vdC operation, we have that $\partial_4\partial_2\partial_2 \mathbf{p} = (q_1'', \ldots, q_7'')$ is a tuple of seven polynomials, $q_1'', \ldots, q_7'' \colon \mathbb{Z}^4 \to \mathbb{Z}^d$, given by

$$\begin{aligned} q_1''(n, h_1, h_2, h_3) &= 2b_{1,2}nh_1 + 2(b_{1,2} - b_{2,2})nh_2 + 2(b_{1,2} - b_{2,2})nh_3 + r_1''(h_1, h_2, h_3), \\ q_2''(n, h_1, h_2, h_3) &= 2(b_{1,2} - b_{2,2})nh_2 + 2(b_{1,2} - b_{2,2})nh_3 + r_2''(h_1, h_2, h_3), \\ q_3''(n, h_1, h_2, h_3) &= 2b_{1,2}nh_1 + 2(b_{1,2} - b_{2,2})nh_3 + r_3''(h_1, h_2, h_3), \\ q_4''(n, h_1, h_2, h_3) &= 2(b_{1,2} - b_{2,2})nh_3 + r_4''(h_1, h_2, h_3), \end{aligned}$$

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$$q_5''(n, h_1, h_2, h_3) = 2b_{1,2}nh_1 + 2(b_{1,2} - b_{2,2})nh_2 + r_5''(h_1, h_2, h_3),$$
$$q_6''(n, h_1, h_2, h_3) = 2(b_{1,2} - b_{2,2})nh_2 + r_6''(h_1, h_2, h_3),$$
$$q_7''(n, h_1, h_2, h_3) = 2b_{1,2}nh_1 + r_7''(h_1, h_2, h_3),$$

where $r_i'' \colon \mathbb{Z}^3 \to \mathbb{Z}^d$, $1 \leq i \leq 7$, are polynomials in h_1, h_2, h_3, and we removed one essentially constant polynomial in Step 2 of the vdC operation. It is clear that $\deg(\partial_4\partial_2\partial_2\mathbf{p}) = 1$.

The vdC operation provides us with a non-degenerate tuple, the value $S(\cdot, \cdot)$ of which satisfies the following.

PROPOSITION 4.8. [7, Proposition 4.10] *Let $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system, $A = (L, s, k, \mathbf{g}, \mathbf{q})$ a PET-tuple, and $q \colon (\mathbb{Z}^L)^{s+1} \to \mathbb{Z}^d$ a polynomial. Then, $\partial_q A$ is non-degenerate and $S(A, 2\tau) \leq 4^\tau S(\partial_q A, \tau)$ for every $\tau \in \mathbb{N}_0$.*

The following crucial result (cf. [7, Theorem 4.2]) shows that when we start with a PET-tuple which is 1-standard for a function, then, after finitely many vdC operations, we arrive at a new PET-tuple of degree 1 which is still 1-standard for the same function, so we can then use some Host–Kra seminorm to bound the lim sup of the average of interest.

THEOREM 4.9. *Let $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system and $f \in L^\infty(\mu)$. Let $A = (L, s, k, \mathbf{g}, \mathbf{q})$ be a non-degenerate PET-tuple which is 1-standard for f. Then, there exist $\rho_1, \ldots, \rho_t \in \mathbb{N}$, for some $t \in \mathbb{N}_0$ depending only on $\deg(A), L, s, k$, such that for all $1 \leq t' \leq t$, $\partial_{\rho_{t'}} \ldots \partial_{\rho_1} A$ is a non-degenerate PET-tuple which is still 1-standard for f, and that $\partial_{\rho_{t'-1}} \ldots \partial_{\rho_1} A \to \partial_{\rho_{t'}} \ldots \partial_{\rho_1} A$ is 1-inherited. Moreover, $\deg(\partial_{\rho_t} \ldots \partial_{\rho_1} A) = 1$.*

Remark 4.10. The dependence of t on $\deg(A), L, s, k$ was not stated in [7, Theorem 4.2], but it follows immediately from its proof. Also, $\partial_{\rho_t} \ldots \partial_{\rho_1} A$ is understood as A if $t = 0$. Notice also that in Theorem 4.9, we need every step of the vdC operation to be 1-inherited. Otherwise, we would have to either drop f or combine it with other functions in some of the induction steps; in the end, that would prevent us from obtaining an upper bound for $S(A, 1)$ by some Host–Kra seminorm of f. In [7, Theorem 4.2], the PET tuple is not required to be 1-standard nor 1-inherited; this comes at no extra cost as the polynomials chosen at each step to run the vdC operation are of minimum degree.

If A is an m-standard PET-tuple for the function f, then, by rearranging the terms if necessary, one can get a new tuple A' which is 1-standard for f with $S(A, \tau) = S(A', \tau)$. However, if A is semi-standard but not standard for f, then the PET-induction does not work well enough to provide an upper bound for $S(A, \tau)$ in terms of some Host–Kra seminorm of f. To overcome this difficulty one follows [7]. More specifically, using [7, Proposition 6.3], which is a 'dimension-increment' argument, A can be transformed into a new PET-tuple which is 1-standard for f (at the cost of increasing the dimension from L to $2L$ which is harmless for our approach). So, following this procedure, for any fixed function f, we may assume without loss of generality that the corresponding polynomial iterate p is of maximum degree, making the PET-tuple, after potential rearrangement of its terms, 1-standard for f. A combination of the previous results will allow us to obtain the required upper bound for each function.

5. *Finding a characteristic factor*

This lengthy section is dedicated to proving Theorem 2.11. To this end, we need to show two intermediate results, that is, Propositions 5.2 and 5.4, which improve two technical results from [7], namely [7, Proposition 5.6] and [7, Proposition 5.5], respectively.

Recall that for a subset A of $\mathbb{Q}^d$, $G(A) = \mathrm{span}_{\mathbb{Q}}\{a \in A\} \cap \mathbb{Z}^d$. Let $G'(A) := \mathrm{span}_{\mathbb{Z}}\{a \in A\}$.

Convention 5.1. For the rest of the paper, for every $\mathbf{u}_i = (u_{i,1}, \ldots, u_{i,L}) \in (\mathbb{Q}^d)^L$, $1 \leq i \leq r$, we denote $G(\mathbf{u}_1, \ldots, \mathbf{u}_r) := G(\{u_{i,j} \colon 1 \leq i \leq r, 1 \leq j \leq L\})$ and $G'(\mathbf{u}_1, \ldots, \mathbf{u}_r) := G'(\{u_{i,j} \colon 1 \leq i \leq r, 1 \leq j \leq L\})$.

The first result, which enhances [7, Proposition 5.6], gives a bound for the average of interest by finite step seminorms (recall Convention 3.3 for notions on Host–Kra seminorms). To pass from infinite step seminorms to finite step ones, we use the implications of Propositions 3.6 and 3.8.

PROPOSITION 5.2. (Bounding averaged Host–Kra seminorms by a single one) *Let $s, s', t, L \in \mathbb{N}$ and $\mathbf{c}_m \colon (\mathbb{Z}^L)^s \to (\mathbb{Z}^d)^L$, $1 \leq m \leq t$, be polynomials with $\mathbf{c}_m \neq \mathbf{0}$ given by*

$$\mathbf{c}_m(h_1, \ldots, h_s) = \sum_{a_1,\ldots,a_s \in \mathbb{N}_0^L, |a_1|+\cdots+|a_s| \leq s'} h_1^{a_1} \ldots h_s^{a_s} \cdot \mathbf{u}_m(a_1, \ldots, a_s) \tag{17}$$

for some

$$\mathbf{u}_m(a_1, \ldots, a_s) = (u_{m,1}(a_1, \ldots, a_s), \ldots, u_{m,L}(a_1, \ldots, a_s)) \in (\mathbb{Q}^d)^L.$$

Denote

$$H_m := G(\{u_{m,i}(a_1, \ldots, a_s) \colon a_1, \ldots, a_s \in \mathbb{N}_0^L, 1 \leq i \leq L\}).$$

There exists $D \in \mathbb{N}_0$ depending only on s, s', L, t such that for every $\mathbb{Z}^d$-system $(X, \mathcal{B}, \mu, (T_g)_{g \in \mathbb{Z}^d})$ and every $f \in L^\infty(\mu)$,

$$\overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s \in \mathbb{Z}^L} |||f|||_{\{G'(\mathbf{c}_m(h_1,\ldots,h_s))\}_{1 \leq m \leq t}} = 0 \ \text{ if } \ |||f|||_{H_1^{\times D},\ldots,H_t^{\times D}} = 0. \tag{18}$$

Remark 5.3. H_m is dependent only on those $u_{m,i}(a_1, \ldots, a_s)$ with $|a_1| + \cdots + |a_s| \leq s'$.

We start by explaining the idea behind Proposition 5.2 with an example, which also illustrates how Proposition 5.2 improves [7, Propositions 5.5 and 5.6].

Example 5.4. Let $p_1, p_2 \colon \mathbb{Z} \to \mathbb{Z}^2$ be polynomials given by $p_1(n) = (n^2 + n, 0)$ and $p_2(n) = (0, n^2)$, and $(X, \mathcal{B}, \mu, (T_g)_{g \in \mathbb{Z}^2})$ be a $\mathbb{Z}^2$-system. Consider the following expression:

$$\sup_{\substack{(I_N)_{N \in \mathbb{N}} \\ \text{Følner seq.}}} \overline{\lim_{N \to \infty}} \|\mathbb{E}_{n \in I_N} T_{p_1(n)} f_1 \cdot T_{p_2(n)} f_2\|_2.$$

Put $e_1 = (1, 0)$, $e_2 = (0, 1)$, and $e = (1, -1)$. Similarly to the computation in [7, Second part of computations for Example 1], we get

$$\sup_{\substack{(I_N)_{N\in\mathbb{N}}\\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} T_{p_1(n)} f_1 \cdot T_{p_2(n)} f_2\|_2^8 \leq C \cdot \overline{\mathbb{E}}^{\square}_{\mathbf{h}\in\mathbb{Z}^3} |||f_1|||_{G'(\mathbf{c}_1(\mathbf{h})),\ldots,G'(\mathbf{c}_7(\mathbf{h}))}, \quad (19)$$

where C is a universal constant, and

$$\begin{aligned}
\mathbf{c}_1(h_1, h_2, h_3) &= -2h_1 e_1,\\
\mathbf{c}_2(h_1, h_2, h_3) &= 2h_2 e,\\
\mathbf{c}_3(h_1, h_2, h_3) &= -2h_1 e_1 + 2h_2 e,\\
\mathbf{c}_4(h_1, h_2, h_3) &= 2h_3 e,\\
\mathbf{c}_5(h_1, h_2, h_3) &= -2h_1 e_1 + 2h_3 e,\\
\mathbf{c}_6(h_1, h_2, h_3) &= 2(h_2 + h_3)e,\\
\mathbf{c}_7(h_1, h_2, h_3) &= -2h_1 e_1 + 2(h_2 + h_3)e.
\end{aligned}$$

Using [7, Proposition 5.6], one can show that if $|||f_1|||_{e_1^{\times D}, e^{\times D}} = 0$ for all $D \in \mathbb{N}$, then the right-hand side of equation (19) is 0. In this paper, we strengthen this result by only assuming that $|||f_1|||_{e_1^{\times D}, e^{\times D}} = 0$ for some $D \in \mathbb{N}$.

Indeed, $Z_{\{G'(\mathbf{c}_i(\mathbf{h}),\mathbf{c}_j(\mathbf{h}'))\}_{1\leq i,j\leq 7}} = Z_{\{G(\mathbf{c}_i(\mathbf{h}),\mathbf{c}_j(\mathbf{h}'))\}_{1\leq i,j\leq 7}}$ by Proposition 3.1. Using Proposition 3.8 (for $I = [-N, N]^3$, letting $N \to \infty$) and Corollary 3.7, we have that the right-hand side of equation (19) is 0 if

$$\|\mathbb{E}(f | Z_{\{G(\mathbf{c}_i(\mathbf{h}),\mathbf{c}_j(\mathbf{h}'))\}_{1\leq i,j\leq 7}})\|_2^2 = 0$$

for a density 1 subset of $(h, h') \in (\mathbb{Z}^3)^2$. Indeed, Proposition 3.8 implies that the right-hand side of equation (19) is 0 if $\overline{\mathbb{E}}^{\square}_{\mathbf{h},\mathbf{h}'\in\mathbb{Z}^3} \|\mathbb{E}(f | Z_{\{G(\mathbf{c}_i(\mathbf{h}))+G(\mathbf{c}_j(\mathbf{h}'))\}_{1\leq i,j\leq 7}})\|_2^2 = 0$. However, by Remark 3.10, $G(\mathbf{c}_i(\mathbf{h})) + G(\mathbf{c}_j(\mathbf{h}'))$ is a finite index subgroup of $G(\mathbf{c}_i(\mathbf{h}), \mathbf{c}_j(\mathbf{h}'))$, so we can use Lemma 3.9(iv) to conclude that $\|\mathbb{E}(f | Z_{\{G(\mathbf{c}_i(\mathbf{h}))+G(\mathbf{c}_j(\mathbf{h}'))\}_{1\leq i,j\leq 7}})\|_2^2 = 0$ for a density 1 subset of $(h, h') \in (\mathbb{Z}^3)^2$, which implies that the average over h, h' is 0 too.

However, for 'almost all' $\mathbf{h}, \mathbf{h}' \in \mathbb{Z}^3$, the group $G(\mathbf{c}_i(\mathbf{h}), \mathbf{c}_j(\mathbf{h}'))$ equals $\mathbb{Z}e_1$ if $i = j = 1$, $\mathbb{Z}e$ if $i, j \in \{2, 4, 6\}$ and $\mathbb{Z}^2$ otherwise. So, $Z_{\{G(\mathbf{c}_i(\mathbf{h}),\mathbf{c}_j(\mathbf{h}'))\}_{1\leq i,j\leq 7}}$ is contained in $Z_{e_1, e^{\times 9}, (\mathbb{Z}^2)^{\times 39}}$, which is contained in $Z_{e_1^{\times 25}, e^{\times 25}}$ by Proposition 3.1. Hence, the right-hand side of equation (19) is 0 if $|||f_1|||_{e_1^{\times 25}, e^{\times 25}} = 0$.

We now prove the general case.

