BULL. AUSTRAL. MATH. SOC. 04A30
VOL. 25 (1982), 143-145.

LOCALIZATION OF PATAI'S THEOREM ON ALEPHS

FREDERICK BAGEMIHL

If μ and γ are ordinal numbers such that $2^{\aleph_{\mu+\alpha}} = \aleph_{\mu+\alpha+\gamma}$ for every $\alpha \leq \gamma \cdot 2$, then $\gamma < \omega$. This localizes a result due to Patai.

According to Patai [*1*, Theorem XIV], if the equality $2^{\aleph_\alpha} = \aleph_{\alpha+\rho}$ holds for *every* ordinal number α, then $\rho < \omega$. We shall show that the same conclusion can be drawn from the assumption that the equality holds when α ranges over merely an arbitrary, but sufficiently long, interval.

THEOREM. *If* μ *and* γ *are ordinal numbers such that*

(1) $$2^{\aleph_{\mu+\alpha}} = \aleph_{\mu+\alpha+\gamma}$$

for every $\alpha \leq \gamma \cdot 2$, *then* $\gamma < \omega$.

Proof. Assume the conclusion to be false. Then $\gamma = \lambda + n$, where λ is a limit number and $n < \omega$, so that (1) holds for every $\alpha \leq \lambda + \lambda + n$.

We show first that, for every $\xi < \lambda$,

(2) $$2^{\aleph_{\mu+\lambda+\xi}} = 2^{\aleph_{\mu+\lambda}} .$$

For suppose this to be false. Then there exists a smallest ordinal number β satisfying $0 < \beta < \lambda$ such that

(3) $$2^{\aleph_{\mu+\lambda+\beta}} > 2^{\aleph_{\mu+\lambda}} .$$

Received 24 August 1981.

Note that

(4) $$|\beta| \leq |\lambda| \leq |\omega_\lambda| = \aleph_\lambda \leq \aleph_{\mu+\lambda} ,$$

where $|\beta|$ denotes the cardinality of β . If β is a limit number, then by the minimal property of β ,

$$2^{\aleph_{\mu+\lambda+\beta}} = 2^{\aleph'} = \prod_{\xi<\beta} 2^{\aleph_{\mu+\lambda+\xi}} = \prod_{\xi<\beta} 2^{\aleph_{\mu+\lambda}} = (2^{\aleph_{\mu+\lambda}})^{|\beta|} = 2^{\aleph_{\mu+\lambda}} ,$$

where $\aleph' = \sum_{\xi<\beta} \aleph_{\mu+\lambda+\xi}$, which contradicts (3). If β is isolated (a case neglected by Patai), then by the minimal property of β ,

(5) $$2^{\aleph_{\mu+\lambda+(\beta-1)}} = 2^{\aleph_{\mu+\lambda}} < 2^{\aleph_{\mu+\lambda+\beta}} .$$

According to (1),

$$2^{\aleph_{\mu+\lambda+(\beta-1)}} = \aleph_{\mu+\lambda+(\beta-1)+\lambda+n} ,$$

whereas

$$2^{\aleph_{\mu+\lambda+\beta}} = \aleph_{\mu+\lambda+\beta+\lambda+n} = \aleph_{\mu+\lambda+(\beta-1)+1+\lambda+n} = \aleph_{\mu+\lambda+(\beta-1)+\lambda+n} ,$$

so that

$$2^{\aleph_{\mu+\lambda+(\beta-1)}} = 2^{\aleph_{\mu+\lambda+\beta}} ,$$

which contradicts (5). Hence (2) is true for every $\xi < \lambda$.

According to (1),

$$2^{\aleph_{\mu+\lambda}} = \aleph_{\mu+\lambda+\gamma} = \aleph_{\mu+\lambda+\lambda+n} = \aleph_{\mu+\lambda\cdot 2+n} ,$$

whereas

$$2^{\aleph_{\mu+\lambda\cdot 2}} = \aleph_{\mu+\lambda\cdot 2+\gamma} = \aleph_{\mu+\lambda\cdot 2+\lambda+n} = \aleph_{\mu+\lambda\cdot 3+n} ,$$

and consequently

(6) $$2^{\aleph_{\mu+\lambda}} < 2^{\aleph_{\mu+\lambda\cdot 2}} .$$

But in view of (2) and (4),

https://doi.org/10.1017/S0004972700005128 Published online by Cambridge University Press

$$2^{\aleph_{\mu+\lambda\cdot 2}} = 2^{\aleph''} = \prod_{\xi<\lambda} 2^{\aleph_{\mu+\lambda+\xi}} = \prod_{\xi<\lambda} 2^{\aleph_{\mu+\lambda}} = \left(2^{\aleph_{\mu+\lambda}}\right)^{|\lambda|} = 2^{\aleph_{\mu+\lambda}} ,$$

where $\aleph'' = \sum_{\xi<\lambda} \aleph_{\mu+\lambda+\xi}$, which contradicts (6).

Our assumption is therefore untenable, and the theorem is true.

Reference

[1] L. Patai, "Untersuchungen über die Alefreihe", *Math. und naturw. Berichte aus Ungarn* 37 (1930), 127-142.

Department of Mathematics,
University of Wisconsin-Milwaukee,
Milwaukee,
Wisconsin 53201,
USA.

https://doi.org/10.1017/S0004972700005128 Published online by Cambridge University Press