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Introduction and Problem Formulation

Mathematics is the key and door to the sciences

– Galileo Galilei (1564–1642)

1.1 History, Background and Rationale

In examining the dynamics of any physical system, the concept of stability becomes relevant only after first establishing the possibility of equilibrium. Once this step has been taken, the concept of stability becomes pervasive, regardless of the actual system being probed. As expressed by Betchov & Criminale (1967), stability can be defined as the ability of a dynamical system to be immune to small disturbances. It is clear that the disturbances need not necessarily be small in magnitude and therefore may become amplified. As such, there is a departure from the state of equilibrium. Should no equilibrium be possible, then it can already be concluded that the particular system in question is statically unstable and the dynamics is a moot point.

Such tests for stability can be and are made in any field, such as mechanics, astronomy, electronics and biology, for example. In each case from this list, there is a common thread in that only a finite number of discrete degrees of freedom are required to describe the motion and there is only one independent variable. Like tests can be made for problems in continuous media but the number of degrees of freedom becomes infinite and the governing equations are now partial differential equations instead of the ordinary variety. Thus, conclusions are harder to obtain in any general manner, but it is not impossible. In fact, successful analysis of many such systems has been made and this has been particularly true in fluid mechanics. This premise is even more so today because there are far more advanced means of computation available to

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supplement analytical techniques. Likewise the means for experimentation has improved in profound ways and will be highlighted throughout the text in validation of the theoretical and computational results.

Fundamentally, there is no difficulty in presenting the problem of stability in fluid mechanics. The governing Navier–Stokes continuum equations for the conservation of momentum and mass that is often expressed by constraints, such as incompressibility that requires the fluid velocity to be solenoidal in a somewhat general sense, are the tools of the science. A specific flow is then fully determined by satisfying the boundary conditions that must be met for that flow. Other considerations involve the importance of the choice of the coordinate system that is best to describe the flow envisioned and whether or not there is any body force, say. Then, the important first step is to identify a flow that is in equilibrium. For this purpose, a flow that is in equilibrium need not necessarily be time independent, but the system is no longer accelerated due to the balance of all forces. For such flows meeting these conditions very few, if any, remain that have not been theoretically evaluated using this approach, but, because the governing equations of motion are a set of nonlinear partial differential equations, the results are most often the result of approximations. Nevertheless these flows are well established, many have been experimentally confirmed, and they are all laminar. In addition, a few exact solutions of the governing equations are known. In such cases, where more complex physics is entailed, such as compressibility or electrical conductivity of the fluid, similar arguments can be made and results have been equally obtainable.

Essentially there are three major categories of base mean flows, namely: (a) flows that are parallel or almost parallel; (b) flows with curved streamlines and; (c) flows where the mean flow has a zero value. Examples of the parallel variety are channel flows, such as plane Couette and Poiseuille flows where the flows are confined by two solid boundaries. There is one mean component for the mean velocity and it is a function of the coordinate that defines the locations of the boundaries. In a polar coordinate system, pipe flow is another example of note. Almost parallel flows are of two main categories: (i) free shear flows, such as the jet, wake and mixing layer where there are no solid boundaries in the flow and (ii) the flat-plate boundary layer where there is but one solid boundary. In these terms, (i) and (ii) have two components for the mean velocity, and they are both functions of the coordinate in the direction of the flow as well as the one that defines the extent of the flow. In Cartesian terms, if U and V are the mean velocity components in the x and y spatial directions, respectively, then almost parallel assumes that $V \ll U$ and that the variation of U with respect to the downstream variable x is weak. Group (b) has flows such as that between concentric circular cylinders (Taylor problem) or flow on concave walls (Görtler problem). The cases where there is no mean flow (e.g., Rayleigh problem, Bénard cells) are simply special cases of the more general picture. Whether from the point of view of the physics or the mathematics needed to make analyses, each of these prototypes has its own unique features and it is the stability of the system that is the question to be answered. It should be clear that the actual causes of any resulting instability will vary as well.

It should be again stressed that, regardless of the methods required for obtaining any mean flow, they are laminar and are in equilibrium or near equilibrium. But, unfortunately, just as the adage states, "turbulence is the rule and not the exception to fluid motion." In other words, laminar flows are extremely hard to maintain; transition to turbulence will occur in the short or the long time. One need only to observe the flow over the wings of an airplane, the meandering of a river, the outflow from the garden hose or the resulting flow behind bluff bodies in both the laboratory and in nature to witness this predominance first hand. Laminar flow is orderly, can be well predicted and is most generally desired. The illustrations of Figs. 1.1, 1.2 and 1.3 vividly demonstrate the more-than-subtle differences for these two flows in the boundary layer setting. Benefits of laminar flow include less drag and reduced acoustics when compared to the turbulent state. Figure 1.1 shows the clean streamline pattern over a flat plate, reminiscent of laminar flow, whereas Fig. 1.2 shows the random turbulent boundary layer over a segment of the same flat plate. Although transition occurs via a different mechanism on a rotating cone, Fig. 1.3 shows the entire set of fluid states whereby the flow is laminar at the apex of the cone. The focus of this text becomes clear as the flow is disturbed and "transitions" to a state between laminar and turbulent. Finally, the flow is fully random, chaotic or what is called turbulent. Contrary to the benefits of laminar flow, a case where a benefit from turbulent flow would be desired over laminar is mixing, for example. The goal of predicting or even approximating the process of transition has been a stated goal throughout the history of fluid mechanics and, it was once thought, stability analysis would be able to do this. Any success has been limited but stability analysis can explain – for almost all of the major cases – why a basic flow cannot be maintained indefinitely.

Although the main focus of the text is on the mathematics of predicting flow instabilities, the classical experiments of Reynolds (1883) are introduced in Fig. 1.4, which shows the circular pipe flow experiment. Note the very raw experimental setup of the era compared with modern-day more advanced laboratory systems. Figure 1.5 shows the classical experiment due for flow in a circular pipe whereby dye was inserted and the mean flow run at different values through a number of pipe diameters. This was an extremely important series of experiments to modern day fluid mechanics, so it is worth revisiting

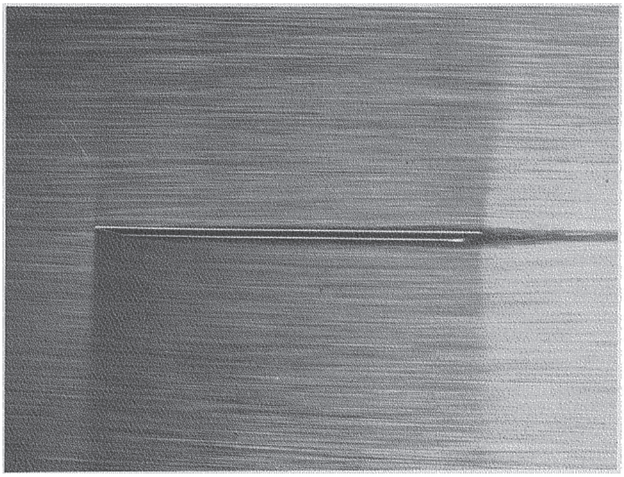

Figure 1.1 Laminar boundary layer on a flat plate (Werlé, 1974).