Proof of Proposition 5.2. Since $|||f|||_{G'(\mathbf{c}_1(h_1,\ldots,h_s))} \leq |||f|||_{G'(\mathbf{c}_1(h_1,\ldots,h_s)),G'(\mathbf{c}_1(h_1,\ldots,h_s))}$ by Proposition 3.1, duplicating $\mathbf{c}_1$ if necessary, we may assume without loss of generality that $t \geq 2$. We may also assume that $\|f\|_{L^\infty(\mu)} \leq 1$. Following our notational convention, denote $\mathbf{h} := (h_1, \ldots, h_s)$. Using Proposition 3.8 and Corollary 3.7 for $I = ([-N, N]^L)^s$, and then letting $N \to \infty$, for all $W = 2^w$, $w \in \mathbb{N}$, we have that

$$\begin{aligned}
&(\overline{\mathbb{E}}^{\square}_{\mathbf{h}\in(\mathbb{Z}^L)^s} |||f|||_{\{G'(\mathbf{c}_m(\mathbf{h}))\}_{1\leq m\leq t}})^{2^t W}\\
&\quad\leq (\overline{\mathbb{E}}^{\square}_{\mathbf{h}\in(\mathbb{Z}^L)^s} \|\mathbb{E}(f | Z_{\{G'(\mathbf{c}_m(\mathbf{h}))\}_{1\leq m\leq t}})\|_2^2)^W \qquad (20)\\
&\quad\leq \overline{\mathbb{E}}^{\square}_{\mathbf{h}^1,\ldots,\mathbf{h}^W\in(\mathbb{Z}^L)^s} \|\mathbb{E}(f | Z_{\{G'(\mathbf{c}_{m_1}(\mathbf{h}^1))+\cdots+G'(\mathbf{c}_{m_W}(\mathbf{h}^W))\}_{1\leq m_1,\ldots,m_W\leq t}})\|_2^2.
\end{aligned}$$

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We claim that this last expression equals 0 if

$$\|\mathbb{E}(f|Z_{\{G(\mathbf{c}_{m_1}(\mathbf{h}^1),\ldots,\mathbf{c}_{m_W}(\mathbf{h}^W))\}_{1\le m_1,\ldots,m_W\le t}})\|_2^2=0$$

for a density 1 set of $(\mathbf{h}^1,\ldots,\mathbf{h}^W)\in(\mathbb{Z}^L)^{sW}$. Note that this is equivalent to

$$\|\mathbb{E}(f|Z_{\{G'(\mathbf{c}_{m_1}(\mathbf{h}^1),\ldots,\mathbf{c}_{m_W}(\mathbf{h}^W))\}_{1\le m_1,\ldots,m_W\le t}})\|_2^2=0$$

for a density 1 set of $(\mathbf{h}^1,\ldots,\mathbf{h}^W)\in(\mathbb{Z}^L)^{sW}$, since by Proposition 3.1(v) and (vi), noting that $t\ge 2$, for every $(\mathbf{h}^1,\ldots,\mathbf{h}^W)\in\Omega$, we have that $Z_{\{G(\mathbf{c}_{m_1}(\mathbf{h}^1),\ldots,\mathbf{c}_{m_W}(\mathbf{h}^W))\}_{1\le m_1,\ldots,m_W\le t}}=Z_{\{G'(\mathbf{c}_{m_1}(\mathbf{h}^1),\ldots,\mathbf{c}_{m_W}(\mathbf{h}^W))\}_{1\le m_1,\ldots,m_W\le t}}$.

Analogously to Example 5.4 via Remark 3.10, and Lemma 3.9(iv), this last condition implies that the last line of equation (20) is equal to 0.

Let w be the smallest integer such that $2^w\ge t(s'+1)^{sL}$ and Ω the set of $(\mathbf{h}^1,\ldots,\mathbf{h}^W)\in(\mathbb{Z}^{sL})^W$ such that for every $1\le m_1,\ldots,m_W\le t$, the group $G(\mathbf{c}_{m_1}(\mathbf{h}^1),\ldots,\mathbf{c}_{m_W}(\mathbf{h}^W))$ contains at least one of $H_1,\ldots,H_t$. Then, $Z_{\{G(\mathbf{c}_{m_1}(\mathbf{h}^1),\ldots,\mathbf{c}_{m_W}(\mathbf{h}^W))\}_{1\le m_1,\ldots,m_W\le t}}$ is a factor of $Z_{H_1^{\times D},\ldots,H_t^{\times D}}$ for $D:=t^W$, and thus $\mathbb{E}(f|Z_{\{G(\mathbf{c}_{m_1}(\mathbf{h}^1),\ldots,\mathbf{c}_{m_W}(\mathbf{h}^W))\}_{1\le m_1,\ldots,m_W\le t}})=0$ since $|||f|||_{H_1^{\times D},\ldots,H_t^{\times D}}=0$. So, to show that the first line of equation (20) is equal to 0, it suffices to show that Ω is of upper density 1.

Let $\tilde{\mathbf{h}}:=(\mathbf{h}^1,\ldots,\mathbf{h}^W)\in(\mathbb{Z}^{sL})^W$ and $1\le m_1,\ldots,m_W\le t$. By the pigeonhole principle, at least $(s'+1)^{sL}$ many of the $m_1,\ldots,m_W$ take the same value. Assume that $m_{i_1}=\cdots=m_{i_{W'}}=m$, where $W'\le(s'+1)^{sL}$ is the number of $a_1,\ldots,a_s\in\mathbb{N}_0^L$ with $|a_1|+\cdots+|a_s|\le s'$. Note that W' depends only on s',s and L. Write $\mathbf{h}^i:=(h_{i,1},\ldots,h_{i,s})\in(\mathbb{Z}^L)^s$ and consider the $W'\times W'$ matrix

$$A_{i_1,\ldots,i_{W'}}(\tilde{\mathbf{h}}):=(h_{i_j,1}^{a_1}\cdots h_{i_j,s}^{a_s})_{a_1,\ldots,a_s\in\mathbb{N}_0^L,|a_1|+\cdots+|a_s|\le s',1\le j\le W'}.$$

If $\det(A_{i_1,\ldots,i_{W'}}(\tilde{\mathbf{h}}))\ne 0$, then by the definition of $\mathbf{c}_m(\mathbf{h})$ in equation (17) and knowledge of linear algebra, each vector in H_m can be written as a linear combination of $\mathbf{c}_m(\mathbf{h}^{i_1}),\ldots,\mathbf{c}_m(\mathbf{h}^{i_{W'}})$ with rational coefficients. So, using Convention 5.1, we have that

$$G(\mathbf{c}_{m_1}(\mathbf{h}^1),\ldots,\mathbf{c}_{m_W}(\mathbf{h}^W))\supseteq G(\mathbf{c}_m(\mathbf{h}^{i_1}),\ldots,\mathbf{c}_m(\mathbf{h}^{i_{W'}}))\supseteq H_m.$$

In conclusion, $\tilde{\mathbf{h}}\in\Omega$ if for all $1\le i_1<\cdots<i_{W'}\le W$, $\det(A_{i_1,\ldots,i_{W'}}(\tilde{\mathbf{h}}))\ne 0$.

Thus, it suffices to show that for all $1\le i_1<\cdots<i_{W'}\le W$, the set of $\tilde{\mathbf{h}}\in(\mathbb{Z}^{sL})^W$ with $\det(A_{i_1,\ldots,i_{W'}}(\tilde{\mathbf{h}}))=0$ is of density 0. We may assume without loss of generality that $i_1=1,\ldots,i_{W'}=W'$. Note that $\det(A_{1,\ldots,W'}(\tilde{\mathbf{h}}))$ is a polynomial in $h_{i,j}$, $1\le i\le W'$, $1\le j\le s$. Looking at the term $h_{1,1}^{(s,0,\ldots,0)}h_{2,2}^{(s,0,\ldots,0)}\cdots h_{W',W'}^{(s,0,\ldots,0)}$, we see that $\det(A_{1,\ldots,W'}(\tilde{\mathbf{h}}))$ is a non-constant polynomial. Therefore, the set of solutions to $\det(A_{1,\ldots,W'}(\tilde{\mathbf{h}}))=0$ is of 0 density by Lemma 3.11, which completes the argument. □

The second statement, which strengthens [7, Proposition 5.5], is the following (see Definition 2.1 for the various notions appearing in the statement).

PROPOSITION 5.5. (Bounding the average by averaged Host–Kra seminorms) *Let* $d,k,K,L\in\mathbb{N}$, $\mathbf{p}=(p_1,\ldots,p_k)$, $p_1,\ldots,p_k\colon\mathbb{Z}^L\to\mathbb{Z}^d$ *be a family of essentially distinct polynomials of degrees at most* K, *with* $p_i(n)=\sum_{v\in\mathbb{N}_0^L,|v|\le K}b_{i,v}n^v$ *for* *some* $b_{i,v} \in \mathbb{Q}^d$. *There exist* $s, s', t_1, \ldots, t_k \in \mathbb{N}$, *depending only on* d, k, K, L, *and polynomials* $\mathbf{c}_{i,m} : (\mathbb{Z}^L)^s \to (\mathbb{Z}^d)^L$, $1 \leq i \leq k$, $1 \leq m \leq t_i$, *with* $\mathbf{c}_{i,m} \not\equiv \mathbf{0}$, *such that all the following hold.*

(i) *(Control of the coefficients) Each* $\mathbf{c}_{i,m}$ *is of the form*

$$\mathbf{c}_{i,m}(h_1, \ldots, h_s) = \sum_{a_1,\ldots,a_s \in \mathbb{N}_0^L, |a_1|+\cdots+|a_s| \leq s'} h_1^{a_1} \cdots h_s^{a_s} \cdot \mathbf{v}_{i,m}(a_1, \ldots, a_s)$$

for some $\mathbf{v}_{i,m}(a_1, \ldots, a_s) = (v_{i,m,1}(a_1, \ldots, a_s), \ldots, v_{i,m,L}(a_1, \ldots, a_s)) \in (\mathbb{Q}^d)^L$, *which is a polynomial function in terms of the coefficients of* p_i, $1 \leq i \leq k$, *and whose degree depends only on* d, k, K, L.

In addition, denoting

$$H_{i,m} := G(\{v_{i,m,j}(a_1, \ldots, a_s) : a_1, \ldots, a_s \in \mathbb{N}_0^L, 1 \leq j \leq L\}),$$

we have that each $H_{i,m}$ *contains one of* $G_{i,j}(\mathbf{p})$ *for some* $0 \leq j \leq k$, $j \neq i$.

(ii) *(Control of the average) For every* $\mathbb{Z}^d$*-system* $(X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ *and every* $f_1, \ldots, f_k \in L^\infty(\mu)$ *bounded by* 1, *we have that*

$$\sup_{\substack{(I_N)_{N\in\mathbb{N}} \\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \left\| \mathbb{E}_{n\in I_N} \prod_{i=1}^{k} T_{p_i(n)} f_i \right\|_2^{2^{t_0}}$$
$$\leq C \cdot \min_{1\leq i\leq k} \overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s\in\mathbb{Z}^L} |||f_i|||_{(G'(\mathbf{c}_{i,m}(h_1,\ldots,h_s)))_{1\leq m\leq t_i}}, \quad (21)$$

where t_0 *and* $C > 0$ *are constants depending only on* $\mathbf{p}$.

Remark 5.6. Both t_0 and C depend on d, k, L, and the highest degree of $p_1, \ldots, p_k$. More specifically, t_0 can be chosen to be the maximum number of vdC operations we have to perform, for each i, for the PET tuple to be non-degenerate, 1-standard for f_i, and with degree equal to 1.

Proposition 5.5 improves on [7, Proposition 5.5] as the description of the subgroup $H_{i,m}$ is much more precise than that of the set $U_{i,r}(a_1, \ldots, a_s)$ defined in the latter. The rest of this section is devoted to proving Proposition 5.5, which is the most technical result of this paper.

Next, we introduce some convenient notation. Let $\mathbf{q} = (q_1, \ldots, q_\ell)$ be a tuple of polynomials $q_i : (\mathbb{Z}^L)^{s+1} \to \mathbb{Z}^d$, $1 \leq i \leq \ell$, where

$$q_i(n; h_1, \ldots, h_s) = \sum_{b,a_1,\ldots,a_s\in\mathbb{N}_0^L} h_1^{a_1} \ldots h_s^{a_s} n^b \cdot u_i(b; a_1, \ldots, a_s)$$

for some $u_i(b; a_1, \ldots, a_s) \in \mathbb{Q}^d$, $1 \leq i \leq \ell$. Then, by writing

$$\mathbf{u}(b; a_1, \ldots, a_s) := (u_1(b; a_1, \ldots, a_s), \ldots, u_\ell(b; a_1, \ldots, a_s)) \in (\mathbb{Q}^d)^\ell,$$

we can express $\mathbf{q}$ as

$$\mathbf{q}(n; h_1, \ldots, h_s) = \sum_{b,a_1,\ldots,a_s\in\mathbb{N}_0^L} h_1^{a_1} \ldots h_s^{a_s} n^b \cdot \mathbf{u}(b; a_1, \ldots, a_s).$$

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We call $\mathbf{u}(b; a_1, \ldots, a_d)$ the *data* of $\mathbf{q}$ at *level* $(b; a_1, \ldots, a_d)$, or simply the *level data* of $\mathbf{q}$.

For the rest of the section, we fix $d, k, K, L \in \mathbb{N}$ and let $\mathbf{p} = (p_1, \ldots, p_k)$ denote a family of essentially distinct polynomials $p_1, \ldots, p_k : \mathbb{Z}^L \to \mathbb{Z}^d$ of degrees at most K, with $p_i(n) = \sum_{v\in\mathbb{N}_0^L, |v|\leq K} b_{i,v} n^v$, where $b_{i,v} \in \mathbb{Q}^d$.

Recalling that $b_{w,v}$, $1 \leq w \leq k$, $v \in \mathbb{N}_0^L$ are the coefficients that arise from the family $\mathbf{p}$ (we also put $b_{0,v} := \mathbf{0} \in \mathbb{Q}^d$ for all $v \in \mathbb{N}_0^L$), for $r \in \mathbb{Q}$, $v \in \mathbb{N}_0^L$, and $0 \leq i \leq k$, we set

$$Q_{r,i,v} := \{r(b_{w,v} - b_{i,v}) : 0 \leq w \leq k\}.$$

One sees that the left-hand side of equation (21) is $S(A, 1)$ (recall equation (16) for the definition), where A is the PET-tuple $(L, 0, k, (f_1, \ldots, f_k), \mathbf{p})$. To prove Proposition 5.5, we first need to perform a series of vdC operations to convert A into a PET-tuple $\partial_{\rho_t} \ldots \partial_{\rho_1} A$ of degree 1, and then compare the coefficients of the polynomials in A with those in $\partial_{\rho_t} \ldots \partial_{\rho_1} A$. Even though the coefficients in the latter are very difficult to compute directly, one can keep track of the connection between them and those of the original polynomial family $\mathbf{p}$. This was first achieved in [7] by introducing an equivalence relation pertaining to the vdC operation (see [7, §5.3] for details). In this paper, we introduce another approach which is more intricate than that used in [7], but that achieves a better tracking of the coefficients, which in turn gives us a stronger upper bound for the multiple averages.

Definition 5.7. (Types and symbols of level data) Fix a tuple $\mathbf{q} = (q_1, \ldots, q_\ell)$ of polynomials with level data $\mathbf{u}$. For all $b, a_1, \ldots, a_s, v \in \mathbb{N}_0^L$, $r \in \mathbb{Q}$, and $0 \leq i \leq k$, we say that $\mathbf{u}(b; a_1, \ldots, a_s)$ is of *type* (r,i,v) if

$$u_1(b; a_1, \ldots, a_s), \ldots, u_\ell(b; a_1, \ldots, a_s) \in Q_{r,i,v}, \quad \text{and}$$
$$u_1(b; a_1, \ldots, a_s) = r(b_{1,v} - b_{i,v}).$$

We say that $\mathbf{u}(b; a_1, \ldots, a_s)$ is *non-trivial* if at least one of $u_m(b; a_1, \ldots, a_s)$ is non-zero.

Let $\mathbf{u}(b; a_1, \ldots, a_s)$ be of type (r, i, v). Suppose that

$$(u_1(b; a_1, \ldots, a_s), \ldots, u_\ell(b; a_1, \ldots, a_s)) = (r(b_{w_1,v} - b_{i,v}), \ldots, r(b_{w_\ell,v} - b_{i,v})),$$

for some $0 \leq w_1, \ldots, w_\ell \leq k$. We call $\mathrm{w} := (w_1, \ldots, w_\ell)$ a *symbol* of $\mathbf{u}(b; a_1, \ldots, a_s)$.

Note that the definition of type and symbol depend on the prefixed polynomial family $\mathbf{p}$. Moreover, if $(w_1, \ldots, w_\ell)$ is a symbol of $\mathbf{u}(b; a_1, \ldots, a_s)$, then so is $(1, w_2, \ldots, w_\ell)$. We also remark that in the special case $\mathbf{p} = \mathbf{q}$ and $s = 0$, all of $\mathbf{u}(b; \emptyset)$ are of type $(1, 0, b)$ and have $(1, \ldots, k)$ as a symbol.

We use the types and symbols of level data to track the coefficients of PET-tuples. We start with an example to illustrate this concept.

Example 5.8. Let $\mathbf{p} = (p_1, p_2)$ be defined as in Example 4.7 and let $\mathbf{q} = \partial_2\mathbf{p} = (q_1, q_2, q_3)$ (recall that by this, we mean the polynomial iterates we get after running the vdC operation, subtracting the second polynomial, p_2). In this case,

$\mathbf{u}(0;1) = (b_{1,1}, b_{2,1}, 0)$ is of type $(1, 0, 1)$ and has symbol $(1, 2, 0)$,
$\mathbf{u}(0;2) = (b_{1,2}, b_{2,2}, 0)$ is of type $(1, 0, 2)$ and has symbol $(1, 2, 0)$,
$\mathbf{u}(1;0) = (b_{1,1} - b_{2,1}, 0, b_{1,1} - b_{2,1})$ is of type $(1, 2, 1)$ and has symbol $(1, 2, 1)$,
$\mathbf{u}(1;1) = (2b_{1,2}, 2b_{2,2}, 0)$ is of type $(2, 0, 2)$ and has symbol $(1, 2, 0)$,
$\mathbf{u}(2;0) = (b_{1,2} - b_{2,2}, 0, b_{1,2} - b_{2,2})$ is of type $(1, 2, 2)$ and has symbol $(1, 2, 1)$.

Definition 5.9. Let S denote the set of all $(a, a') \in \mathbb{N}_0^{2L}$ such that a and a' are both $\mathbf{0}$ or both different than $\mathbf{0}$. Let $\mathbf{q}$ be a polynomial family of degree at least 1. We say that $\mathbf{q}$ satisfies properties (P1)–(P4) if its level data $\mathbf{u}$ satisfy the following conditions.

(P1) For all $a_1, \ldots, a_s, b$, there exist r, i, v such that $\mathbf{u}(b; a_1, \ldots, a_s)$ is of type (r, i, v). Moreover, we may choose the type and symbol for all of $\mathbf{u}(b; a_1, \ldots, a_s)$ in a way such that (P2)–(P4) hold.

(P2) If $\mathbf{u}(b; a_1, \ldots, a_s)$ is of type (r, i, v), then $r = \binom{b+a_1+\cdots+a_s}{a_1,\ldots,a_s}$ and $v = b + a_1 + \cdots + a_s$ (in particular, $r \neq 0$, where, for $v_i = (v_{i,1}, \ldots, v_{i,L}) \in \mathbb{N}_0^L$, $1 \le i \le s$, we denote $\binom{v_1+\cdots+v_s}{v_1,\ldots,v_{s-1}} := \prod_{j=1}^{L} (v_{1,j} + \cdots + v_{s,j})!/v_{1,j}! \ldots v_{s,j}!$).

(P3) Suppose that $\mathbf{u}(b; a_1, \ldots, a_s)$ is of type (r, i, v) and $\mathbf{u}(b'; a_1', \ldots, a_s')$ is of type (r', i', v'). If $(a_1, a_1'), \ldots, (a_s, a_s') \in S$, then $i = i'$ and $\mathbf{u}(b; a_1, \ldots, a_s)$, $\mathbf{u}(b'; a_1', \ldots, a_s')$ share a symbol.

(P4) For every $\mathbf{u}(b; a_1, \ldots, a_s)$, the first coordinate w_1 of its symbol $(w_1, \ldots, w_\ell)$ equals to 1.

For convenience, we say that a PET-tuple $A = (L, s, \ell, \mathbf{g}, \mathbf{q})$ satisfies properties (P1)–(P4) if the polynomial family $\mathbf{q}$ associated to A satisfies properties (P1)–(P4).

Once again, properties (P1)–(P4) are taken with respect to the prefixed polynomial family $\mathbf{p}$. It is obvious that $\mathbf{p}$ itself satisfies properties (P1)–(P4). An important feature of the type and symbol of level data is that properties (P1)–(P4) are preserved under vdC operations.

Example 5.10. We will verify that the polynomial family $\mathbf{q} = \partial_2 \mathbf{p}$ in Example 5.8 satisfies all of the properties (P1)–(P4). Indeed, property (P1) holds as all $\mathbf{u}(0;1)$, $\mathbf{u}(0;2)$, $\mathbf{u}(1;0)$, $\mathbf{u}(1;1)$, $\mathbf{u}(2;0)$ have a type. For all $0 \le a, b \le 2$, $a + b \le 2$, if $\mathbf{u}(b;a)$ is of type (r, i, v), then it is not hard to see that $r = \binom{b+a}{a}$, so property (P2) holds. Property (P3) can be verified by comparing the types and symbols of the pairs $(\mathbf{u}(0,1), \mathbf{u}(0,2))$ and $(\mathbf{u}(1,0), \mathbf{u}(2,0))$. Finally, property (P4) also holds since the first entry of every symbol in $\mathbf{q}$ is 1.

We caution the reader that the symbol and type may not be unique if the coefficients $b_{i,v}$ satisfy some algebraic relations. For example, in Example 5.8, if $b_{1,1} = b_{2,1}$, then both $(1, 2, 0)$ and $(1, 1, 0)$ are symbols of $\mathbf{u}(0;1) = (b_{1,1}, b_{2,1}, 0)$. However, the following result says that there is always a way to choose symbols and types so that properties (P1)–(P4) are preserved under vdC operations.

PROPOSITION 5.11. *Let $A = (L, s, \ell, \mathbf{g}, \mathbf{q})$ be a non-degenerate PET-tuple and $1 \le \rho \le \ell$. Assume that $A \to \partial_\rho A$ is 1-inherited. If A satisfies properties (P1)–(P4), then $\partial_\rho A$ also satisfies properties (P1)–(P4).*

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Proof. Suppose that $\mathbf{q} = (q_1, \ldots, q_\ell)$. Denote $\mathbf{q}^* = (q_1^*, \ldots, q_{2\ell}^*)$, where for all $1 \leq i \leq \ell$,

$$q_i^*(n; h_1, \ldots, h_{s+1}) = q_i(n + h_{s+1}; h_1, \ldots, h_s) - q_\rho(n; h_1, \ldots, h_s)$$

and

$$q_{\ell+i}^*(n; h_1, \ldots, h_{s+1}) = q_i(n; h_1, \ldots, h_s) - q_\rho(n; h_1, \ldots, h_s).$$

Assuming that

$$q_i(n; h_1, \ldots, h_s) = \sum_{b, a_1, \ldots, a_s \in \mathbb{N}_0^L} h_1^{a_1} \cdots h_s^{a_s} n^b \cdot u_i(b; a_1, \ldots, a_s)$$

for all $1 \leq i \leq \ell$, we may write $\mathbf{q}$ as

$$\mathbf{q}(n; h_1, \ldots, h_s) = \sum_{b, a_1, \ldots, a_s \in \mathbb{N}_0^L} h_1^{a_1} \cdots h_s^{a_s} n^b \cdot \mathbf{u}(b; a_1, \ldots, a_s),$$

and define $\mathbf{u}^*(b; a_1, \ldots, a_{s+1})$ in a similar way.

One can immediately check that

$$\begin{aligned} &q_i(n + h_{s+1}; h_1, \ldots, h_s) \\ &\quad = \sum_{b, a_1, \ldots, a_{s+1} \in \mathbb{N}_0^L} h_1^{a_1} \cdots h_{s+1}^{a_{s+1}} n^b \cdot \binom{b + a_{s+1}}{b} u_i(b + a_{s+1}; a_1, \ldots, a_s). \end{aligned}$$

(For $a = (a_1, \ldots, a_L), b = (b_1, \ldots, b_L) \in \mathbb{N}_0^L$, $\binom{a}{b}$ denotes the quantity $\prod_{m=1}^L \binom{a_m}{b_m}$.) Then,

$$u_i^*(b; a_1, \ldots, a_{s+1}) = \begin{cases} u_i(b; a_1, \ldots, a_s) - u_\rho(b; a_1, \ldots, a_s), & a_{s+1} = \mathbf{0}, \\ \binom{b + a_{s+1}}{b} u_i(b + a_{s+1}; a_1, \ldots, a_s), & a_{s+1} \neq \mathbf{0}, \end{cases} \tag{22}$$

and

$$u_{i+\ell}^*(b; a_1, \ldots, a_{s+1}) = \begin{cases} u_i(b; a_1, \ldots, a_s) - u_\rho(b; a_1, \ldots, a_s), & a_{s+1} = \mathbf{0}, \\ \mathbf{0}, & a_{s+1} \neq \mathbf{0}. \end{cases} \tag{23}$$

We first show that $\mathbf{q}$ satisfying properties (P1)–(P4) implies the same for $\mathbf{q}^*$.

Since $\mathbf{q}$ satisfies property (P1), for all $(b; a_1, \ldots, a_s)$, there exists a choice of type

$$(r(b; a_1, \ldots, a_s), i(b; a_1, \ldots, a_s), v(b; a_1, \ldots, a_s))$$

and symbol

$$w(b; a_1, \ldots, a_s) = (w_1(b; a_1, \ldots, a_s), \ldots, w_\ell(b; a_1, \ldots, a_s))$$

such that properties (P2)–(P4) hold. By property (P2), we have that

$$r(b; a_1, \ldots, a_s) = \binom{b + a_1 + \cdots + a_s}{a_1, \ldots, a_s} \text{ and } v(b; a_1, \ldots, a_s) = b + a_1 + \cdots + a_s. \tag{24}$$

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By property (P4), $w_1(b; a_1, \ldots, a_s) = 1$. We have that

$$u_m(b; a_1, \ldots, a_s) = r(b_{w_m(b;a_1,\ldots,a_s),v(b;a_1,\ldots,a_s)} - b_{i(b;a_1,\ldots,a_s),v(b;a_1,\ldots,a_s)}). \tag{25}$$

From now on, we fix some $b, a_1, \ldots, a_{s+1}$ and denote $\mathbf{x} := (b + a_{s+1}; a_1, \ldots, a_s)$. By equations (22), (23), and (25), it is not hard to see that $\mathbf{u}^*(b; a_1, \ldots, a_s, a_{s+1})$ is

$$\begin{cases} \text{of type } (r(\mathbf{x}), w_\rho(\mathbf{x}), v(\mathbf{x})) \text{ and has symbol } (\mathrm{w}(\mathbf{x}), \mathrm{w}(\mathbf{x})), & a_{s+1} = \mathbf{0}, \\ \text{of type } \left(r(\mathbf{x})\binom{b + a_{s+1}}{b}, i(\mathbf{x}), v(\mathbf{x})\right) \text{ and has symbol } (\mathrm{w}(\mathbf{x}), i, \ldots, i), & a_{s+1} \neq \mathbf{0}. \end{cases} \tag{26}$$

So, each $\mathbf{u}^*(b; a_1, \ldots, a_s, a_{s+1})$ has a choice of type and symbol given by equation (26). So $\mathbf{q}^*$ satisfies property (P1). It suffices to show that the type and symbol given by equation (26) satisfies properties (P2)–(P4).

To check property (P2) for $\mathbf{q}^*$, suppose that $\mathbf{u}^*(b; a_1, \ldots, a_{s+1})$ is of type (r, i, v). By equation (26), we have that $\mathbf{u}(b + a_{s+1}; a_1, \ldots, a_s)$ is of type $(r\binom{b+a_{s+1}}{b}^{-1}, i', v) = (r(\mathbf{x}), i(\mathbf{x}), v(\mathbf{x}))$ for some i'. Since $\mathbf{q}$ satisfies property (P2), we have the same for $\mathbf{q}^*$ as it follows from equation (24) that

$$r = \binom{b + a_{s+1}}{b}\binom{(b + a_{s+1}) + a_1 + \cdots + a_s}{a_1, \ldots, a_s} = \binom{b + a_1 + \cdots + a_{s+1}}{a_1, \ldots, a_{s+1}}$$

and

$$v = (b + a_{s+1}) + a_1 + \cdots + a_s = b + a_1 + \cdots + a_{s+1}.$$

To show property (P3), pick any $b', a'_1, \ldots, a'_{s+1}$ with $(a_1, a'_1), \ldots, (a_{s+1}, a'_{s+1}) \in S$. For convenience, denote $\mathbf{x}' := (b' + a'_{s+1}; a'_1, \ldots, a'_s)$. Since $\mathbf{q}$ satisfies property (P3), we have that $i(\mathbf{x}) = i(\mathbf{x}')$ and that $w(\mathbf{x}) = w'(\mathbf{x}')$. Since $(a_{s+1}, a'_{s+1}) \in S$, by equation (26), we have that $\mathbf{q}^*$ satisfies property (P3).

Finally, it is straightforward from equation (26) that property (P4) is also heritable.

Since the polynomials in $\partial_\rho A$ are obtained by removing some terms from the tuple $\mathbf{q}^*$ (but not the first one as $A \to \partial_\rho A$ is 1-inherited), the fact that $\mathbf{q}^*$ satisfies properties (P1)–(P4) implies that $\partial_\rho A$ also satisfies properties (P1)–(P4). □

For the family of essentially distinct polynomials $\mathbf{p} = (p_1, \ldots, p_k)$, Proposition 5.11 implies that $\partial_{i_k} \ldots \partial_{i_1}\mathbf{p}$ satisfies properties (P1)–(P4) for all $k, i_1, \ldots, i_k \in \mathbb{N}$. In the special case when $\partial_{i_k} \ldots \partial_{i_1}\mathbf{p}$ is of degree 1, properties (P1)–(P4) provide us with some information on the seminorm we use to bound $S(\partial_{i_k} \ldots \partial_{i_1} A, 1)$. To be more precise, if properties (P1)–(P4) hold for some non-degenerate $\mathbf{q}$, then there is some connection between the level data of $\mathbf{q}$ and the groups $G_{1,0}(\mathbf{p}), G_{1,2}(\mathbf{p}), \ldots, G_{1,k}(\mathbf{p})$.

PROPOSITION 5.12. *Suppose that properties (P1)–(P4) hold for some non-degenerate* $\mathbf{q}$. *Then for all* $0 \leq m \leq \ell, m \neq 1$, *the group*

$$H_{1,m}(\mathbf{q}) := G(\{u_1(b; a_1, \ldots, a_s) - u_m(b; a_1, \ldots, a_s) \colon (b, a_1, \ldots, a_s) \in (\mathbb{N}_0^L)^{s+1}, b \neq \mathbf{0}\})$$

contains at least one of the groups $G_{1,j}(\mathbf{p}), 0 \leq j \leq k, j \neq 1$.

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We remark that although Proposition 5.12 holds for all non-degenerate $\mathbf{q}$, we will use it for the case $\deg(\mathbf{q}) = 1$. We first give an example to explain the idea behind it.

Example 5.13. Let $\mathbf{p} = (p_1, p_2)$ be as in Example 4.7. Then, $G_{1,0}(\mathbf{p}) = G(b_{1,2})$ and $G_{1,2}(\mathbf{p}) = G(b_{1,2} - b_{2,2})$. Let $\mathbf{u}(b; a)$ be the level data of $\partial_2\mathbf{p}$. Then,

$$\begin{aligned}
&\mathbf{u}(1; 0) = (b_{1,1} - b_{2,1}) \cdot (1, 0, 1) \text{ is of type } (1, 2, 1) \text{ and has symbol } (1, 2, 1),\\
&\mathbf{u}(1; 1) = 2b_{1,2} \cdot (1, 0, 0) + 2b_{2,2} \cdot (0, 1, 0) \text{ is of type } (2, 0, 2) \text{ and has symbol } (1, 2, 0),\\
&\mathbf{u}(2; 0) = (b_{1,2} - b_{2,2}) \cdot (1, 0, 1) \text{ is of type } (1, 2, 2) \text{ and has symbol } (1, 2, 1).
\end{aligned}$$

Here, we will not compute $\mathbf{u}(b; a)$ for $b = 0$ as it is irrelevant to our purposes. It is easy to see that $H_{1,0}(\partial_2\mathbf{p}) = G(b_{1,1} - b_{2,1}, b_{1,2}, b_{1,2} - b_{2,2}) \supseteq G_{1,0}(\mathbf{p}) \cup G_{1,2}(\mathbf{p})$, $H_{1,2}(\partial_2\mathbf{p}) = G(b_{1,1} - b_{2,1}, b_{1,2} - b_{2,2}) \supseteq G_{1,2}(\mathbf{p})$, $H_{1,3}(\partial_2\mathbf{p}) = G(b_{1,2}) = G_{1,0}(\mathbf{p})$. So, Proposition 5.12 holds for $\partial_2\mathbf{p}$.