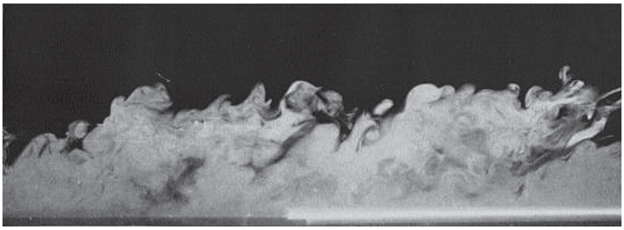

Figure 1.2 Turbulent boundary layer on a flat plate (Reprinted from Falco, 1977 with the permission of AIP Publishing).

these results briefly, as well as the thoughts of Osborne Reynolds. At the beginning of his paper, Reynolds stated the following:

> There appeared to be two ways of proceeding – the one theoretical, the other practical. The theoretical method involved the integration of the equations for unsteady motion in a way that had not been accomplished and which, considering the general intractability of the equations, was not promising. The practical method was to test the relation between U, μ/ρ, and c.

The first way of proceeding – theory – is the primary focus of this text and clearly shows the progress made over time and, with the advent of computers, the equations have become tractable. The second way of proceeding – namely, experimentation – was important to the contemporary scientist because the variation of velocity U, kinematic viscosity μ/ρ, and pipe radius c was the advent of the Reynolds number $Re = \rho U c/\mu$.

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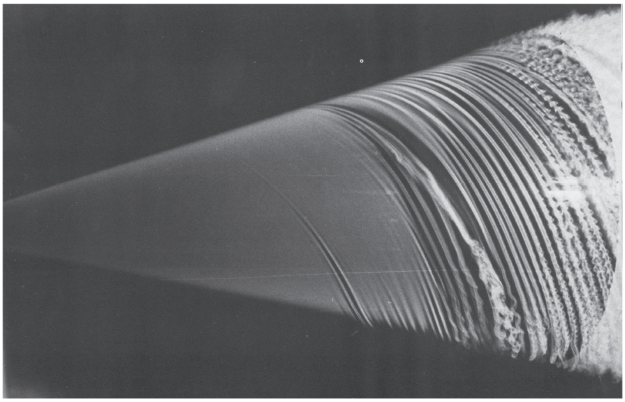

Figure 1.3 Spiral vortices on a cone in rotation with freestream (Kobayashi, Kohama & Kurosawa, 1983, reproduced with permission).

In returning to the discussion of Reynolds' main observations in Fig. 1.4, the original organized parallel laminar flow is seen at several stages with the ultimate breakdown and fully random three-dimensional motion transpiring. At low "Reynolds number," the dye is transported through the pipe evident as a straight line at the top of the image. As the Reynolds number increases, or a critical velocity is reached, Reynolds noted:

> And it was a matter of surprise to me to see the sudden force with which the eddies sprang into existence, showing a highly unstable condition to have existed at the time the steady motion broke down.

As the critical velocity increases, the dye image clearly shows a more random or turbulent pattern. Ironically, this problem is one where stability theory has not been able to make any conclusions whatsoever and remains an enigma in the field. In short, linear theory has been used to investigate this flow in many ways and no solutions that predict instability have been found. This has been found to be true regardless of any added complexities that might be envisioned – for example, axisymmetric versus non-axisymmetric disturbances. Still, it is clear that this flow is unstable.

Drawings of vortices can be traced as far back as those of Leonardo da Vinci that were made in the fifteenth century. The first significant contribution to the theory of hydrodynamic stability is that due to Helmholtz (1868). The principal initial experiments are due to Hagan (1855). Later a major list of contributions can be cited. Reynolds (1883), Kelvin (1880, 1887a,b) and Rayleigh (1879, 1880, 1887, 1892a,b,c, 1895, 1911, 1913, 1914, 1915, 1916a,b) were all ac-

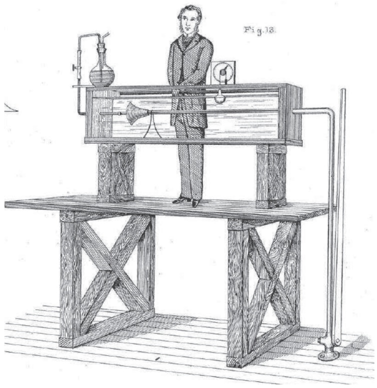

Figure 1.4 Sketch of the Reynolds pipe flow experiment (Reynolds, 1883).

Figure 1.5 Repetition of Reynolds' dye pipe experiment (van Dyke, 1982).

tive in this period. Here, the birth of the Reynolds number as well as the first theorems due to Rayleigh appeared. As has been noted before, Lord Rayleigh was thirty-six when he considered the stability of flames and then published his work on jets. At seventy-two he began to do work in nonlinear stability theory! Unlike Reynolds' pipe experiment, which was intrinsically viscous, the exceptional theoretical work of Kelvin and Rayleigh was done using the inviscid approximation in the analysis.

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Independently, Orr (1907a,b) and Sommerfeld (1908) framed the viscous stability problem. Both were attempting to investigate channel flow, with Orr considering plane Couette flow, and Sommerfeld plane Poiseuille flow. Of course one case is the limit of the other and the combination has led to the Orr–Sommerfeld equation that has become the essential basis in the theory of hydrodynamic stability. But, even here, it should be remembered that it was not until twenty-two years after the derivation of this equation that any solution at all could be produced. Tollmien (1929) calculated the first neutral eigenvalues for plane Poiseuille flow and showed that there was a critical value for the Reynolds number. This work was made possible by the development of Tietjens' functions (Tietjens, 1925) and analysis of Heisenberg (1924), connected with the topic of resistive instability. Romanov (1973) proved theoretically that plane Couette flow is stable. Unlike pipe flow, there is no experimental controversy here. Plane Poiseuille flow, on the other hand, is unstable.

Schlichting (1932a,b, 1933a,b,c, 1934, 1935) continued the work of Tollmien and extended it even further. The combination of these efforts have led to the designation for the oscillations that are now the salient results for the stability of parallel or nearly parallel flows, namely Tollmien–Schlichting waves. It should be noted that such waves correspond to those waves where friction is critical and do not exist for any problem that does not include viscosity and are known to be present only in flows where a solid boundary is present in the flow. Also, in the limit of infinite Reynolds number, the flow is stabilized.

Prandtl (1921–1926, 1930, 1935) was active in problems related to stability in the hopes that the theory might lead to the prediction of transition and the onset of turbulence. As mentioned, to date no such success has been achieved but the effort continues as the understanding makes progress. But, for the first time during this period, a major boost to stability analysis was given by the work of Taylor (1923) where theory was confirmed by his experiment for the case of rotating concentric cylinders. Taylor himself was responsible for this, and the work continues to be a model for understanding the stability of mean flows with curved stream lines.