Let $\mathbf{u}'(b; a_1, a_2)$ denote the level data of $\partial_2\partial_2\mathbf{p}$. Then,

$$\begin{aligned}
&\mathbf{u}'(1; 0, 0) = (b_{1,1} - b_{2,1}) \cdot (1, 1, 1, 1) \text{ is of type } (1, 2, 1) \text{ and has symbol } (1, 1, 1, 1),\\
&\mathbf{u}'(1; 1, 0) = 2(b_{1,2} - b_{2,2}) \cdot (1, 0, 1, 0) - 2b_{2,2} \cdot (0, 1, 0, 1)\\
&\qquad\qquad \text{is of type } (2, 2, 2) \text{ and has symbol } (1, 0, 1, 0),\\
&\mathbf{u}'(1; 0, 1) = 2(b_{1,2} - b_{2,2}) \cdot (1, 1, 0, 0) \text{ is of type } (2, 2, 2) \text{ and has symbol } (1, 1, 2, 2),\\
&\mathbf{u}'(2; 0, 0) = (b_{1,2} - b_{2,2}) \cdot (1, 1, 1, 1) \text{ is of type } (1, 2, 2) \text{ and has symbol } (1, 1, 1, 1).
\end{aligned}$$

(We do not compute the types and symbols for $\mathbf{u}'(b; a_1, a_2)$ for $b = 0$.) It is easy to see that $H_{1,0}(\partial_2\partial_2\mathbf{p}) = G(b_{1,1} - b_{2,1}, b_{1,2} - b_{2,2}) \supseteq G_{1,2}(\mathbf{p})$, $H_{1,2}(\partial_2\partial_2\mathbf{p}) = G(b_{1,2}) = G_{1,0}(\mathbf{p})$, $H_{1,3}(\partial_2\partial_2\mathbf{p}) = G(b_{1,2} - b_{2,2}) \supseteq G_{1,2}(\mathbf{p})$, $H_{1,4}(\partial_2\partial_2\mathbf{p}) = G(b_{1,2}, b_{1,2} - b_{2,2}) \supseteq G_{1,0}(\mathbf{p}) \cup G_{1,2}(\mathbf{p})$. So, Proposition 5.12 holds for $\partial_2\partial_2\mathbf{p}$.

Finally, let $\mathbf{u}''(b; a_1, a_2, a_3)$ denote the level data of $\partial_4\partial_2\partial_2\mathbf{p}$. Then, $\deg(\partial_4\partial_2\partial_2\mathbf{p}) = 1$ and

$$\begin{aligned}
&\mathbf{u}''(1; 0, 0, 0) = (0, 0, 0, 0, 0, 0, 0) \text{ is trivial},\\
&\mathbf{u}''(1; 1, 0, 0) = 2b_{1,2} \cdot (1, 0, 1, 0, 1, 0, 1)\\
&\qquad\qquad \text{is of type } (2, 0, 2) \text{ and has symbol } (1, 0, 1, 0, 1, 0, 1),\\
&\mathbf{u}''(1; 0, 1, 0) = 2(b_{1,2} - b_{2,2}) \cdot (1, 1, 0, 0, 1, 1, 0)\\
&\qquad\qquad \text{is of type } (2, 2, 2) \text{ and has symbol } (1, 1, 2, 2, 1, 1, 2),\\
&\mathbf{u}''(1; 0, 0, 1) = 2(b_{1,2} - b_{2,2}) \cdot (1, 1, 1, 1, 0, 0, 0)\\
&\qquad\qquad \text{is of type } (2, 2, 2) \text{ and has symbol } (1, 1, 1, 1, 2, 2, 2).
\end{aligned}$$

(Once more, we do not compute the types and symbols for $\mathbf{u}''(b; a_1, a_2, a_3)$ for $b = 0$.) It is easy to see that $H_{1,0}(\partial_4\partial_2\partial_2\mathbf{p}) = H_{1,4}(\partial_4\partial_2\partial_2\mathbf{p}) = H_{1,6}(\partial_4\partial_2\partial_2\mathbf{p}) = G(b_{1,2}, b_{1,2} - b_{2,2}) \supseteq G_{1,1}(\mathbf{p})$, $H_{1,2}(\partial_4\partial_2\partial_2\mathbf{p}) = G(b_{1,2}) = G_{1,1}(\mathbf{p})$, and $H_{1,3}(\partial_4\partial_2\partial_2\mathbf{p}) = H_{1,5}(\partial_4\partial_2\partial_2\mathbf{p}) = H_{1,7}(\partial_4\partial_2\partial_2\mathbf{p}) = G(b_{1,2} - b_{2,2}) = G_{1,2}(\mathbf{p})$. So, Proposition 5.12 holds for $\partial_4\partial_2\partial_2\mathbf{p}$.

To briefly explain why Proposition 5.12 holds for Example 5.13, we explain, for convenience, why $H_{1,0}(\mathbf{q})$ contains either $G_{1,0}(\mathbf{p})$ or $G_{1,2}(\mathbf{p})$ for $\mathbf{q} = \partial_2\mathbf{p}, \partial_2\partial_2\mathbf{p}$, and $\partial_4\partial_2\partial_2\mathbf{p}$. Let $\mathbf{u}(b; a_1, \ldots, a_s)$ be level data of type (r, i, v) and symbol w, and $\mathbf{u}(b'; a'_1, \ldots, a'_s)$ be level data of type (r', i', v') and symbol w′. We say that the level data $\mathbf{u}(b'; a'_1, \ldots, a'_s)$ *dominates* (or *strictly dominates*) $\mathbf{u}(b; a_1, \ldots, a_s)$ if $i = i'$, $\mathrm{w} = \mathrm{w}'$ and $|v'| \geq |v|$ (or $|v'| > |v|$, respectively). In Example 5.13, it is not hard to see that for all $b \in \mathbb{N}, a_1, a_2 \in \mathbb{N}_0$, if $\mathbf{u}'(b; a_1, a_2)$ is not of type $(*, *, 2)$, then there exist $b' \in \mathbb{N}, a'_1, a'_2 \in \mathbb{N}_0$ such that $\mathbf{u}'(b'; a'_1, a'_2)$ strictly dominates $\mathbf{u}'(b; a_1, a_2)$ (in this example, $\mathbf{u}'(1; 0, 0)$ is strictly dominated by $\mathbf{u}'(2; 0, 0)$). Similar conclusions hold for $\mathbf{u}(b; a)$ and $\mathbf{u}''(b; a_1, a_2, a_3)$. In other words, the group $H_{1,0}(\mathbf{q})$ must contain the elements of level data of type $(*, *, 2)$, and thus it must contain one of $G_{1,0}(\mathbf{p})$ and $G_{1,2}(\mathbf{p})$.

We are now ready to prove Proposition 5.12. The main point is that given any non-trivial level data $\mathbf{u}(b; a_1, \ldots, a_s)$, we can find another one, $\mathbf{u}(b'; a'_1, \ldots, a'_s)$, which dominates $\mathbf{u}(b; a_1, \ldots, a_s)$ and is of type $(*, *, v)$ with $|v|$ being as large as possible (in this step, we need to exploit the properties (P1)–(P4)). After that, we use the information of the 'top' level data $\mathbf{u}(b'; a'_1, \ldots, a'_s)$ to conclude.

Proof of Proposition 5.12. We start with a claim. Recall that S denotes the set of all $(a, a') \in \mathbb{N}_0^{2L}$ such that a and a' are both $\mathbf{0}$ or both different than $\mathbf{0}$ (check Definition 2.1 for notation).

CLAIM. *Let $d, s \in \mathbb{N}$ and $b, a_1, \ldots, a_s, v \in \mathbb{N}_0^L$. If $|v| \geq |b + a_1 + \cdots + a_s|$, then there exist $b', a'_1, \ldots, a'_s \in \mathbb{N}_0^L$ such that $(a_1, a'_1), \ldots, (a_s, a'_s) \in S$, $|b'| \geq |b|$ and $b' + a'_1 + \cdots + a'_s = v$.*

Proof of the claim. To show the claim, we may first assume that $|v| = |b + a_1 + \cdots + a_s|$. Indeed, if $c := |v| - |b + a_1 + \cdots + a_s| > 0$, then we write $b' = b + (c, 0, \ldots, 0)$. Then, $|b'| > |b|$ and $|v| = |b' + a_1 + \cdots + a_s|$.

It suffices to show that if $|v - (b + a_1 + \cdots + a_s)| > 0$, then there exist $b', a'_1, \ldots, a'_s \in \mathbb{N}_0^L$ such that $(a_1, a'_1), \ldots, (a_s, a'_s) \in S$, $|b'| = |b|$, and $|v - (b' + a'_1 + \cdots + a'_s)| < |v - (b + a_1 + \cdots + a_s)|$. Since $|v| = |b + a_1 + \cdots + a_s|$, $|v - (b + a_1 + \cdots + a_s)|$ is at least 2, there exist $1 \leq i, j \leq L, i \neq j$ such that the tth coordinate of $v - (b + a_1 + \cdots + a_s)$ is at least 1 for $t = i$ and is at most -1 for $t = j$. We may assume without loss of generality that $i = 1$ and $j = 2$. Then we have that $v_1 \geq 1$, and one of $b_2, a_{1,2}, \ldots, a_{s,2}$ is at least 1. If $b_2 \geq 1$, then $b' = b + (1, -1, 0, \ldots, 0) \in \mathbb{N}_0^L$, $a'_i = a_i$, $1 \leq i \leq s$ satisfy the requirement. If one of $a_{i,2}$ is positive, then $b' = b$, $a'_i = a_i + (1, -1, 0, \ldots, 0) \in \mathbb{N}_0^L$, $a'_j = a_j$, $1 \leq j \leq s$, $j \neq i$ satisfy the requirement. This proves the claim. □

Consider the group $H_{1,m}(\mathbf{q})$ for some $0 \leq m \leq \ell, m \neq 1$. Since $\mathbf{q}$ is non-degenerate, there exist some $b, a_1, \ldots, a_s \in \mathbb{N}_0^L$, $|b| \geq 1$ such that $u_1(b; a_1, \ldots, a_s) - u_m(b; a_1, \ldots, a_s) \neq \mathbf{0}$. By property (P1), we may assume that $\mathbf{u}(b; a_1, \ldots, a_s)$ is of type (r, i, v) and has symbol $(w_1, \ldots, w_\ell)$. Since $w_1 = 1$,

$$u_1(b; a_1, \ldots, a_s) - u_m(b; a_1, \ldots, a_s) = r(b_{1,v} - b_{w_m,v}).$$

Recall that for $\mathbf{p} = (p_1, \ldots, p_k)$, $d_{1,0} = \deg(p_1)$ and $d_{1,j} = \deg(p_1 - p_j)$ for $2 \leq j \leq \ell$. Since $u_1(b; a_1, \ldots, a_s) - u_m(b; a_1, \ldots, a_s) \neq \mathbf{0}$, we have that $w_m \neq 1$ and $|v| = |b + a_1 + \cdots + a_s| \leq d_{1,w_m}$.

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By the claim, for all $v' \in \mathbb{N}_0^L$ with $|v'| = d_{1,w_m}$, there exist $b', a'_1, \ldots, a'_s \in \mathbb{N}_0^L$ such that $(a_1, a'_1), \ldots, (a_s, a'_s) \in S$, $|b'| \geq 1$ and $b' + a'_1 + \cdots + a'_s = v'$. By properties (P2) and (P3), $\mathbf{u}(b'; a'_1, \ldots, a'_s)$ is of type (r', i, v'), $r' \neq 0$ and has symbol $(w_1, \ldots, w_\ell)$ (that is, $\mathbf{u}(b'; a'_1, \ldots, a'_s)$ dominates $\mathbf{u}(b; a_1, \ldots, a_s)$). So

$$u_1(b'; a'_1, \ldots, a'_s) - u_m(b'; a'_1, \ldots, a'_s) = r'(b_{1,v'} - b_{w_m,v'}).$$

In other words, for all $v' \in \mathbb{N}^L$ with $|v'| = d_{1,w_m}$, the group $H_{1,m}(\mathbf{q})$ contains a non-zero multiple of $b_{1,v'} - b_{w_m,v'}$. So this group contains $G_{1,w_m}(\mathbf{p})$ and we are done. □

We are now ready to prove Proposition 5.5.

Proof of Proposition 5.5. Let A denote the PET-tuple $(L, 0, k, (f_1, \ldots, f_k), (p_1, \ldots, p_k))$. Then, for all $\tau > 0$,

$$S(A, \tau) = \sup_{\substack{(I_N)_{N\in\mathbb{N}} \\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \left\| \mathbb{E}_{n\in I_N} \prod_{m=1}^{k} T_{p_m(n)} f_m \right\|_{L^2(\mu)}^{\tau}.$$

By assumption, A is non-degenerate. We only prove equation (21) for f_1 as the other cases are similar. We first assume that A is 1-standard for f_1. By Theorem 4.9, there exist $t \in \mathbb{N}_0$, depending only on d, k, K, L, and finitely many vdC operations $\partial_{\rho_1}, \ldots, \partial_{\rho_t}$, $\rho_1, \ldots, \rho_t \neq 1$ such that for all $1 \leq t' \leq t$, $\partial_{\rho_{t'}} \ldots \partial_{\rho_1} A$ is non-degenerate and 1-standard for f_1, and that $\partial_{\rho_{t'-1}} \ldots \partial_{\rho_1} A \to \partial_{\rho_{t'}} \ldots \partial_{\rho_1} A$ is 1-inherited. Moreover, $A' := \partial_{\rho_t} \ldots \partial_{\rho_1} A$ is of degree 1. By Proposition 4.8, $S(A, 2^t) \leq C \cdot S(A', 1)$ for some $C > 0$ that depends only on t. Write

$$S(A', 1) = \overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s\in\mathbb{Z}^L} \sup_{\substack{(I_N)_{N\in\mathbb{N}} \\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \left\| \mathbb{E}_{n\in I_N} \times \prod_{m=1}^{\ell} T_{\mathbf{d}_m(h_1,\ldots,h_s)\cdot n + r_m(h_1,\ldots,h_s)} g_m(x; h_1, \ldots, h_s) \right\|_2$$

for some $s, \ell \in \mathbb{N}$, functions $g_1, \ldots, g_\ell \colon X \times (\mathbb{Z}^L)^s \to \mathbb{R}$, where $g_1(\cdot; h_1, \ldots, h_s) = f_1$ for all $h_1, \ldots, h_s$ and such that each $g_m(\cdot; h_1, \ldots, h_s)$ is an $L^\infty(\mu)$ function bounded by 1, and polynomials $\mathbf{d}_m \colon (\mathbb{Z}^L)^s \to (\mathbb{Z}^d)^L$ and $r_m \colon (\mathbb{Z}^L)^s \to \mathbb{Z}^d$, $1 \leq m \leq \ell$, where the values of $\mathbf{d}_m, r_m$ are vectors with integer coordinates as vdC operations transform integer-valued polynomials to integer-valued polynomials. (Here, $\mathbf{d}_m(h_1, \ldots, h_s) \cdot n := n_1 d_{m,1}(h_1, \ldots, h_s) + \cdots + n_L d_{m,L}(h_1, \ldots, h_s)$, where $n = (n_1, \ldots, n_L)$, $n_i \in \mathbb{Z}$, and $\mathbf{d}_m = (d_{m,1}, \ldots, d_{m,L})$, $d_{m,i} \colon (\mathbb{Z}^L)^s \to \mathbb{Z}^d$.)