The advent of matched asymptotic expansions and singular perturbation analysis brought new vigor to the theory. Lin (1944, 1945) made use of these tools and re did all previous calculations, thereby confirming the earlier results that had been obtained by less sophisticated means. Experiments also gained momentum with the work of Schubauer & Skramstad (1943) in the investigation of the flat-plate boundary layer setting the standard. Here, a vibrating ribbon was employed to simulate a controlled disturbance, that is a Tollmien–Schlichting wave, at the boundary. This method is still employed by many today. Theoretical calculations were confirmed and, equally important, for the first time it became apparent that the value of the critical Reynolds number meant the stability boundary for the onset of unstable Tollmien–Schlichting waves and not the threshold for the onset of turbulence. Figure 1.6, depicting the results of this experiment, is a hallmark in this field. This conclusion has been further substantiated today. For example, Schubauer & Klebanoff (1955, 1956), Klebanoff, Tidstrom & Sargent (1962) and Gaster & Grant (1975) performed even more extensive experiments for the boundary layer.

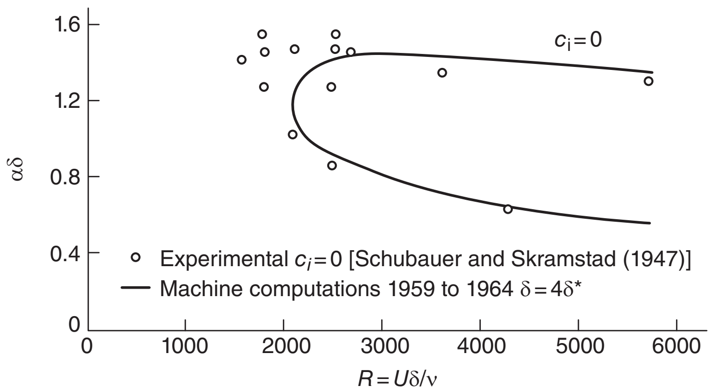

Figure 1.6 Experimental and theoretical stability results for neutral oscillations of the Blasius boundary layer (after Betchov & Criminale, 1967).

Investigating the stability of compressible flows was not done until much later with the theoretical work of Landau (1944), Lees (1947) and Dunn & Lin (1955) being the principal contributors at this time. Physically and mathematically, this is a far more complex problem and, in view of the time span it took to resolve the theory in an incompressible medium, this was understandable. A wide range of problems have been investigated here, including different prototypes and Mach numbers up to hypersonic in value. Likewise, there are experiments that have been done for these flows (see Kendall, 1966).

The use of numerical computation for stability calculations was made with the work of Brown (1959, 1961a,b, 1962, 1965), Mack (1960, 1965a,b) and Kaplan (1964) being the principal contributions. Neutral curves that were previously obtained by asymptotic theory and hand calculations are now routinely determined by numerical treatment of the governing stability equations. Such numerical evaluation has proven to be more efficient and far more accurate than any of the methods employed heretofore. Furthermore, the complete and unsteady nonlinear Navier–Stokes equations are evaluated by the use of high-order numerical methods in tandem with machines that range from the personal computer to supercomputers and the parallel class of machines, which are the standard tool for solving fluid mechanics problems today. By numerical calculations, one of the earliest results for the full Navier–Stokes calculations was obtained by Fromm & Harlow (1963), where the problem of vortex shedding from a vertical flat plate was investigated. Since this time, the complete Navier–Stokes equations are routinely used to study the vortex shedding process. Among others, Lecointe & Piquet (1984), Karniadakis & Triantafyllou (1989) and Mittal & Balachandar (1995), for example, have all numerically solved the full equations in order to investigate instability and vortex shedding from cylinders. A summary of this vortex shedding problem is provided in a review by Williamson (1996).

Effort has been made to assess nonlinearity in stability theory. Meksyn & Stuart (1951), Benney (1961, 1964) and Eckhaus (1962a,b, 1963, 1965) were all early contributors to what is now known as weakly nonlinear theory. Each effort was directed to different aspects of the problems. For example, the nonlinear critical layer, development of longitudinal or streamwise vortices in the boundary layer and the possibility of a limiting amplitude for an amplifying disturbance were examined. The role of streamwise vorticity in the breakdown from laminar to turbulent flow has recently been explored using the complete Navier–Stokes equations. For this purpose, Fasel (1990), Fasel & Thumm (1991), Schmid & Henningson (1992a,b) and Joslin, Streett & Chang (1993) have introduced oblique wave pairs at amplitudes ranging from very small to finite values. The interaction of such oblique waves leads to dominant streamwise vortex structure. When the waves have small amplitudes, the disturbances first amplify but then decay at some further downstream location. When finite, the nonlinear interactions of the vortex and the oblique waves result in breakdown.

Since the experimental setting for probing in this field is almost unequivocally one where any disturbance changes in space and only oscillates in time, thought has been given to the question of spatial instability so that theory may be more compatible with experimental data. The problem can be posed in very much the same way as the temporal one, but the equations must be adapted for this purpose (e.g., see Section 1.8 for the discussion based on Gaster (1965a,b)). This is true even if the problem is governed by the linear equations. Direct numerical simulation also has major complexities when computations are made in this way. Nevertheless, this is done. For this purpose, reference to the summaries of Kleiser & Zang (1991) and Liu (1998) can be made where the use of direct numerical simulation for many instability problems has been given. More specifically, among this vast group, Wray & Hussaini (1984) and Spalart & Yang (1987) both investigated the breakdown of the flat-plate boundary layer by use of a temporal numerical code. In other words, an initial value problem was prescribed at time $t = 0$, and the disturbance developed for later times. By contrast, when a spatial code is employed, and initial values are given at a fixed location and then the development thereafter downstream, the work of Fasel (1976), Murdock (1977), Spalart (1989), Kloker & Fasel (1990), Rai & Moin (1991a,b) and Joslin, Streett & Chang (1992, 1993) should be noted. For three-dimensional mean flows, where cross flow disturbances are present, Spalart (1990), Joslin & Streett (1994) and Joslin (1995a) studied the breakdown process by means of direct numerical simulation.

Stability theory uses perturbation analysis in order to test whether or not the equilibrium flow is unstable. Consider the flows that are incompressible, time independent and parallel or almost parallel by defining the mean state as

$$\underline{U} = (U(y), 0, 0); \quad P$$

in Cartesian coordinates where $U(y)$ is in the x-direction with y the coordinate that defines the variation of the mean flow, z is in the transverse direction and P is the mean pressure. For some flows, such as that of channel flow, this result is exact; for the case of the boundary layer or one of the free shear flows, then this is only approximate but, as already mentioned, the U-component of the velocity, $U \gg V$ and $U \gg W$, as well as U varying only weakly with x, and hence the designation of almost parallel flow. In this configuration, both x and z range from minus to plus infinity with y giving the location of the solid boundaries, if there are any. P is the mean pressure and the density is taken as constant.

Now assume that there are disturbances to this flow that are fully three-dimensional and hence

$$\underline{u} = (U(y) + \tilde{u}, \tilde{v}, \tilde{w}); \qquad p = P + \tilde{p}$$

can be written for the velocity and pressure of the instantaneous flow. By assuming that the products of the amplitudes (defined nondimensionally with the measure in terms of the mean flow) of the perturbations, as well as the products of the perturbations with the spatial derivatives of the perturbations, are small, then, by subtracting the mean value terms from the combined flow, a set of linear equations can be found and are dimensionally

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0, \tag{1.1}$$

for incompressibility, and

$$\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{dU}{dy}\tilde{v} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x} + \nu \nabla^2 \tilde{u}, \tag{1.2}$$

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$$\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial y} + \nu\nabla^2\tilde{v}, \tag{1.3}$$

$$\frac{\partial \tilde{w}}{\partial t} + U\frac{\partial \tilde{w}}{\partial x} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial z} + \nu\nabla^2\tilde{w}, \tag{1.4}$$

for the momenta, where ρ is the density of the fluid, ν is the kinematic viscosity and the three-dimensional Laplace operator is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Hereafter, $()' = d/dy$, $()'' = d^2/dy^2$, etc. It is more prudent to nondimensionalize the equations and this will be done eventually but, for the purposes of the discussion of the basic concepts, they will here be considered dimensionally throughout this chapter. When nondimensionalization has been done in this case, all quantities are redefined and the coefficient of viscosity is replaced with the reciprocal of the Reynolds number, defined in terms of the chosen length and velocity scales of the particular flow.

1.2 Initial-Value Concepts and Stability Bases

At this stage a temporal initial-value, spatial boundary-value problem has been prescribed and must be solved in order to determine whether or not the given flow is unstable. In this respect, it is well defined but, as will be seen, there are many difficulties in actually performing this task. There is, of course, more than one definition for stability that can be used, but the major concern is whether or not the behavior of the disturbances causes an irreversible alteration in the mean flow. In short, if, as time advances from the initial instant, there is a return to the basic state, then the flow is considered stable. There are various ways that instability can occur but it is first essential to understand what means are possible for solving these problems in order that any decision can be made. At the outset it can already be seen that the order of the system is higher than the traditional second-order boundary value problems of mathematical physics. As a result, some of the classic methods of exploration are of limited value; others that may be used require extensions or alterations in order to be employed here.

Any velocity vector field can be decomposed into its solenoidal, rotational and harmonic components. For the problems being discussed here there is no solenoidal part due to the fact that the fluid is incompressible and $\nabla \cdot \underline{u} = 0$. On physical grounds the rotational part of the velocity corresponds to the perturbation vorticity with the harmonic portion related to the pressure. This analogy

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makes for better interpretation of the physics for, even though the boundary conditions must be cast in terms of the velocity, the initial specification can be considered as that of vorticity. In this respect, each of the mean flows that has been cited, when the governing equations are written in terms of the vorticity, the vorticity is essentially a quantity that is diffused or advected from what it was initially and the velocity profile is the result of this action. The same reasoning can be made for the perturbation field.

The reasoning for the decomposition of the velocity can be best understood by actually using the definitions for the divergence and the curl. First, operate on (1.2) to (1.4) by taking the divergence, and use (1.1) to give

$$\nabla^2 \tilde{p} = -2\rho U' \frac{\partial \tilde{v}}{\partial x}. \tag{1.5}$$

The relation (1.5) is an equation for the perturbation pressure and has an inhomogeneous term that is effectively a source for the pressure, due to the interaction of the fluctuating and mean strain rates. When neither is strained, then the pressure is harmonic. If the velocity had not been solenoidal, then factors relating to the compressibility of the fluid would come into play.

Now, the definitions of the perturbation vorticity components are

$$\tilde{\omega}_x = \frac{\partial \tilde{w}}{\partial y} - \frac{\partial \tilde{v}}{\partial z}, \tag{1.6}$$

$$\tilde{\omega}_y = \frac{\partial \tilde{u}}{\partial z} - \frac{\partial \tilde{w}}{\partial x}, \tag{1.7}$$

and

$$\tilde{\omega}_z = \frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y}, \tag{1.8}$$

respectively, since $\underline{\omega} = \nabla \times \underline{u}$.

By using these definitions and the operation of the curl on the same set of equations for the momenta, the following are obtained:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\tilde{\omega}_x - \nu\nabla^2\tilde{\omega}_x = -U'\frac{\partial \tilde{w}}{\partial x} = \Omega_z\frac{\partial \tilde{w}}{\partial x}, \tag{1.9}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\tilde{\omega}_y - \nu\nabla^2\tilde{\omega}_y = -U'\frac{\partial \tilde{v}}{\partial z} = \Omega_z\frac{\partial \tilde{v}}{\partial z}, \tag{1.10}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\tilde{\omega}_z - \nu\nabla^2\tilde{\omega}_z = -U'\frac{\partial \tilde{w}}{\partial z} + U''\tilde{v} = \Omega_z\frac{\partial \tilde{w}}{\partial z} - \Omega_z'\tilde{v}, \tag{1.11}$$

where $\Omega_z = -dU/dy$ is the single component of the mean vorticity and is in the z-direction. Each of these equations has the expected transport by the mean velocity and diffusion but, in case there is also an inhomogeneous term that is due to the interaction of the fluctuating strain and the mean vorticity. Just as in the pressure relation, these interactions are needed for any generation of the respective fluctuating component. But, it is important to note, such generation here is due to three-dimensionality for, if there was neither the $\tilde{w}$-component of the velocity nor the spatial dependence in the transverse z-direction, as it would be for the two-dimensional problem, then the fluctuating vorticity components, except for $\tilde{\omega}_z$, could only be advected and diffused regardless of any initial input.

In order to seek a solution for this problem, the number of equations needs to be reduced. There are several ways of doing this but one in particular is more than efficient. From kinematics it can be shown that

$$\nabla^2 \tilde{v} = \frac{\partial \tilde{\omega}_z}{\partial x} - \frac{\partial \tilde{\omega}_x}{\partial z}. \tag{1.12}$$

Thus, by differentiating (1.9) by z and (1.11) by x and combining equations using (1.12), then

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 \tilde{v} - \nu \nabla^4 \tilde{v} = U'' \frac{\partial \tilde{v}}{\partial x} = -\Omega_z' \frac{\partial \tilde{v}}{\partial x} \tag{1.13}$$

can be obtained and, although still in a partial differential equation form, it is the Orr–Sommerfeld equation of stability theory. It is fortuitous that this equation uncouples in such a way as to only be fourth order and fully homogeneous in the $\tilde{v}$ dependent variable. The solution of (1.13) is then to be used in (1.10) for the solution of $\tilde{\omega}_y$. In like manner, the results found for $\tilde{\omega}_y$ are combined with $\tilde{v}$ and the problem is complete when these are used in (1.7) together with (1.1) to determine $\tilde{u}$ and $\tilde{w}$. Finally, $\tilde{p}$ can be evaluated from one of the momenta, (1.2) to (1.4). If the initial data and boundary conditions are satisfied, the problem is complete and the query as to stability can now be answered.