Let $\mathbf{c}_1 = -\mathbf{d}_1$ and $\mathbf{c}_m = \mathbf{d}_m - \mathbf{d}_1$ for $m \neq 1$. Since A' is non-degenerate, we have that $\mathbf{c}_1, \ldots, \mathbf{c}_s \not\equiv \mathbf{0}$. Similar to the proof of [**7**, Proposition 6.1] (see also [**17**, Proposition 1]), if $\ell \geq 2$, we also have that

$$\begin{aligned} S(A', 1) &\leq C' \cdot \overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s\in\mathbb{Z}^L} |||T_{r_1(h_1,\ldots,h_s)} f_1|||_{\{G'(\mathbf{c}_i(h_1,\ldots,h_s))\}_{1\leq i\leq \ell}} \\ &= C' \cdot \overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s\in\mathbb{Z}^L} |||f_1|||_{\{G'(\mathbf{c}_i(h_1,\ldots,h_s))\}_{1\leq i\leq \ell}} \end{aligned}$$

for some $C' > 0$ depending only on ℓ. If $\ell = 1$, using the mean ergodic theorem (see for example [**7**, Theorem 2.3]) and [**7**, Lemma 2.4(iv), (vi)], we have

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$$\begin{aligned}S(A',1) &= \overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s\in\mathbb{Z}^L}\|\mathbb{E}(T_{r_1(h_1,\ldots,h_s)}f_1|\mathcal{I}(\mathbf{c}_1(h_1,\ldots,h_s)))\|_2\\ &= \overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s\in\mathbb{Z}^L}\|\mathbb{E}(f_1|\mathcal{I}(\mathbf{c}_1(h_1,\ldots,h_s)))\|_2\\ &= \overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_s\in\mathbb{Z}^L}|||f_1|||_{G'(\mathbf{c}_1(h_1,\ldots,h_s))}.\end{aligned}$$

Combining this with the fact that $S(A,2^t)\leq C\cdot S(A',1)$, we get equation (21).

We now consider the groups $H_{1,m}$, $0\leq m\leq \ell$, $m\neq 1$. Suppose that

$$\mathbf{c}_m(h_1,\ldots,h_s)=\sum_{a_1,\ldots,a_s\in\mathbb{N}_0^L} h_1^{a_1}\ldots h_s^{a_s}\cdot\mathbf{v}_m(a_1,\ldots,a_s)$$

and

$$\mathbf{d}_m(h_1,\ldots,h_s)=\sum_{a_1,\ldots,a_s\in\mathbb{N}_0^L} h_1^{a_1}\ldots h_s^{a_s}\cdot\mathbf{u}_m(a_1,\ldots,a_s)$$

for some vectors $\mathbf{u}_m(a_1,\ldots,a_s)=(u_{m,1}(a_1,\ldots,a_s),\ldots,u_{m,L}(a_1,\ldots,a_s))$, and $\mathbf{v}_m(a_1,\ldots,a_s)=(v_{m,1}(a_1,\ldots,a_s),\ldots,v_{m,L}(a_1,\ldots,a_s))\in(\mathbb{Q}^d)^L$ with all but finitely many terms being zero for each m. Obviously A satisfies properties (P1)–(P4) (recall that the first coordinate of a symbol can always be chosen to be 1 by the comment after Definition 5.7). $A'=\partial_{\rho_t}\ldots\partial_{\rho_1}A$ satisfies, by Proposition 5.11, properties (P1)–(P4). Since $\deg(A')=1$, by Proposition 5.12, each of

$$\begin{aligned}&G(\{v_{1,j}(a_1,\ldots,a_s)\colon (a_1,\ldots,a_s)\in(\mathbb{N}_0^L)^s, 1\leq j\leq L\})\\ &\quad= G(\{u_{1,j}(a_1,\ldots,a_s)\colon (a_1,\ldots,a_s)\in(\mathbb{N}_0^L)^s, 1\leq j\leq L\})=H_{1,0},\end{aligned}$$

and

$$\begin{aligned}&G(\{v_{m,j}(a_1,\ldots,a_s)\colon (a_1,\ldots,a_s)\in(\mathbb{N}_0^L)^s\})\\ &\quad= G(\{u_{1,j}(a_1,\ldots,a_s)-u_{m,j}(a_1,\ldots,a_s)\colon (a_1,\ldots,a_s)\in(\mathbb{N}_0^L)^s\})\\ &\quad= H_{1,m},\ 2\leq m\leq \ell,\end{aligned}$$

contains some of the groups $G_{1,j}(\mathbf{p})$, $0\leq j\leq k$, $j\neq 1$.

Next we assume that A is not 1-standard for f_1. In this case, we need to invoke a 'dimension increment' argument to convert A to be 1-standard for f_1. We may assume without loss of generality that p_k has the highest degree. Since A is semi-standard for f_1, by [7, Proposition 6.3], there exists a PET-tuple $A'=(2L,0,\ell,\mathbf{p}',\mathbf{g})$ which is non-degenerate and 1-standard for f_1 such that $S(A,2\tau)\leq S(A',\tau)$ for all $\tau>0$. Moreover, $\mathbf{p}'$ is obtained by selecting some polynomials from the family

$$\mathbf{q}:=(p_1(n)-p_k(n'),\ldots,p_k(n)-p_k(n'),p_1(n')-p_k(n'),\ldots,p_{k-1}(n')-p_k(n'))$$

with $2L$-dimensional variables (n,n'), where $p_1(n)-p_k(n')$ is selected in $\mathbf{p}'$ and is associated to f_1. It is not hard to check that $G_{1,j}(\mathbf{q})=G_{1,j}(\mathbf{p})$ for $0\leq j\leq k$, $j\neq 1$. Moreover, for $1\leq j\leq k-1$, $G_{1,k+j}(\mathbf{q})=G_{1,0}(\mathbf{p})+G_{j,0}(\mathbf{p})\supseteq G_{1,0}(\mathbf{p})$ if $d_{1,0}=d_{j,0}$, $G_{1,k+j}(\mathbf{q})=G_{1,0}(\mathbf{p})=G_{1,j}(\mathbf{q})$ if $d_{1,0}>d_{j,0}$, and $G_{1,k+j}(\mathbf{q})=G_{j,0}(\mathbf{p})=G_{1,j}(\mathbf{q})$ if $d_{1,0}<d_{j,0}$. In other words, each $G_{1,j}(\mathbf{q})$, and thus each $G_{1,j}(\mathbf{p}')$, contains some $G_{1,j'}(\mathbf{p})$. Applying the previous conclusion to A', we are done.

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Finally, the fact that each $\mathbf{v}_{i,m}(a_1, \ldots, a_s)$ is a polynomial function in terms of the coefficients of p_i, $1 \le i \le k$, whose degree depends only on d, k, K, L, follows easily from the polynomial nature of the vdC operations. □

We now have all the ingredients needed to prove Theorem 2.11. In fact, Theorem 2.11 has a proof similar to that of [7, Theorem 5.1].

Proof of Theorem 2.11. By Propositions 5.5 and 5.2, and the definition of Host–Kra characteristic factors, the left-hand side of equation (10) is 0 if for some $1 \le i \le k$, f_i is orthogonal to $Z_{\{H_{i,m}\}_{1\le m\le t_i}^{\times D_i}}(\mathbf{X})$ for some $t_i, D_i \in \mathbb{N}$, where $H_{i,m}$ is defined as in Proposition 5.5. By Proposition 5.5, $H_{i,m}$ is contained in one of $G_{i,j}(\mathbf{p})$, $0 \le j \le k$, $j \ne i$. Using Proposition 3.1(v), if some f_i, $1 \le i \le k$, is orthogonal to $Z_{\{G_{i,j}(\mathbf{p})\}_{0\le j\le k, j\ne i}^{\times D_i t_i}}(\mathbf{X})$, then it is also orthogonal to $Z_{\{H_{i,m}\}_{1\le m\le t_i}^{\times D_i}}(\mathbf{X})$, and thus the left-hand side of equation (10) is 0.

The 'in particular' part follows from Corollary 3.2. □

Remark 5.14. We remark that the number D derived in Theorem 2.11 is not optimal.

To see this, recall that this number indicates the step of the nilsequence in the splitting results. For multicorrelation sequences with general polynomial iterates, this D can be taken to be equal to the number of vdC operations we have to perform for all the iterates to become constant (e.g. [10] via [13], and [25] via [10]). At this point, a word of caution is necessary for the approach of this paper. Specifically, while the number D in Theorem 2.11 can still be chosen to be the number of transformations in the case of linear iterates (given that there is no dependence on h—the variables arising from the vdC operations), in the general case, the picture is quite different. By carefully tracking the constants that appear in Propositions 5.2 and 5.4, D can be chosen to be the maximum of $t_i s_i^{[t_i(s_i'+1)^{s_i L}]+1}$, $1 \le i \le k$, where s_i' is the degree of $\mathbf{p}$, t_i is the number of terms remaining when $\mathbf{p}$ is converted to a linear family which is 1-standard for f_i for the first time, and s_i is t_i plus the number of vdC operations needed to convert $\mathbf{p}$ in such a way. (The details are left to the interested readers.)

6. *Proof of main results*

Using Theorem 2.11, we prove in this section Theorems 2.2, 2.5, and 2.9.

Proof of Theorem 2.2. We follow and adapt the proof strategy in [28, §3]. To avoid confusion, we use $|||\cdot|||$ to denote the Host–Kra seminorm on the $\mathbb{Z}^d$-system $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$, and $|||\cdot|||'$ to denote the Host–Kra seminorm on the $\mathbb{Z}^d$-system $(X^2, \mathcal{B}^2, \mu^{\otimes 2}, (S_n)_{n\in\mathbb{Z}^d})$, where $S_n = T_n \times T_n$.

Let $(I_N)_{N\in\mathbb{N}}$ be a Følner sequence in $\mathbb{Z}^L$. Note that

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \left| \int_X f_0 \cdot T_{p_1(n)} f_1 \cdots T_{p_k(n)} f_k \, d\mu \right|^2$$
$$= \lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \int_{X^2} f_0 \otimes \bar{f}_0 \cdot \prod_{i=1}^{k} S_{p_i(n)}(f_i \otimes \bar{f}_i) \, d(\mu \times \mu). \tag{27}$$

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By Theorem 2.11, there exists $D' \in \mathbb{N}$ such that for all $1 \leq i \leq k$, equation (27) equals to 0 if $|||f_i \otimes \bar{f}_i|||'_{\{\tilde{G}_{i,j}(\mathbf{p})\}^{\times D'}_{0\leq j\leq k, j\neq i}} = 0$, where $\tilde{G}_{i,j}(\mathbf{p})$ is the group action generated by S_n for all $n \in G_{i,j}(\mathbf{p})$. (Strictly speaking, $G_{i,j}(\mathbf{p})$ and $\tilde{G}_{i,j}(\mathbf{p})$ are the same subgroup of $\mathbb{Z}^d$. We distinguish these two notions to indicate that $G_{i,j}(\mathbf{p})$ and $\tilde{G}_{i,j}(\mathbf{p})$ are attached to the distinct group actions $(T_n)_{n\in G_{i,j}(\mathbf{p})}$ and $(S_n)_{n\in G'_{i,j}(\mathbf{p})}$.) Using Lemma 3.4, $|||f_i \otimes \bar{f}_i|||'_{\{\tilde{G}_{i,j}(\mathbf{p})\}^{\times D'}_{0\leq j\leq k, j\neq i}}$ is bounded by $|||f_i|||^2_{\{G_{i,j}(\mathbf{p})\}^{\times D'}_{0\leq j\leq k, j\neq i}, \mathbb{Z}^d}$, which is equal to $|||f_i|||^2_{(\mathbb{Z}^d)^{\times(D+1)}}$ with $D = kD'$ by our ergodicity assumptions and Corollary 3.2.

Therefore, for $1 \leq i \leq k$, we have

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \int_{X^2} (f_0 \otimes \bar{f}_0) \cdot \prod_{i=1}^{k} S_{p_i(n)}(f_i \otimes \bar{f}_i)\, d(\mu \times \mu) = 0 \text{ if } |||f_i|||_{(\mathbb{Z}^d)^{\times(D+1)}} = 0. \tag{28}$$

Then, equation (28) and Theorem 3.5 imply that the sequence

$$\Lambda(n) := a(n) - \int_X f_0 \cdot T_{p_1(n)}\mathbb{E}(f_1 \mid Z_{(\mathbb{Z}^d)^{\times(D+1)}}(\mathbf{X})) \cdots T_{p_k(n)}\mathbb{E}(f_k \mid Z_{(\mathbb{Z}^d)^{\times(D+1)}}(\mathbf{X}))\, d\mu \tag{29}$$

is a nullsequence.

Let $\varepsilon > 0$. By our assumption, $\mathbf{X}$ is ergodic. The factor $Z_{(\mathbb{Z}^d)^{\times(D+1)}}(\mathbf{X})$, via Theorem 3.5, is an inverse limit of D-step nilsystems. Thus, there exists a factor of $Z_{(\mathbb{Z}^d)^{\times(D+1)}}(\mathbf{X})$ with the structure of a D-step nilsystem $(\tilde{X}, \mathcal{B}(\tilde{X}), \mu_{\tilde{X}}, T_1, \ldots, T_d)$, on which each T_i acts as a niltranslation by an element $a_i \in \tilde{X}$, such that for $\tilde{f}_i = \mathbb{E}(f_i \mid \tilde{X})$ and $\vec{a} := (a_1, \ldots, a_d)$, we have

$$\left| \int_X f_0 \cdot \prod_{i=1}^{k} T_{p_i(n)}\mathbb{E}(f_i \mid Z_{(\mathbb{Z}^d)^{\times(D+1)}}(\mathbf{X}))\, d\mu - \int_{\tilde{X}} \tilde{f}_0 \cdot \prod_{i=1}^{k} \vec{a}_{p_i(n)} \tilde{f}_i\, d\mu_{\tilde{X}} \right| < \varepsilon$$

for all $n \in \mathbb{Z}^L$, where, if $p_i = (p_{i,1}, \ldots, p_{i,d})$, then $\vec{a}_{p_i(n)}$ denotes the niltranslation by the element $(a_1^{p_{i,1}(n)}, \ldots, a_d^{p_{i,d}(n)})$. Therefore, there exists a nullsequence Λ such that

$$\left\| a(n) - \left(\int_{\tilde{X}} \tilde{f}_0 \cdot \vec{a}_{p_1(n)} \tilde{f}_1 \cdots \vec{a}_{p_k(n)} \tilde{f}_k\, d\mu_{\tilde{X}} + \Lambda(n) \right) \right\|_{\ell^\infty(\mathbb{Z}^L)} < \varepsilon. \tag{30}$$

A standard approximation argument allows us to assume without loss of generality that $\tilde{f}_1, \ldots, \tilde{f}_k \in C(\tilde{X})$ in equation (30). Applying [**28**, Theorem 2.5] to the nilmanifold $\tilde{X}^k$, the diagonal subnilmanifold $\{(x, \ldots, x) : x \in \tilde{X}\}$, the polynomial sequence $(\vec{a}_{p_1(n)}, \ldots, \vec{a}_{p_k(n)})$, and the function $f(x_1, \ldots, x_k) = \tilde{f}_1(x_1) \cdots \tilde{f}_k(x_k) \in C(\tilde{X}^k)$, we obtain that the sequence

$$\psi(n) := \int_{\tilde{X}} \tilde{f}_0 \cdot \vec{a}_{p_1(n)} \tilde{f}_1 \cdots \vec{a}_{p_k(n)} \tilde{f}_k\, d\mu_{\tilde{X}}$$

is a sum of a D-step nilsequence and a nullsequence.