One last observation should be noted here. Equation (1.10) is actually the Squire equation that is known to accompany that of Orr–Sommerfeld. In this form, however, the dependent variable is the component of the vorticity that is perpendicular to the x–z plane and is only of interest in the full three-dimensional perturbation problem, strictly speaking. The importance of this cannot be stressed enough, for it leads to the understanding of the physics of the problem and details of the flow. It is not necessary if only the stability of the flow is the requirement. This equation also provides the other two orders of the anticipated sixth-order system. Unlike (1.13) though, it is not homogeneous.

1.3 Classical Treatment: Modal Expansions

The traditional classical method for solving (1.13) for $\tilde{v}$ is by modal expansion (normal modes). First, it is recognized that the coefficients in (1.13) are functions of y only. Therefore, since the extent of the planes perpendicular to y defined by the x, z spatial variables is doubly infinite, $\tilde{v}$ can be Fourier transformed in these two variables. Accordingly, define

$$\check{v}(\alpha, y, \gamma, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{v}(x, y, z, t) e^{i(\alpha x + \gamma z)} dx dz. \tag{1.14}$$

With this step, the governing equation remains a partial differential equation in terms of the variables y and t, but the far-field boundary conditions in x and z, namely boundedness as $x, z \to \pm\infty$, are satisfied by the rigid conditions for Fourier transforms with α and γ, both real and which are the streamwise and spanwise wavenumbers. At this point, since the problem is linear, it would be natural to reduce the equation even further by employing a Laplace transform in time so that an ordinary differential equation for $\check{v}$ results. This procedure will be reserved until later for it deserves its own treatment. Suffice it for the moment to note that this has been done by Gustavsson (1979). Instead, the classical method for solution has been made by assuming that the time dependence can be separated from that of y. Thus,

$$\check{v}(\alpha, y, \gamma, t) = \sum_{n=0}^{\infty} \hat{v}_n(\alpha, y, \gamma) e^{-i\omega_n t} \tag{1.15}$$

is taken and, as noted, (1.15) should be the infinite sum of all such model solutions. Moreover, ω_n is taken as a complex frequency with a positive imaginary part indicating an unstable mode. The substitution of (1.15) into (1.13) after the Fourier decomposition that is prescribed by (1.14) has been done, then the Orr–Sommerfeld equation is reduced to that of an ordinary differential equation for $\hat{v}$. Solutions are then required to meet the boundary conditions at the respective locations marked in terms of the y-variable; at y_1 and y_2, say. First, this means that $\hat{v}$ must satisfy conditions at y_1 and y_2. In Fourier space, the equivalent of (1.1) is $i(\alpha\hat{u} + \gamma\hat{w}) = \hat{v}'$, where $\hat{u}$ and $\hat{w}$ are defined in exactly the same manner as was done in (1.14) for $\check{v}$ as well as the solution form of (1.15). Thus, the conditions for $\hat{u}$ and $\hat{w}$ are now in terms of the first derivative of $\hat{v}$. The combination leads to the result that ω_n is a function of α, γ and kinematic viscosity (equivalently, the Reynolds number) of the flow for all n. From the point of view of the Laplace transform method, such solutions would be tantamount to finding poles in the complex Laplace space. But, in this way, the determination using (1.15) is more direct. Provided there are homogeneous boundary

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conditions in y, then the problem is that of the eigenvalue, eigenfunction variety with ω_n the eigenvalue. But, here the analogy to classical homogeneous eigenvalue–eigenfunction problems ends. First, as already noted, this differential equation is fourth order rather than second. Also, it is not self-adjoint, has a small parameter (reciprocal of the Reynolds number that is large compared to one) multiplying the highest derivative, thereby constituting a singular perturbation problem analytically or a stiff problem numerically. In short, neither the analysis nor the numerics are straightforward. Both of these topics will be treated in more detail. On the other hand, if only the question of stability is to be answered, then only *one* unstable eigenvalue need be found. But, no details of any specific initial-value problem or a determination of the full dynamics of any disturbance will follow in this way and it is not necessary for such a stability decision. But, if the modal expansions are to be used for this purpose, then all modes must be known and this includes those that are damped. The transient period becomes critical and it cannot be evaluated without this information. This topic will be presented in detail in Chapter 8 but, for now, it must be indicated that, among other things, it relates in part to the boundary conditions of the problem. Doubly bounded flows such as the channels have a different foundation in terms of the mathematics than those that have only one boundary (the boundary layer), or those without boundaries whatsoever as the jet, wake and mixing layer.

An alternative to using Fourier transformations is to use normal modes. Because the disturbances are described by linear equations, solutions can be sought in the following form

$$\tilde{v}(x,y,z,t) = \hat{v}(y)e^{i(\alpha x+\gamma z-\omega t)}. \tag{1.16}$$

This method is commonly referred to as the normal mode approach, and is tantamount to using Fourier transformations.

After making the substitution given by (1.16), the familiar Orr–Sommerfeld equation is found and is, for each mode,

$$(\alpha U - \omega)\Delta\hat{v} - \alpha U''\hat{v} = -i\nu\Delta^2\hat{v}, \tag{1.17}$$

where

$$\Delta = \frac{d^2}{dy^2} - \widetilde{\alpha}^2, \tag{1.18}$$

and

$$\widetilde{\alpha}^2 = \alpha^2 + \gamma^2, \tag{1.19}$$

with $\widetilde{\alpha}$ the scalar polar wavenumber in the α-γ plane of Fourier space. This form of the equation offers some interesting properties. This is best seen by

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returning to the original set of equations (1.1) to (1.4) and making the transformation on all of the dependent variables, and then using the modal form of solution (1.16).

The ordinary differential equations that are obtained by the prescribed operations are

$$i(\alpha\hat{u}+\gamma\hat{w})+\hat{v}'=0, \tag{1.20}$$

$$i(\alpha U-\omega)\hat{u}+U'\hat{v}=-\frac{i\alpha}{\rho}\hat{p}+\nu\Delta\hat{u}, \tag{1.21}$$

$$i(\alpha U-\omega)\hat{v}=-\frac{1}{\rho}\hat{p}'+\nu\Delta\hat{v}, \tag{1.22}$$

and

$$i(\alpha U-\omega)\hat{w}=-\frac{i\gamma}{\rho}\hat{p}+\nu\Delta\hat{w}. \tag{1.23}$$

Squire (1933) introduced what should be properly called an equivalent transformation. And, once this is done, the very useful Squire theorem in stability theory emerges. For this purpose, let

$$\widetilde{\alpha}\bar{u}=\alpha\hat{u}+\gamma\hat{w}, \tag{1.24}$$

$$\widetilde{\alpha}\bar{w}=-\gamma\hat{u}+\alpha\hat{w}. \tag{1.25}$$

Just as $\widetilde{\alpha}$ was the polar variable in the α-γ plane, it should be clear that $\bar{u}$ in (1.24) is the fluctuating component parallel to the wavenumber vector and $\bar{w}$ of (1.25) is in the angular direction. With the polar angle $\varphi=\tan^{-1}(\gamma/\alpha)$ defined in the plane, we see that $\overline{u}$,$\overline{w}$ are therefore the polar components of the velocity in the α-γ plane. With the use of these definitions the set of equations (1.20) to (1.23) can be combined to give

$$i\widetilde{\alpha}\bar{u}+\hat{v}'=0, \tag{1.26}$$

$$i\widetilde{\alpha}(\alpha U-\omega)\bar{u}+\alpha U'\hat{v}=-\frac{i\widetilde{\alpha}^2}{\rho}\hat{p}+\nu\widetilde{\alpha}\Delta\bar{u}, \tag{1.27}$$

and

$$i(\alpha U-\omega)\hat{v}=-\frac{1}{\rho}\hat{p}'+\nu\Delta\hat{v}. \tag{1.28}$$

Thus, if the additional changes of variables,

$$\bar{v}=\hat{v}, \tag{1.29}$$

$$\frac{\bar{p}}{\widetilde{\alpha}}=\frac{\hat{p}}{\alpha} \tag{1.30}$$

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are used along with

$$\omega = \alpha c, \tag{1.31}$$

$$\widetilde{c} = c, \tag{1.32}$$

where $\widetilde{c}$ or c is the phase speed, then equations (1.27) and (1.28) read

$$i\widetilde{\alpha}(U - \widetilde{c})\bar{u} + U'\bar{v} = -\frac{i\widetilde{\alpha}}{\rho}\bar{p} + \frac{\nu\widetilde{\alpha}}{\alpha}\Delta\bar{u} \tag{1.33}$$

and

$$i\widetilde{\alpha}(U - \widetilde{c})\bar{v} = -\frac{1}{\rho}\bar{p}' + \frac{\nu\widetilde{\alpha}}{\alpha}\Delta\bar{v}. \tag{1.34}$$

Clearly these equations are analogous to those of a purely two-dimensional system except, that is, for the factor that multiplies the coefficient of viscosity. When the pressure is eliminated between (1.33) and (1.34) then

$$(U - \widetilde{c})\Delta\bar{v} - U''\bar{v} = -i\frac{\widetilde{\nu}}{\widetilde{\alpha}}\Delta^2\bar{v} \tag{1.35}$$

with

$$\widetilde{\nu} = \frac{\nu\widetilde{\alpha}}{\alpha} = \frac{\nu}{\cos\varphi} \tag{1.36}$$

becomes the Orr–Sommerfeld equation in this notation. From (1.35), the well-known Squire theorem can now be identified. Except for the viscosity, the equations governing a three-dimensional and a two-dimensional perturbation are the same. The relation (1.36), when written in terms of the nondimensional Reynolds number, Re, is simply $\widetilde{Re} = Re\cos\varphi$; $Re = \rho U_0 L/\mu = U_0 L/\nu$ with U_0 and L as characteristic scales of the mean flow. Now, as was shown in Fig. 1.5 or 1.6, there is a minimum Reynolds number for the onset of instability. Although this result is for the flat-plate boundary layer, it is also true for plane Poiseuille flow. Consequently, by use of the Squire transformation, the Squire theorem can be noted. The minimum Reynolds number for instability will be higher for an oblique three-dimensional wave than for a purely two-dimensional wave. Note that this statement does not rule out the possibility that, for a high enough values of the Reynolds number, an unstable oblique oscillation is possible even though the purely two-dimensional one that has the same value of α is damped. This last point is one referred to by Watson (1960) as well as Betchov & Criminale (1967) but has not been exploited to date.

The equation for the other component of the polar wave velocity is found directly by combining the definition (1.25) with operations on the appropriate equation and is

$$i(\alpha U - \omega)\bar{w} - \nu\Delta\bar{w} = \sin\varphi U'\bar{v}, \tag{1.37}$$

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which is nothing more than

$$i(\alpha U - \omega)\hat{\omega}_y - \nu \Delta \hat{\omega}_y = -i\gamma U' \bar{v} = -i\tilde{\alpha} \sin\varphi U' \bar{v} \tag{1.38}$$

when written using normal modes, as can be seen by taking normal modes of (1.7) and using (1.21) and (1.23). Regardless of the choice, this equation has become known as the Squire equation in stability theory. It is important to notice that the inhomogeneous term depends upon the solution for $\bar{v}$ and has the factor that is a measure of the obliquity of the wave. This term has been referred to as "lift up" by several authors and is attributed to Landahl (1980). When the angle of obliquity is perpendicular to the direction of the flow $(\varphi = \pi/2)$, then the mean flow no longer has any influence and the equation, in this limit, can be solved exactly for $\bar{v}$ even without the assumption of modes.

The completion of the problem from this basis can be made by (i) solving for $\bar{v}$ from (1.35); (ii) determining $\bar{u}$ from the condition of incompressibility,

$$i\tilde{\alpha}\bar{u} = -\bar{v}';$$

(iii) solving for $\bar{w}$ from (1.37) and then inverting the transformations given by (1.24) and (1.25), namely

$$\hat{u} = \cos\varphi \bar{u} - \sin\varphi \bar{w}, \tag{1.39}$$

and

$$\hat{w} = \sin\varphi \bar{u} + \cos\varphi \bar{w} \tag{1.40}$$

to obtain the original Cartesian velocity components. And, as has already been shown, the pressure can be subsequently determined. From this summary, it can be seen that the central part of the analysis clearly rests with the success of solving both the Orr–Sommerfeld and the Squire equations if a full examination is desired. This is far different than merely determining whether or not the flow is stable.

1.4 Transient Dynamics

By comparison, the transient portion of the dynamics of perturbations has only relatively recently become a topic of some importance in stability theory. On the one hand, because of the many complexities in the mathematics and the lack of adequate computing in the early stages of the development, it was practically impossible to actually accomplish this task. At the same time, traditional thought on this matter did not indicate that this aspect could have any bearing on the ultimate behavior, and was simply ignored. Today, it is now quite clear that the results of stability calculations in the modal form are really more for the purpose of predicting the asymptotic fate of any disturbance, and the transient dynamics can have and do lead to events that make this part of the problem even more of interest than it ever was.

It can be recalled that the leading equations to be used in the stability analysis have different properties than those that are more common in initial-value, boundary value problems. For iteration, the principal ones, namely the Orr–Sommerfeld equation, is fourth order and is not self-adjoint. Thus, for a specific initial-value designation, there is the question of exactly how to express arbitrary functions or even what set of functions are to be used for expansion of these given functions. The Orr–Sommerfeld equation does not have a set of known functions. Of course, there are means to form inner products (cf. Drazin & Reid, 1984) in this case and therefore all constants needed can be evaluated. But, it is only the channel flows that have a complete set of eigenfunctions (cf. DiPrima & Habetler, 1969) so long as the problem is viscous. Inviscidly, there is only a continuous spectrum (Case, 1960a, 1961; Criminale, Long & Zhu, 1991). The boundary layer (Mack, 1976) and the free shear flows have been shown to have only a finite number of such modes. But, regardless, the fact that there must be a continuous spectrum to make the problem complete is already a recognition to the salient fact that there can be temporal behavior that is algebraic rather than just exponential.