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Therefore, for each $\varepsilon > 0$, we can find a D-step nilsequence ψ (the one described above), a nullsequence Λ, and a bounded sequence δ with $\|\delta\|_{\ell^\infty(\mathbb{Z}^L)} \leq \varepsilon$, such that

$$a(n) = \psi(n) + \Lambda(n) + \delta(n). \tag{31}$$

For each $l \in \mathbb{N}$, consider the decomposition $a = \psi_l + \Lambda_l + \delta_l$, where $\|\delta_l\|_{\ell^\infty(\mathbb{Z}^L)} < 1/l$. For $r \neq l$, we have

$$|\psi_l(n) - \psi_r(n)| = |(\Lambda_l(n) - \Lambda_r(n)) + (\delta_l(n) - \delta_r(n))|. \tag{32}$$

Now, $\lim_{|I_N|\to\infty} 1/|I_N| \sum_{n\in I_N} |\Lambda_l(n) - \Lambda_r(n)| = 0$ and $\sup_{n\in\mathbb{Z}^L} |\delta_r(n) - \delta_l(n)| \leq 1/l + 1/r$. Therefore,

$$|\psi_l(n) - \psi_r(n)| \leq \frac{2}{l} + \frac{2}{r} \tag{33}$$

for all $n \in \mathbb{Z}^L$ except potentially a subset $A \subseteq \mathbb{Z}^L$, with its characteristic function, $\mathbb{1}_A(n)$, being a nullsequence. For each $l, r \in \mathbb{N}$, the sequence $\psi_l(n) - \psi_r(n)$ is a nilsequence, so, by the same argument in the proof of [28, Theorem 3.1], it follows that the inequality in equation (33) must, in fact, hold for all $n \in \mathbb{Z}^L$. Hence, the sequence $(\psi_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $\ell^\infty(\mathbb{Z}^L)$ that consists of D-step nilsequences, and since we already showed that $(\delta_r)_{r\in\mathbb{N}}$ is a Cauchy sequence in $\ell^\infty(\mathbb{Z}^L)$ converging to a nullsequence, the conclusion follows. □

Remark 6.1. It is worth noting that if the polynomials $p_1, \ldots, p_k$ are linear, then there is an easier proof of Theorem 2.2, where one has $D = k$. The reason is that, instead of Theorem 2.11, one can use [17, Proposition 1] or [7, Proposition 6.1] to improve the right-hand side of equation (28) to $|||f_i|||^2_{(\mathbb{Z}^d)^{\times(k+1)}}$.

Next, we provide the proof of Theorem 2.9. To this end, we recall a definition from [4] which is adapted from [11].

Definition. [4] We say that a collection of mappings $a_1, \ldots, a_k : \mathbb{Z}^d \to \mathbb{Z}^d$ is:

(i) *good for seminorm estimates for the system* $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ *along a Følner sequence* $(I_N)_{N\in\mathbb{N}}$ *of* $\mathbb{Z}^d$, if there exists $M \in \mathbb{N}$ such that if $f_1, \ldots, f_k \in L^\infty(\mu)$ and $\|f_\ell\|_{(\mathbb{Z}^d)^{\times M}} = 0$ for some $\ell \in \{1, \ldots, k\}$, then

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \prod_{i=1}^{\ell} T_{a_i(n)} f_i = 0,$$

where the convergence takes place in $L^2(\mu)$;

(ii) *good for equidistribution for the system* $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ *along a Følner sequence* $(I_N)_{N\in\mathbb{N}}$ *of* $\mathbb{Z}^d$, if for all $\alpha_1, \ldots, \alpha_k \in \mathrm{Spec}((T_n)_{n\in\mathbb{Z}^d})$, not all of them trivial, we have

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \exp(\alpha_1(a_1(n)) + \cdots + \alpha_k(a_k(n))) = 0,$$

where

$$\begin{aligned}\operatorname{Spec}((T_n)_{n\in\mathbb{Z}^d}) &:= \{\alpha \in \operatorname{Hom}(\mathbb{Z}^d, \mathbb{T})\colon T_n f = \exp(\alpha(n)) f, n \in \mathbb{Z}^d, \\ &\qquad \text{for some non-zero } f \in L^2(\mu)\} \quad \text{and} \\ \exp(x) &:= e^{2\pi i x} \quad \text{for all } x \in \mathbb{R}.\end{aligned}$$

Proof of Theorem 2.9. Fix a Følner sequence $(I_N)_{N\in\mathbb{N}}$ of $\mathbb{Z}^L$. We wish to show that

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \prod_{i=1}^{k} T_{p_i(n)} f_i = \prod_{i=1}^{k} \int_X f_i \, d\mu$$

for all $f_1, \ldots, f_k \in L^\infty(\mu)$.

We first consider the case $L = d$ to use [4, Theorem 3.9]. To this end, it suffices to show that $\mathbf{p}$ is good for seminorm estimates and good for equidistribution.

Since $G_{i,j}(\mathbf{p})$ is ergodic for μ for all $0 \leq i, j \leq k, i \neq j$, applying Theorem 2.11, we get that $\mathbf{p}$ is good for seminorm estimates. Moreover, $\mathbf{X}$ is an ergodic system.

Suppose, for the sake of contradiction, that $\mathbf{p}$ is not good for equidistribution. Then there exist $\alpha_1, \ldots, \alpha_k \in \operatorname{Spec}((T_n)_{n\in\mathbb{Z}^d})$, not all of them trivial, such that

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \exp(\alpha_1(p_1(n)) + \cdots + \alpha_k(p_k(n))) = c \tag{34}$$

for some $c \neq 0$. For $1 \leq i \leq k$, since $\alpha_i \in \operatorname{Spec}((T_n)_{n\in\mathbb{Z}^d})$, there exists some non-zero $f_i \in L^2(\mu)$ such that $T_n f_i = \exp(\alpha_i(n)) f_i$ for all $n \in \mathbb{Z}^d$. Since $\mathbf{X}$ is ergodic, we have that $|f_i|$ is a non-zero constant μ-almost everywhere. Using equation (34), we have

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \bigotimes_{i=1}^{k} T_{p_i(n)} f_i = \lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \bigotimes_{i=1}^{k} \exp(\alpha_i(p_i(n))) f_i = c \bigotimes_{i=1}^{k} f_i \not\equiv 0.$$

However, since at least one of $\alpha_1, \ldots, \alpha_k$ is non-trivial, we have that $\int_{X^k} \bigotimes_{i=1}^{k} f_i \, d\mu^{\otimes k} = \prod_{i=1}^{k} \int_X f_i \, d\mu = 0$, which contradicts condition (ii). Therefore, $\mathbf{p}$ is good for equidistribution.

Assume now that $L < d$. Let $(I'_N)_{N\in\mathbb{N}}$ be the Følner sequence of $\mathbb{Z}^d$ given by $I'_N := I_N \times [-N, N]^{d-L}$. Let $p'_1, \ldots, p'_k\colon \mathbb{Z}^d \to \mathbb{Z}^d$ be polynomials given by $p'_i(n, m) := p_i(n)$ for all $n \in \mathbb{Z}^L$ and $m \in \mathbb{Z}^{d-L}$. Put $\mathbf{p}' := (p'_1, \ldots, p'_k)$. It is not hard to see that $G_{i,j}(\mathbf{p}) = G_{i,j}(\mathbf{p}')$ for all $0 \leq i, j \leq k, i \neq j$. Moreover, since $(T_{p_1(n)} \times \cdots \times T_{p_k(n)})_{n\in\mathbb{Z}^L}$ is ergodic for $\mu^{\otimes k}$, so is $(T_{p'_1(n)} \times \cdots \times T_{p'_k(n)})_{n\in\mathbb{Z}^d}$. By the $d = L$ case, we have that

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \prod_{i=1}^{k} T_{p_i(n)} f_i = \lim_{N\to\infty} \frac{1}{|I'_N|} \sum_{n\in I'_N} \prod_{i=1}^{k} T_{p'_i(n)} f_i = \prod_{i=1}^{k} \int_X f_i \, d\mu.$$

Finally, we assume that $L > d$. Let $(S_{(n,m)})_{n\in\mathbb{Z}^d, m\in\mathbb{Z}^{L-d}}$ be a $\mathbb{Z}^L$ action on $(X, \mathcal{B}, \mu)$ such that $S_{(n,m)} = T_n$. Let $p'_1, \ldots, p'_k\colon \mathbb{Z}^L \to \mathbb{Z}^L$ be polynomials given by $p'_i(n) := (p_i(n), 0, \ldots, 0)$ for all $n \in \mathbb{Z}^L$, where the last $L - d$ entries are zero. Denote $\mathbf{p}' := (p'_1, \ldots, p'_k)$. By definition, for all $0 \leq i, j \leq k, i \neq j$, the group $G_{i,j}(\mathbf{p}')$ consists of

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elements of the form $(n, 0, \ldots, 0) \in \mathbb{Z}^L$, $n \in G_{i,j}(\mathbf{p})$ (with respect to the $\mathbb{Z}^L$-system $(X, \mathcal{B}, \mu, (S_n)_{n\in\mathbb{Z}^L})$). By the construction of $(S_n)_{n\in\mathbb{Z}^L}$, ergodicity of $G_{i,j}(\mathbf{p})$ with respect to the $\mathbb{Z}^d$-system $(X, \mathcal{B}, \mu, (T_n)_{n\in\mathbb{Z}^d})$ implies ergodicity of $G_{i,j}(\mathbf{p}')$ with respect to the $\mathbb{Z}^L$-system $(X, \mathcal{B}, \mu, (S_n)_{n\in\mathbb{Z}^L})$. Moreover, since $S_{p'_i(n)} = S_{(p_i(n),0,\ldots,0)} = T_{p_i(n)}$ for all $n \in \mathbb{Z}^L$, we have that $(S_{p'_1(n)} \times \cdots \times S_{p'_k(n)})_{n\in\mathbb{Z}^L} = (T_{p_1(n)} \times \cdots \times T_{p_k(n)})_{n\in\mathbb{Z}^L}$ is ergodic for $\mu^{\otimes k}$. By the $d = L$ case, we have that

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \prod_{i=1}^{k} T_{p_i(n)} f_i = \lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} \prod_{i=1}^{k} S_{p'_i(n)} f_i = \prod_{i=1}^{k} \int_X f_i \, d\mu,$$

which completes the proof. □

Finally, we prove Theorem 2.5. We start by proving that condition (C1) implies condition (C2). In fact, we show the following more general result.

PROPOSITION 6.2. *Let $d, k, L \in \mathbb{N}$, $q_1, \ldots, q_k \colon \mathbb{Z}^L \to \mathbb{Z}^d$ be polynomials, and $\mathbf{X} = (X, \mathcal{B}, \mu, (T_g)_{g\in\mathbb{Z}^d})$ be a $\mathbb{Z}^d$-system. Suppose that $(T_{q_1(n)}, \ldots, T_{q_k(n)})_{n\in\mathbb{Z}^L}$ is jointly ergodic for μ. Then for all $1 \le i, j \le k$, $i \ne j$, we have that $(T_{q_i(n)-q_j(n)})_{n\in\mathbb{Z}^L}$ is ergodic for μ.*

Furthermore, if there exist polynomials $p_1, \ldots, p_k \colon \mathbb{Z}^L \to \mathbb{Z}$ and $v_1, \ldots, v_k \in \mathbb{Z}^d$ such that $q_i(n) = p_i(n)v_i$ for all $1 \le i \le k$, then $(T_{p_1(n)v_1} \times \cdots \times T_{p_k(n)v_k})_{n\in\mathbb{Z}^L}$ is ergodic for $\mu^{\otimes k}$.

Proof. The sequence $(T_{q_i(n)-q_j(n)})_{n\in\mathbb{Z}^L}$ is ergodic for μ for all $1 \le i \ne j \le k$ by an argument similar to that given in the proof of [7, Proposition 5.3], so we choose to omit the details.

We now assume that $q_i(n) = p_i(n)v_i$ for all $1 \le i \le k$ and show that $(T_{p_1(n)v_1} \times \cdots \times T_{p_k(n)v_k})_{n\in\mathbb{Z}^L}$ is ergodic for $\mu^{\otimes k}$. It suffices to show that for all $f_i \in L^\infty(\mu)$, $1 \le i \le k$ with $\prod_{i=1}^k \int_X f_i \, d\mu = 0$, we have that

$$\sup_{\substack{(I_N)_{N\in\mathbb{N}} \\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \left\| \mathbb{E}_{n\in I_N} \bigotimes_{i=1}^{k} T_{p_i(n)v_i} f_i \right\|_{L^2(\mu^{\otimes k})} = 0. \tag{35}$$

CLAIM. *If $\mathbb{E}(f_i | Z_{\mathbb{Z}^d,\mathbb{Z}^d}(\mathbf{X})) = 0$ for some $1 \le i \le k$, then equation (35) holds.*

Proof of the claim. We may assume without loss of generality that $\deg(p_1) \ge \deg(p_2) \ge \cdots \ge \deg(p_k)$. Suppose that we have shown that for some $1 \le k_0 \le k$, equation (35) holds if $\mathbb{E}(f_i | Z_{\mathbb{Z}^d,\mathbb{Z}^d}(\mathbf{X})) = 0$ for some $1 \le i \le k_0 - 1$, where the case $k_0 = 1$ is understood to be always true. It suffices to show that equation (35) holds if $\mathbb{E}(f_{k_0} | Z_{\mathbb{Z}^d,\mathbb{Z}^d}(\mathbf{X})) = 0$.

By the induction hypothesis, we may assume without loss of generality that f_i is $Z_{\mathbb{Z}^d,\mathbb{Z}^d}(\mathbf{X})$-measurable for all $1 \le i \le k_0 - 1$. By [7, Lemma 2.7], we can approximate each f_i in $L^2(\mu)$ by an eigenfunction of $\mathbf{X}$. By multi-linearity, we may assume without loss of generality that each f_i, $1 \le i \le k_0 - 1$, is a non-constant eigenfunction of $\mathbf{X}$ given by $T_n f_i = \exp(\lambda_i(n)) f_i$ for all $n \in \mathbb{Z}^d$ and some group homomorphism $\lambda_i \colon \mathbb{Z}^d \to \mathbb{R}$, with $f_i(x) \ne 0$ μ-almost every $x \in X$. Then, the left-hand side of equation (35) is equal to

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$$\sup_{\substack{(I_N)_{N\in\mathbb{N}} \\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \left\| \mathbb{E}_{n\in I_N} \exp(P(n)) \bigotimes_{i=1}^{k_0-1} f_i \bigotimes_{i=k_0}^{k} T_{p_i(n)v_i} f_i \right\|_{L^2(\mu^{\otimes k})}, \tag{36}$$

where $P(n) := \sum_{i=1}^{k_0-1} \lambda_i(v_i)p_i(n)$. Denote $P(n) = \sum_{j=0}^{\deg(p_1)} Q_j(n)$, where Q_j is a homogeneous polynomial of degree j for all $0 \leq j \leq \deg(p_1)$. Then

$$\Delta^K P(n, h_1, \ldots, h_K) = \sum_{j=K}^{\deg(p_1)} \Delta^K Q_j(n, h_1, \ldots, h_K). \tag{37}$$

We first consider the case where $Q_j(n) \notin \mathbb{Q}[n]$ for some $\deg(p_{k_0}) + 1 \leq j \leq \deg(p_1)$. In this case, let $K = \deg(p_{k_0})$ in equation (37). Since $\Delta^K p_i(n, h_1, \ldots, h_K)$ is constant in n for all $k_0 \leq i \leq k$, by Lemma 4.3, to show that equation (36) is 0, it suffices to show that

$$\overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_K\in\mathbb{Z}^L} \sup_{\substack{(I_N)_{N\in\mathbb{N}} \\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} |\mathbb{E}_{n\in I_N} \exp(\Delta^K P(n, h_1, \ldots, h_K))| = 0 \tag{38}$$

(see Definition 3.13 for the definition of the polynomial $\Delta^K P$).