The use of the Laplace transform in time to transform the partial differential equations to ones that are but ordinary has been made by Gustavsson (1979) as an alternative to modal expansions for initial-value problems. In this way the problem is completely specified and, in principle, can be made tractable. Unfortunately, only general properties can actually be found using this approach, since the ordinary differential equation that must be solved is the same as the Orr–Sommerfeld. However, the important algebraic behavior is shown to exist along with the exponential modes, and is due to the existence of a continuous spectrum, because there must be branch cuts as well as poles when the inversion to real time is to be made. This method also closes the gap for those flows where there is the lack of modes for the arbitrary initial-value problem, and the continuous spectrum, together with the discrete modes, allows for arbitrary expansions. This approach, where both the discrete and continuous spectra are used, has been well described by Grosch & Salwen (1978) and Salwen & Grosch (1981). There is no continuous spectrum for viscous channel flows since, as stated, the modes form a complete set with the problem confined to one of finiteness where there is normally only a discrete spectrum.

Then, there is yet another way in which algebraic behavior can arise. This can be seen by referring to the Squire equation where there is the one inhomogeneous term that is proportional to the normal velocity component, that is, the term attributed to lift-up. This equation, unlike that of the homogeneous Orr–Sommerfeld equation, can be resonant if there is a matching of the frequencies of the respective modes of the normal velocity with the dependent variable of this equation. This phenomenon has been shown to be possible for plane channel flow by Benney & Gustavsson (1981) but, it was concluded, resonance is not possible for the boundary layer. The case for resonance in the free shear flows is yet to be determined.

Exactly how dominant the algebraic behavior might be depends upon the particular problem and, to some extent, whether or not the problem is treated with or without viscosity. For any of the cases where there is the existence of a continuous spectrum it should be noted that perturbations can increase algebraically to quite large amplitudes before any exponentially growing mode supersedes its progress. The algebraic growth is ultimately damped by viscous action if viscosity is included in the problem. Otherwise, for some problems, the portion that grows algebraically can do so without bound and thus the assumption of linearity is overcome long before the dominance of any exponential growth. Thus, the concept of stability needs to be put in the proper context and it would be better to ask such questions as the existence of (a) optimum or maximum growth of disturbances or (b) behavior of the relative components of the perturbation velocity or vorticity, for example. Such undertakings have been and are continuing to be made.

1.5 Asymptotic Behavior

As has been stated, one answer to the question of whether or not a given flow is stable is to determine whether or not there is at least one eigenmode that results in exponential growth. Then, regardless of the time scale, there will eventually be an unlimited increase of the perturbation amplitude and the flow cannot in any way be stable. And, this may be possible with or without any early transient algebraic development. In short, it is the long time limit that must now be found. For this purpose there are numerous numerical schemes that can be used to make the determination in a reasonably efficient manner. The question of the many or infinite number of modes does not actually need to be answered for only one growing mode is required to answer the question. Typical results of this strategy results in an eigenvalue expression that has the complex frequency as a function of the polar wavenumber, angle of obliquity and the Reynolds number if viscous forces are included. Or, because of the Squire transformation from the Cartesian to the polar wavenumber variables, the determination of these values can be made without resorting to the oblique angle value. If the behavior for three-dimensionality is desired, it can be inferred from the equivalent two-dimensional data by use of the transformation as was shown by Watson (1960) or Betchov & Criminale (1967), for example.

There are other means of assessing the asymptotic fate of a particular initial input in more detail. For example, in order to predict the complete spatial behavior of the initial distribution, then the Fourier transforms that were made in the x and z variables must be inverted for this purpose. In the asymptotic state any transient response has long been exceeded by the exponential modal behavior and thus the leading behavior of these double integrals is exponential in time and can be evaluated by the method of steepest descent. In this way, the general features of the evolving disturbance can be predicted as well as the maximum amplitude. Such features include the location and distribution of the maximum part of the evolving disturbance or, as is better known, the description of the wave packet. A very early attempt for this kind of analysis for a localized disturbance in the laminar boundary layer was made by Criminale & Kovasznay (1962) and it was found that a wave packet ultimately was formed with the wave fronts swept back (three-dimensional) and the wavenumbers and frequencies those of the band of amplified Tollmien–Schlichting waves. The relative widths of the packet could also be determined by this method. The important point here is that modal expansions can and do provide the critical information required for the asymptotic behavior.

1.6 Role of Viscosity

The role of viscosity in the stability of parallel or almost parallel flows has two parts and is both the cause of the instability and has the role of damping as well. This scenario is, in many ways, unique in fluid mechanics but the phenomenon is known to exist in other fields. It is best explained by analogy. First, as in many other physical problems, viscous forces do ultimately act as damping but not necessarily at all times or in all situations in certain flows. An unstable Tollmien–Schlichting wave not only requires viscosity to be unstable but have only been shown to exist only in the presence of solid boundaries.

As suggested, an explanation as to why viscosity is destabilizing can best be illustrated by analogy. Such an analogy was suggested by Betchov & Criminale (1967) and it remains valid today. For an oscillator with mass m and a linear restoring force proportional to k but with a time delay τ, the equation of motion

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can be written as

$$m\frac{d^2x(t)}{dt^2} + kx(t-\tau) = 0. \tag{1.41}$$

Then, for small values of the delay, τ, (1.41) takes the form

$$m\frac{d^2x(t)}{dt^2} - \tau k\frac{dx(t)}{dt} + kx(t) = 0. \tag{1.42}$$

Thus, it is clear from this result that such action is destabilizing and it is essentially a question of phasing. Although the conclusions to be drawn from (1.42) may appear simple minded, it expresses the elements required. In the more subtle arguments that will be used to demonstrate this point more precisely, it will be expressed in terms of Reynolds stress and interaction with the mean flow but it is the certain phasing that must be correct in order for there to be an instability in the flow.

Still, there are many problems that can be investigated without viscosity and, historically, this is exactly what was done. Most notably the contributions of Rayleigh (1879, 1880, 1887, 1892a,b,c) were all made by inviscid analysis. And, interestingly enough, when investigating these problems, Rayleigh only examined two-dimensional perturbations. The Squire transformation and theorem that demonstrates that the two-dimensional problem is all that need be done in order to determine the stability came much later. Except for the much earlier work that is now referenced as Kelvin–Helmholtz (Helmholtz, 1868; Kelvin, 1871), this work provided much of the important bases that remains even to this day in the field of hydrodynamic stability. There are several theorems due to Rayleigh that are important both for the mathematics and to the understanding of the physics of such flows and this is true even if viscous effects are retained. However, the flows that were extensively examined in this manner by Rayleigh were those of the jet, wake and the mixing layer.

One need only to return to the fundamental equations, (1.33)–(1.35), which were expressed as an equivalent two-dimensional system, as $\gamma = 0$ for no z-variation for true two-dimensionality, and neglect the viscous terms to derive the Rayleigh equation. This is straightforward where the pressure is eliminated and it is found that

$$(U-c)\left(\hat{v}'' - \alpha^2\hat{v}\right) - U''\hat{v} = 0. \tag{1.43}$$

Unlike the Orr–Sommerfeld equation, (1.43) is second order and, although not self-adjoint, it can easily be so constructed and the more conventional rules for boundary-value problems can be used. Unfortunately, there is no set of known functions for this equation save for some special $U(y)$ distributions and, for the initial-value part of the problem, a continuous spectrum must be added since there are only a finite number of discrete eigenmodes. If one is interested in the full three-dimensional problem, then the equivalent Squire equation must be included.

Comparison of (1.43)–(1.35), say, tacitly reveals another point that is well known in the theory. Just as the ignoring of viscous effects is tantamount to lowering the order of the governing equation and thereby making it singular, the Rayleigh equation can also be singular if $(U - c) = 0$ somewhere in the flow. For this to be true, then c must be purely real and thus the interpretation is that the phase speed for the mode, c_r, is equal to the value of the mean flow at some y-location in the flow. Likewise, this implies that the flow is neutrally stable in this case. Exploitation of this fact is the basis for many of the theorems due to Rayleigh. It is also part of the reasoning for the emergence of a continuous spectrum of eigenvalues, as demonstrated by Case (1960a, 1961). Chapter 2 is devoted to inviscid problems.

1.7 Geometries of Relevance

The mean flows envisioned and described have been at least tacitly assumed to be those that are two-dimensional and unidirectional, whether or not they are the true or approximate solutions to the Navier–Stokes equations. *Under these restrictions, the channel, flat-plate boundary layer and free shear flows of the jet, wake and mixing layer are classical problems studied under this approximation*. Cartesian coordinates describe these situations quite well. But there is no reason that the parallel flow that exists in a round pipe or the wake or jet of a round nozzle cannot be explored in the same manner. Flow along curved walls is another similar analogy. Then, there is the flow that can exist between concentric cylinders. All of these are important and can be investigated but the respective governing equations in these cases are better cast in terms of polar or other coordinates. Such action has been taken and these problems generate still more surprises for the chain of logical thought. In short, a great deal of the physics can be transposed to the new geometries but the results are not nearly so satisfying. Some of the failures can be explained but some are still enigmas; others yield even more salient conclusions. Examples for each of such flows will be examined in detail.

1.8 Spatial Stability Bases

As has been suggested, the major experiments that have been done in the investigation of stability in flat-plate boundary layers for example, does not, strictly speaking, correspond to the theory defined by a temporal initial-value problem. Instead, exploration was made by introducing a disturbance at an initial x-location upstream in the flow. Then, subsequent measurements are made downstream from this location in order to determine the resulting flow. By definition, this is a spatial initial-value problem. The behavior in time is simply periodic and neither decreases nor increases. From the definition given by (1.16), ω must be purely real. As a result, some alteration in the formulation must be made so that this formulation has merit.

The set of equations, (1.20)–(1.23), are still valid. However, these were developed with the understanding that the respective wavenumbers, α and γ, were real. For the spatial problem, these parameters are to be complex. But, if an initial value for the relevant quantity is to be given as a function of y and z at $x = x_0$, γ must be real in order to satisfy the far-field boundary conditions as $z \to \pm\infty$. In like manner, the integral that defines the limits for the x-variable in (1.14) would only be for $x > x_0$, much in the manner of a Laplace transform in time with $t > 0$. The net result leads to an amended Orr–Sommerfeld equation in the sense that α is the eigenvalue with γ, ω and the Reynolds number as parameters. Also, in this case, $\alpha_i < 0$ implies instability. The Squire transformation is still permitted and the theorem is valid since it implies the neutral locus where $\alpha_i = \omega_i = 0$.

A general problem can be constructed that combines both the temporal and the spatial bases. The boundary value requirements remain the same but now the resulting dispersion relation can be seen to be one where there are two complex variables, namely ω and α. The wavenumber γ remains real and the Reynolds number is again a parameter if the system is taken as viscous. It is for this reason that Gaster (1965a,b) offered an alternative when the question of spatial stability was originally proposed. In like manner, Briggs (1964) used this formulation in studying instabilities in plasmas. Here, solutions were sought by use of the normal modes decomposition, and the dispersion relation that is developed once the boundary conditions have been met is taken as a function of the two complex variables ω and α. For many problems as the boundary layer, the amplification rates are small and this allowed Gaster to make a local Laurent expansion and use the Cauchy–Riemann relations of complex variable theory to establish a correspondence between the temporal values that were already computed and the spatial quantities that were unknown. These relations have proven to be of major importance. When amplification rates are large, however, care must be taken and the complex eigenvalues must be computed directly, as shown by Betchov & Criminale (1966) for jet and wake problems. Mattingly & Criminale (1972) extended this work and added experimental confirmation results as well. Direct numerical calculations have also been made for the boundary layer by Kaplan (1964), Raetz (1964) and Wazzan, Okamura & Smith (1966) and the agreement with the analytical continuation method of Gaster was shown to be quite accurate.

Once the spatial-temporal problem is established, then other issues must be considered. Briefly stated, means of instability – now known as convective and absolute instabilities – have been identified when viewed from the spatial initial-value construction. These concepts relate to the fact that, for convective instability, at a fixed spatial location, amplification can occur and then pass as it is convected downstream. Absolute instability is one that, when amplification has begun, it does not cease and local break down is inevitable. Chapter 4 will discuss such problems in detail.

So to move from this historical perspective of hydrodynamic instabilities, the discussion will move from the simpler problems and mathematical approximations to solving the full Navier–Stokes equations in later chapters. Chapter 2 will begin by introducing dimensionless equations and moving directly into the temporal instability formulation for inviscid incompressible equations, followed in Chapter 3 by viscous incompressible flows. Chapters 4 and 5 will discuss the spatial stability of incompressible and compressible flows, respectively. Chapters 6, 7 and 8 will introduce centrifugal, geophysical and transient dynamics, respectively. The remaining chapters move beyond classical linear hydrodynamic instability to nonlinear model interactions. Chapter 9 will primarily focus on nonlinear modes pertaining to the flat plate. Chapter 10 will look at how instabilities are initiated (i.e., receptivity) and methods of predicting the transition from laminar to turbulent flow. Chapters 11 and 12 will briefly discuss direct numerical simulation and control within the framework of the unsteady Navier–Stokes equations. Finally, the text will close with the challenging discussion of how experiments have evolved from the early days of observations by Reynolds (1833) to our ability to make measurements of these "infinitesimal" disturbances – called hydrodynamic instabilities.