As $Q_j(n) \notin \mathbb{Q}[n]$ for some $\deg(p_{k_0}) + 1 \leq j \leq \deg(p_1)$, Lemma 3.14 implies that $\Delta^K Q_j(\cdot, h_1, \ldots, h_K) \notin \mathbb{Q}[n]$ for a set of $(h_1, \ldots, h_K)$ of density 1. By Weyl's criterion and equation (37), we have that equation (38) holds and thus equation (35) holds.

We now consider the case where $Q_j(n) \in \mathbb{Q}[n]$ for all $K + 1 \leq j \leq \deg(p_1)$. Let $P'(n) = \sum_{j=0}^{K} Q_j(n)$. It is not hard to see that there exists $Q \in \mathbb{N}$ such that for all $r \in \{0, \ldots, Q-1\}^L$ and $n \in \mathbb{Z}^L$, we have that

$$P(Qn + r) - P'(Qn + r) = P(r) - P'(r).$$

By equation (36), to show equation (35), it suffices to show that for all $r \in \{0, \ldots, Q-1\}^L$, we have that

$$\sup_{\substack{(I_N)_{N\in\mathbb{N}} \\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \left\| \mathbb{E}_{n\in I_N} \exp(P'(Qn + r)) \bigotimes_{i=k_0}^{k} T_{p_i(Qn+r)v_i} f_i \right\|_{L^2(\mu^{\otimes t})} = 0, \tag{39}$$

where $t = k - k_0 + 1$. Fix $r \in \{0, \ldots, Q-1\}^L$ and set $R(n) = P'(Qn + r)$. Let $p\colon \mathbb{Z}^L \to \mathbb{Z}^{dt}$ be the polynomial given by

$$p(n) = (p_i(Qn + r)v_i)_{k_0\leq i\leq k}.$$

Let $(X^t, \mathcal{B}^t, \mu^t, (S_g)_{g\in\mathbb{Z}^{dt}})$ be the $\mathbb{Z}^{dt}$-system such that

$$S_{(u_i)_{k_0\leq i\leq k}} := \prod_{i=k_0}^{k} T_{u_i}$$

for all $u_i \in \mathbb{Z}^d$, $k_0 \leq i \leq k$, and denote $f := \bigotimes_{i=k_0}^{k} f_i$. We may then rewrite the left-hand side of equation (39) as

$$\sup_{\substack{(I_N)_{N\in\mathbb{N}} \\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} \exp(R(n)) S_{p(n)} f\|_{L^2(\mu^{\otimes t})}. \tag{40}$$

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For $K = \deg(p_{k_0}) - 1$, to show that equation (40) is zero, it suffices, by Lemma 4.3, to show

$$\overline{\mathbb{E}}^{\square}_{\mathbf{h}=(h_1,\ldots,h_K)\in(\mathbb{Z}^L)^K} \sup_{\substack{(I_N)_{N\in\mathbb{N}} \\ \text{Følner seq.}}} \overline{\lim_{N\to\infty}} \|\mathbb{E}_{n\in I_N} \exp(\Delta^K R(n,\mathbf{h}))S_{\Delta^K p(n,\mathbf{h})} f\|_{L^2(\mu^{\otimes t})} = 0. \tag{41}$$

By assumption, $\Delta^K R(n, h_1, \ldots, h_K)$ is of degree 1 in the variable n. Since $\deg(p) = \deg(p_{k_0}) \geq \deg(p_i)$ for all $i \geq k_0$, $\Delta^K p(n, h_1, \ldots, h_K)$ is also of degree 1 in the variable n. We may thus assume that

$$\Delta^K p(n, h_1, \ldots, h_K) = ((c_i(h_1, \ldots, h_K) \cdot n + c'_i(h_1, \ldots, h_K))v_i)_{k_0\leq i\leq k},$$

for some polynomials $c_{k_0}, \ldots, c_k \colon \mathbb{Z}^{LK} \to \mathbb{Z}^L$ and $c'_{k_0}, \ldots, c'_k \colon \mathbb{Z}^{LK} \to \mathbb{Z}$. Write $\mathbf{c}(h_1, \ldots, h_K) := (c_i(h_1, \ldots, h_K)v_i)_{k_0\leq i\leq k}$ (which is viewed as a t-tuple of L-tuple of vectors in $\mathbb{Z}^d$). If we write

$$c_i(h_1, \ldots, h_K) = (c_{i,1}(h_1, \ldots, h_K), \ldots, c_{i,L}(h_1, \ldots, h_K))$$

for some $c_{i,j}(h_1, \ldots, h_K) \in \mathbb{Z}$, then, by definition, $G(\mathbf{c}(h_1, \ldots, h_K))$ is the subgroup of $\mathbb{Z}^{dt}$ generated by the elements

$$(c_{k_0,j}(h_1, \ldots, h_K)v_{k_0}, \ldots, c_{k,j}(h_1, \ldots, h_K)v_k),\ 1 \leq j \leq L.$$

By Lemma 4.4, the left-hand side of equation (41) is bounded by a constant multiple of

$$(\overline{\mathbb{E}}^{\square}_{h_1,\ldots,h_K\in\mathbb{Z}^L} |||f_{k_0}|||^4_{G(c_{k_0}(h_1,\ldots,h_K)v_{k_0})^{\times 2}})^{1/4}, \tag{42}$$

where $G(c_{k_0}(h_1, \ldots, h_K)v_{k_0})$ is the subgroup of $\mathbb{Z}^d$ generated by the elements

$$c_{k_0,1}(h_1, \ldots, h_K)v_{k_0}, \ldots, c_{k_0,L}(h_1, \ldots, h_K)v_{k_0},$$

that is, the entries of $c_{k_0}(h_1, \ldots, h_K)v_{k_0}$. For any $u_{k_0} \in G(c_{k_0}(h_1, \ldots, h_K)v_{k_0})$, note that u_{k_0} is a rational multiple of v_{k_0}. So, if $c_{k_0}(h_1, \ldots, h_K) \neq \mathbf{0}$, then $G(c_{k_0}(h_1, \ldots, h_K)v_{k_0}) = G(v_{k_0})$.

Since $(T_{p_i(n)v_i})_{n\in\mathbb{Z}^L}$ is ergodic for μ, we have that T_{v_i} is ergodic for μ. As $\mathbb{E}(f_{k_0}|Z_{\mathbb{Z}^d,\mathbb{Z}^d}(\mathbf{X})) = 0$, by [7, Lemma 2.4], we have that

$$|||f_{k_0}|||_{G(c_{k_0}(h_1,\ldots,h_K)v_{k_0})^{\times 2}} = |||f_{k_0}|||_{v_{k_0}^{\times 2}} = |||f_{k_0}|||_{(\mathbb{Z}^d)^{\times 2}} = 0$$

whenever $c_{k_0}(h_1, \ldots, h_K) \neq \mathbf{0}$. Since $K = \deg(p_{k_0}) - 1 = \deg(p_{k_0}(Q \cdot +r)) - 1$, it is easy to see that $c_{k_0} \not\equiv \mathbf{0}$. By [7, Lemma 2.11], the set of such $(h_1, \ldots, h_K)$ is of density 1. So, averaging over all $h_1, \ldots, h_K \in \mathbb{Z}^L$, we have that equation (42) is 0. This finishes the proof of the claim. □

Using the claim, it suffices to prove equation (35) under the assumption that all f_i terms are measurable with respect to $Z_{(\mathbb{Z}^d)^{\times 2}}(\mathbf{X})$. By [7, Lemma 2.7], we can approximate each f_i in $L^2(\mu)$ by an eigenfunction of $\mathbf{X}$. By multilinearity, we may assume without loss of generality that each f_i is a non-constant eigenfunction of $\mathbf{X}$ satisfying $T_n f_i = \exp(\lambda_i(n)) f_i$ for all $n \in \mathbb{Z}^d$, for some group homomorphism $\lambda_i \colon \mathbb{Z}^d \to \mathbb{R}$, with

$f_i(x) \neq 0$ μ-almost every $x \in X$. Then, since $(T_{p_i(n)v_i} : 1 \leq i \leq k)_{n \in \mathbb{Z}^L}$ is jointly ergodic for μ, for any Følner sequence $(I_N)_{N \in \mathbb{N}}$ of $\mathbb{Z}^d$,

$$0 = \prod_{i=1}^{k} \int_X f_i \, d\mu = \lim_{N \to \infty} \mathbb{E}_{n \in I_N} \prod_{i=1}^{k} T_{p_i(n)v_i} f_i$$
$$= \bigg(\lim_{N \to \infty} \mathbb{E}_{n \in I_N} \prod_{i=1}^{k} \exp(\lambda_i(p_i(n)v_i)) \bigg) \prod_{i=1}^{k} f_i.$$

This implies that $\lim_{N \to \infty} \mathbb{E}_{n \in I_N} \prod_{i=1}^{k} \exp(\lambda_i(p_i(n)v_i)) = 0$, so,

$$\lim_{N \to \infty} \mathbb{E}_{n \in I_N} \bigotimes_{i=1}^{k} T_{p_i(n)v_i} f_i = \bigg(\lim_{N \to \infty} \mathbb{E}_{n \in I_N} \prod_{i=1}^{k} \exp(\lambda_i(p_i(n)v_i)) \bigg) \bigotimes_{i=1}^{k} f_i.$$

This finishes the proof. □

We are now ready to complete the proof of Theorem 2.5.

Proof of Theorem 2.5. Using Proposition 6.2, we have that condition (C1) implies condition (C2). It is obvious that condition (C2) implies condition (C2'). So, it suffices to show that condition (C2') implies condition (C1).

It is not hard to see that we may assume without loss of generality that $p_i(0) = 0$ for all $1 \leq i \leq k$. By Theorem 2.9, to show that $(T_{p_i(n)v_i} : 1 \leq i \leq k)_{n \in \mathbb{Z}^L}$ is jointly ergodic for μ, it suffices to show that $G_{i,j}(\mathbf{p})$ is ergodic for μ for all $0 \leq i, j \leq k, i \neq j$. Fix any such pair (i, j). We may assume without loss of generality that $i \neq 0$. If $j = 0$, then by subcondition (ii), $(T_{p_i(n)v_i})_{n \in \mathbb{Z}^L}$ is ergodic for μ. So, T_{v_i} is ergodic for μ, and thus $G_{i,0}(\mathbf{p}) = G(v_i)$ is ergodic for μ. Hence, we may now assume that $j \neq 0$.

Assume first that $\deg(p_i) = \deg(p_j)$. By assumption, either v_i and v_j are linearly dependent, or p_i and p_j are linearly dependent.

If v_i and v_j are linearly dependent over $\mathbb{Z}$, then we may assume without loss of generality that $v_i = av$ and $v_j = bv$ for some $a, b \in \mathbb{Q}$ and $v \in \mathbb{Z}^d$. By subcondition (i), $(T_{(ap_i(n)-bp_j(n))v})_{n \in \mathbb{Z}^L}$ is ergodic for μ, which implies that $G(v)$ is ergodic for μ. However, $G_{i,j}(\mathbf{p})$ is a group generated by some elements which are linear combinations of v_i and v_j, thus multiples of v. Since $\mathbf{p}$ is non-degenerate, $G_{i,j}(\mathbf{p})$ is not the trivial group. It follows that $G_{i,j}(\mathbf{p}) = G(v)$, so the group $G_{i,j}(\mathbf{p})$ is ergodic for μ.

If p_i and p_j are linearly dependent over $\mathbb{Z}$, then we may assume without loss of generality that $p_i = ap$ and $p_j = bp$ for some $a, b \in \mathbb{Q}$ and polynomial p. By subcondition (i), $(T_{(p(n)(av_i - bv_j))})_{n \in \mathbb{Z}^L}$ is ergodic for μ, which implies that $G(av_i - bv_j)$ is ergodic for μ. However, $G_{i,j}(\mathbf{p})$ is a group generated by some elements which are multiples of $av_i - bv_j$. Since $\mathbf{p}$ is non-degenerate, $G_{i,j}(\mathbf{p})$ is not the trivial group. It follows that $G_{i,j}(\mathbf{p}) = G(av_i - bv_j)$, so the group $G_{i,j}(\mathbf{p})$ is ergodic for μ.

Finally, we consider the case when $\deg(p_i) \neq \deg(p_j)$. We may further assume without loss of generality that $\deg(p_i) > \deg(p_j)$. In this case, $G_{i,j}(\mathbf{p}) = G_{i,0}(\mathbf{p})$, which we have shown is ergodic for μ. □

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7. *Potential future directions*

We close this article with two potential future directions regarding the splitting of multicorrelation sequences. The first one is for integer polynomial iterates under no assumptions on the transformations other than commutativity (see Theorem 7.1 for a special case of two terms).

The second one pertains to potential results analogous to Theorem 2.2 for iterates of the form $[p_i(n)]$, $1 \le i \le k$, where $p_i = (p_{i,1}, \ldots, p_{i,d}) : \mathbb{Z}^L \to \mathbb{R}^d$ are vectors of real valued polynomials. (Here, for $x = (x_1, \ldots, x_L) \in \mathbb{R}^L$, we write $[x] := ([x_1], \ldots, [x_L])$, where $[\cdot]$ is the floor function. In fact, one can consider any combination of rounding functions, that is, floor, ceiling, or closest integer.)

7.1. *The two-term case with no ergodicity assumptions.* Given the results in the [6, Appendix], we are able to obtain the following splitting result for two commuting transformations without any ergodicity assumptions.

THEOREM 7.1. *Let $(X, \mathcal{B}, \mu, T, S)$ be a measure-preserving system with $TS = ST$. Let $f_0, f_1, f_2 \in L^\infty(\mu)$ and $p \in \mathbb{Z}[n]$ with degree $K \ge 2$. Then, the multicorrelation sequence*

$$a(n) := \int_X f_0 \cdot T^n f_1 \cdot S^{p(n)} f_2 \, d\mu$$

can be decomposed as a sum of a uniform limit of K-step nilsequences plus a nullsequence.

Proof. Setting $F_i = f_i \otimes \bar{f}_i$, $i = 0, 1, 2$, and $\tilde{\mu} = \mu \times \mu$, we have that

$$\frac{1}{N} \sum_{n=1}^{N} |a(n)|^2 = \frac{1}{N} \sum_{n=1}^{N} \int_{X^2} F_0 \cdot (T \times T)^n F_1 \cdot (S \times S)^{p(n)} F_2 \, d\tilde{\mu}. \tag{43}$$

Using [6, Theorem A.3], we get that the rational Kronecker factor is characteristic for the averages appearing in equation (43). (The *rational Kronecker factor* is the smallest sub-σ-algebra of $\mathcal{B}$ that makes all functions with finite orbit in $L^2(\mu)$ under the transformation T measurable.) Consequently, we may replace f_1 by $P_c f_1$ and f_2 by $Q_c f_2$ in $a(n)$ up to a nullsequence, where P_c denotes the orthogonal projection onto the compact component of the splitting associated to T, and Q_c that associated to S. (Here we make use of the Hilbert space splitting of $L^2(\mu)$ into its compact and weakly mixing components for a given unitary operator. The seeds for these results are already present in the work of Koopman and von Neumann [23]. These were later generalized by Jacobs, Glicksberg, and de Leeuw. See [8, §16.3] for a more modern treatment.) Thus, the sequence

$$a(n) - \int_X f_0 \cdot T^n P_c f_1 \cdot S^{p(n)} Q_c f_2 \, d\mu$$

is a nullsequence. Let $\varepsilon > 0$ and choose $h_1, \ldots, h_k, g_1, \ldots, g_k \in L^2(\mu)$ such that $Th_i = \lambda_i h_i$ and $Sg_i = \rho_i g_i$ (for some $\lambda_1, \ldots, \lambda_k, \rho_1, \ldots, \rho_k \in \mathbb{C}$ of absolute value 1) as well as $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{C}$ such that

$$\bigg| \int_X f_0 \cdot T^n P_c f_1 \cdot S^{p(n)} Q_c f_2 \, d\mu - \int_X f_0 \cdot T^n \sum_{i=1}^k a_i h_i \cdot S^{p(n)} \sum_{j=1}^k b_j g_j \, d\mu \bigg| < \varepsilon.$$

Observe that

$$\int_X f_0 \cdot T^n \sum_{i=1}^k a_i h_i \cdot S^{p(n)} \sum_{j=1}^k b_i g_i \, d\mu = \int_X f_0 \cdot \sum_{i=1}^k a_i \lambda_i^n h_i \cdot \sum_{j=1}^k b_j \rho_j^{p(n)} g_j \, d\mu$$
$$= \sum_{i,j=1}^k \bigg(a_i b_j \int_X f_0 \cdot h_i \cdot g_j \, d\mu \bigg) \lambda_i^n \rho_j^{p(n)},$$

which is a K-step nilsequence. Applying the same argument as in the proof Theorem 2.2, we deduce the decomposition result. The rest of the details are omitted for the sake of brevity. □

It is natural to ask whether a result analogous to Theorem 7.1 holds for longer expressions (potentially via a generalization of the results in the [**6**, Appendix]), and with more general polynomial iterates, even without necessarily assuming they have distinct degrees. Thus, we state the following problem.

Problem 1. Obtain decomposition results of the form 'uniform limit of nilsequences plus a nullsequence' for multicorrelation sequences with (integer) polynomial iterates for general systems under no ergodicity assumptions on the transformations.

7.2. *Integer part polynomial iterates.* With a, by now, standard argument (introduced in [**5**, **30**] for a single term, extended for two terms in [**35**], and further developed in [**24**, **25**, **27**]), one has, for the vectors of real polynomials $p_i = (p_{i,1}, \ldots, p_{i,d})$, that the expression

$$\frac{1}{|I_N|} \sum_{n \in I_N} \prod_{i=1}^k T_{[p_i(n)]} f_i = \frac{1}{|I_N|} \sum_{n \in I_N} \prod_{i=1}^k \prod_{j=1}^d T_j^{[p_{i,j}(n)]} f_i \tag{44}$$

is close to

$$\frac{1}{|I_N|} \sum_{n \in I_N} \prod_{i=1}^k \prod_{j=1}^d S_j^{p_{i,j}(n)} g_i, \tag{45}$$

where S_j terms are $\mathbb{R}$-flows on an 'extension system' Y of X, and the functions g_i are extensions of the f_i. (We say that a jointly measurable family $(S_t)_{t \in \mathbb{R}^d}$ of measure-preserving transformations on a probability space is an $\mathbb{R}^d$*-action* (*flow*), if it satisfies $S_{t+r} = S_t \circ S_r$ for all $t, r \in \mathbb{R}^d$—see [**25**] for details.) As an application of Theorem 2.2, one can prove splitting theorems for $\mathbb{R}^d$-actions on the extension system.

Indeed, consider the multicorrelation sequence

$$\int_X f_0 \cdot S_{p_1(n)} f_1 \cdots S_{p_k(n)} f_k \, d\mu, \tag{46}$$

where S is a measure-preserving $\mathbb{R}^d$-action on the probability space $(X, \mathcal{B}, \mu)$, $f_0, f_1, \ldots, f_k \in L^\infty(X)$, and $p_1, \ldots, p_k \colon \mathbb{Z} \to \mathbb{R}^d$ a non-degenerate family of polynomials of degree at most K with $p_i(n) = \sum_{h=0}^{K} a_{i,h}n^h$, $a_{i,h} \in \mathbb{R}^d$. (We address the $L = 1$ case for simplicity; following the same argument, one can similarly get the corresponding result for the general case of L-variable polynomials by using an ordering on the parameters, e.g. $n_1 > \ldots > n_L$.) Then

$$S_{p_i(n)} = S_{\sum_{h=0}^{K} a_{i,h}n^h} = \prod_{h=0}^{K} (S_{a_{i,h}})^{n^h}.$$

Note that $S_{a_{i,h}}$, $1 \leq i \leq k$, $1 \leq h \leq K$ generate a $\mathbb{Z}^{kK}$-action on $(X, \mathcal{B}, \mu)$. For convenience, set p_0 to be the constant zero polynomial. For $0 \leq i, j \leq k$, $i \neq j$, let $D_{i,j}$ be the largest integer h so that $S_{a_{i,D_{i,j}} - a_{j,D_{i,j}}} \neq \mathrm{id}$. This transformation will be denoted by $R_{i,j}$. By Theorem 2.2, one can show the desired splitting result for the sequence in equation (46), if the transformations $R_{i,j}$, $0 \leq i, j \leq k$, $i \neq j$ are all ergodic (as $\mathbb{Z}$-actions on the extension system Y).

Unfortunately, even though we have the previous result for flows, the error term that arises from the approximation of equation (44) by equation (45) prevents us from getting the conclusion of Theorem 2.2 for multicorrelation sequences of the form

$$\int_X f_0 \cdot T_{[p_1(n)]} f_1 \cdots T_{[p_k(n)]} f_k \, d\mu.$$

(To this day, only splittings of the form nilsequence plus an error term that is small in uniform density are known for this class of multicorrelation sequences (for this, see [**25**]). One is referred to [**27**] for averages along primes for the error term—in this last reference, only single variable real polynomials were considered. Using the multivariable approach of [**12**] instead of [**10**], one immediately gets the aforementioned result for integer part, or indeed for combinations of any other rounding functions, multivariable real polynomial iterates.)

Remark 7.2. It is important to stress that, for integer part real polynomial iterates, one does not expect to have the desired multicorrelation splitting in general. The next example shows that, even for $k = 1$, an ergodic system, and linear iterates, it can be too much to hope for.

Example 7.3. Following [**27**, Example 7], let $X = \mathbb{T} := \mathbb{R}/\mathbb{Z}$, $T(x) = x + 1/\sqrt{2}$, $p(n) = \sqrt{2}n$, $f_0(x) = e(x)$ and $f_1(x) = e(-x)$, where $e(x) := e^{2\pi i x}$. Then, we have that

$$\int_X f_0 \cdot T^{[p(n)]} f_1 \, d\mu = \int_X e(x) e\bigg(-x - \frac{1}{\sqrt{2}}[\sqrt{2}n] \bigg) \, dx = e\bigg(-\frac{1}{\sqrt{2}}[\sqrt{2}n] \bigg) = e\bigg(\frac{1}{\sqrt{2}}\{\sqrt{2}n\} \bigg),$$

which cannot be written as a uniform limit of nilsequences and a nullsequence.

Remark 7.4. One may think that the fact that $\sqrt{2}$ and $1/\sqrt{2}$ are not linearly independent over $\mathbb{Q}$ is behind the impossibility of the splitting in the example above. However, a closer examination of the proof given in [**27**] shows that this is not the case, and that the failure extends quite generally.

Indeed, we can imitate the example quoted above as follows. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure-preserving system with non-trivial irrational spectrum. Let $f_1 : X \to \mathbb{S}^1$

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be an eigenfunction of T with eigenvalue $e^{2\pi i\beta}$, with $\beta \in \mathbb{R} \setminus \mathbb{Q}$. Put $f_0 = \bar{f}_1$. Then, if we consider the multicorrelation sequence

$$a(n) := \int_X f_0 \cdot T^{[\alpha n]} f_1 \, d\mu,$$

with the choices made above, we observe that, in fact, $a(n) = e^{2\pi i[\alpha n]\beta}$. The same argument as in [27, Example 7] shows that $a(n)$ cannot be written as a uniform limit of nilsequences plus a nullsequence.

However, if we postulate very strong assumptions on our transformations, we do have the desired decomposition results. For example (see [25]), if $T_1, \ldots, T_k$ are commuting weakly mixing transformations on $(X, \mathcal{B}, \mu)$, $q_i(n) = p_i(n)e_i$, $1 \le i \le k$, where $p_i : \mathbb{Z} \to \mathbb{R}$ are real polynomials of distinct, positive degrees, and $f_1, \ldots, f_k \in L^\infty(\mu)$, then we have that

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} T_{[q_1(n)]} f_1 \cdots T_{[q_k(n)]} f_k = \prod_{i=1}^{k} \int_X f_i \, d\mu.$$

Hence, for any $f_0 \in L^\infty(\mu)$, the multicorrelation sequence

$$\int_X f_0 \cdot T_{[q_1(n)]} f_1 \cdots T_{[q_k(n)]} f_k \, d\mu$$

can be written as a sum of a constant (that is, a 0-step nilsequence) and a nullsequence.

We conclude this article with the following problem that arises naturally.

Problem 2. Let $d, k, K, L \in \mathbb{N}$, $p_1, \ldots, p_k : \mathbb{Z}^L \to \mathbb{R}^d$ be a non-degenerate family of polynomials of degree at most K, $(X, \mathcal{B}, \mu, T_1, \ldots, T_d)$ a measure-preserving system, and $f_0, \ldots, f_k \in L^\infty(\mu)$. Find conditions, on the p_i and/or the $\mathbb{Z}^d$-action T that is defined by the T_i, so that the multicorrelation sequence

$$\int_X f_0 \cdot T_{[p_1(n)]} f_1 \cdots T_{[p_k(n)]} f_k \, d\mu$$

can be decomposed as a sum of a uniform limit of D-step nilsequences and a nullsequence.

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References

[1] D. Berend and V. Bergelson. Jointly ergodic measure-preserving transformations. *Israel J. Math.* **49**(4) (1984), 307–314.

[2] V. Bergelson. Weakly mixing PET. *Ergod. Th. & Dynam. Sys.* **7**(3) (1987), 337–349.

[3] V. Bergelson, B. Host and B. Kra. Multiple recurrence and nilsequences. *Invent. Math.* **160**(2) (2005), 261–303, with an appendix by I. Ruzsa.

[4] A. Best and A. Ferré Moragues. Polynomial ergodic averages for countable ring actions. *Discrete Contin. Dyn. Syst.* **42**(7) (2022), 3379–3413.

[5] M. Boshernitzan, R. L. Jones and M. Wierdl. Integer and fractional parts of good averaging sequences in ergodic theory. *Convergence in Ergodic Theory and Probability*. Eds. V. Bergelson, P. March and J. Rosenblatt. Walter de Gruyter & Co., Berlin, 1996, pp. 117–132.
[6] Q. Chu, N. Frantzikinakis and B. Host. Ergodic averages of commuting transformations with distinct degree polynomial iterates. *Proc. Lond. Math. Soc. (3)* **102**(5) (2011), 801–842.
[7] S. Donoso, A. Koutsogiannis and W. Sun. Seminorms for multiple averages along polynomials and applications to joint ergodicity. *J. Anal. Math.* **146** (2022), 1–64.
[8] T. Eisner, B. Farkas, M. Haase and R. Nagel. *Operator Theoretic Aspects of Ergodic Theory (Graduate Texts in Mathematics, 272)*. Springer, Cham, 2015.
[9] A. Ferré Moragues. Properties of multicorrelation sequences and large returns under some ergodicity assumptions. *Discrete Contin. Dyn. Syst.* **41**(6) (2021), 2809–2828.
[10] N. Frantzikinakis. Multiple correlation sequences and nilsequences. *Invent. Math.* **202**(2) (2015), 875–892.
[11] N. Frantzikinakis. Joint ergodicity of sequences. *Adv. Math.* **417** (2023), 108918.
[12] N. Frantzikinakis and B. Host. Weighted multiple ergodic averages and correlation sequences. *Ergod. Th. & Dynam. Sys.* **38**(1) (2018), 81–142.
[13] N. Frantzikinakis, B. Host and B. Kra. The polynomial multidimensional Szemerédi theorem along shifted primes. *Israel J. Math.* **194**(1) (2013), 331–348.
[14] N. Frantzikinakis and B. Kra. Polynomial averages converge to the product of integrals. *Israel J. Math.* **148** (2005), 267–276.
[15] H. Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. Anal. Math.* **31** (1977), 204–256.
[16] J. Griesmer. Ergodic averages, correlation sequences, and sumsets. *Doctoral Dissertation*, Ohio State University, 2009. OhioLINK Electronic Theses and Dissertations Center, http://rave.ohiolink.edu/etdc/view?acc_num=osu1243973834.
[17] B. Host. Ergodic seminorms for commuting transformations and applications. *Studia Math.* **195**(1) (2009), 31–49.
[18] B. Host and B. Kra. Nonconventional ergodic averages and nilmanifolds. *Ann. of Math. (2)* **161**(1) (2005), 397–488.
[19] B. Host and B. Kra. Uniformity seminorms on ℓ^∞ and applications. *J. Anal. Math.* **108** (2009), 219–276.
[20] M. C. R. Johnson. Convergence of polynomial ergodic averages of several variables for some commuting transformations. *Illinois J. Math.* **53**(3) (2009), 865–882 (2010).
[21] D. Karageorgos and A. Koutsogiannis. Integer part independent polynomial averages and applications along primes. *Studia Math.* **249**(3) (2019), 233–257.
[22] A. Khintchine. The method of spectral reduction in classical dynamics. *Proc. Natl. Acad. Sci. USA* **19**(5) (1933), 567–573.
[23] B. O. Koopman and J. von Neumann. Dynamical systems of continuous spectra. *Proc. Natl. Acad. Sci. USA* **18**(3) (1932), 255–263.
[24] A. Koutsogiannis. Closest integer polynomial multiple recurrence along shifted primes. *Ergod. Th. & Dynam. Sys.* **38**(2) (2018), 666–685.
[25] A. Koutsogiannis. Integer part polynomial correlation sequences. *Ergod. Th. & Dynam. Sys.* **38**(4) (2018), 1525–1542.
[26] A. Koutsogiannis. Multiple ergodic averages for variable polynomials. *Discrete Contin. Dyn. Syst.* **42**(9) (2022), 4637–4668.
[27] A. Koutsogiannis, A. Le, J. Moreira and F. K. Richter. Structure of multicorrelation sequences with integer part polynomial iterates along the primes. *Proc. Amer. Math. Soc.* **149**(1) (2021), 209–216.
[28] A. Leibman. Multiple polynomial correlation sequences and nilsequences. *Ergod. Th. & Dynam. Sys.* **30**(3) (2010), 841–854.
[29] A. Leibman. Nilsequences, null-sequences, and multiple correlation sequences. *Ergod. Th. & Dynam. Sys.* **35**(1) (2015), 176–191.
[30] E. Lesigne. On the sequence of integer parts of a good sequence for the ergodic theorem. *Comment. Math. Univ. Carolin.* **36**(4) (1995), 737–743.
[31] W. Sun. Weak ergodic averages over dilated curves. *Ergod. Th. & Dynam. Sys.* **41**(2) (2021), 606–621.
[32] T. Tao and J. Teräväinen. The structure of logarithmically averaged correlations of multiplicative functions, with applications to the Chowla and Elliott conjectures. *Duke Math. J.* **168**(11) (2019), 1977–2027.
[33] T. Tao and T. Ziegler. Concatenation theorems for anti-Gowers-uniform functions and Host–Kra characteristic factors. *Discrete Anal.* **13** (2016), article no. 13.
[34] M. Walsh. Norm convergence of nilpotent ergodic averages. *Ann. of Math. (2)* **175**(3) (2012), 1667–1688.
[35] M. Wierdl. Personal communication with the third author, 2015.
[36] T. Ziegler. Nilfactors of $\mathbb{R}^m$-actions and configurations in sets of positive upper density in $\mathbb{R}^m$. *J. Anal. Math.* **99** (2006), 249–266.