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# COUNTEREXAMPLES TO A CONJECTURE FOR NEUTRAL EQUATIONS

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ABSTRACT. A collection of examples of first order linear neutral differential delay equations having a nonoscillatory solution with lim sup $= \infty$ and lim inf $= 0$ at $\infty$ is given. This disproves a recent conjecture about the asymptotic behavior of solutions to such equations.

In a paper in 1986, Grammatikopoulos, Grove and Ladas [3] proved some asymptotic properties of nonoscillatory solutions of the first order linear differential delay equation

$$\frac{d}{dt}[y(t)+py(t-\tau)]+qy(t-\sigma)=0 \tag{1}$$

where $q \neq 0, p, \tau$ and $\sigma$ are real constants. The asymptotic behavior of solutions of (1) in several cases involving various sign conditions on $q, \tau, p$ and $p-1$ was left unresolved in [3], but two conjectures covering these unresolved cases were given in that paper. Before stating these conjectures, we observe that $y$ satisfies (1) if and only if $-y$ satisfies (1). Thus we can without loss of generality assume that a nonoscillatory solution of (1) is eventually positive, *i.e.*, is positive on $[t_0, \infty)$ for some real number $t_0$.

CONJECTURE 1. *Suppose* $p < 0$ *and* $q\tau < 0$. *Then* $\lim_{t\to\infty} y(t) = \infty$ *or* $\lim_{t\to\infty} y(t) = 0$ *for every eventually positive solution of (1).*

CONJECTURE 2. *Suppose* $q < 0$. *If (i)* $p = 1$ *or (ii)* $p > 1$ *and* $\tau > 0$, *then* $\lim_{t\to\infty} y(t) = \infty$ *for every eventually positive solution of (1).*

Recently in [4], Conjecture 1 was proved as was Conjecture 2(i). In addition, Conjecture 2(ii) was shown to hold in case any one of the following three conditions is satisfied:

$$-q\tau < \ln p \text{ and } \sigma \geq 0, \text{ or}$$
$$-q\tau < p\ln p \text{ and } \sigma \geq \tau, \text{ or}$$
$$\sigma \geq 0,\ 1+q\tau \geq 0,\ p \geq 2 \text{ and } 1+q\tau+p-2 > 0.$$

The purpose of this note is to show that in general Conjecture 2(ii) is false. Let $\alpha$ be a positive real number. Put $p = e^{\alpha}$, $q = -2\alpha e^{\alpha}$ and $\tau = \sigma = 1$. With these choices the characteristic equation for (1) becomes

$$\lambda(1+e^{\alpha-\lambda}) = 2\alpha e^{\alpha-\lambda}. \tag{2}$$

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$\lambda = \alpha$ is clearly a positive root of this equation. To find nonreal roots with real part $\alpha$, let $\lambda = \alpha + i\beta$. Putting this expression for $\lambda$ in (2) and equating real and imaginary parts gives the two equations

$$(3) \qquad \alpha = -\frac{\beta(1+\cos\beta)}{\sin\beta} \text{ and } \alpha = -\frac{\beta\sin\beta}{1-\cos\beta}$$

which are equivalent. Note that $\frac{d\alpha}{d\beta} = \frac{(1+\cos\beta)(\beta-\sin\beta)}{\sin^2\beta} \geq 0$, so $\alpha$ is an increasing function of $\beta$ on each of the intervals $\big(2(k-1)\pi, 2k\pi\big)$ for each positive integer $k$ and has vertical asymptotes at $\beta = 2k\pi$. Also, $\lim_{\beta\to 2k\pi^-}\alpha(\beta) = \infty$, $\lim_{\beta\to 2k\pi^+}\alpha(\beta) = -\infty$, $\alpha\big((2k+1)\pi\big) = 0$ and $\lim_{\beta\to 0^+}\alpha(\beta) = -2$. Thus for any $\alpha > 0$ there are unique numbers $\beta_0 = \beta_0(\alpha) \in (\pi, 2\pi)$ and $\beta_1 = \beta_1(\alpha) \in (3\pi, 4\pi)$ so that $\alpha = g(\beta_1) = g(\beta_0)$ where $g(\beta) = -\beta(\sin\beta)/(1-\cos\beta)$. This means that $\alpha$, $\alpha \pm i\beta_0$ and $\alpha \pm i\beta_1$ are roots of the characteristic equation (2). Consequently

$$(4) \qquad y(t) = e^{\alpha t}(2 - \cos\beta_0 t - \cos\beta_1 t)$$

is a solution of (1). Clearly $y(t) \geq 0$ and $\limsup_{t\to\infty} y(t) = \infty$ for any choice of $\beta_0$ and $\beta_1$. $y(t) > 0$ for all $t \geq 0$ if and only if $\beta_1/\beta_0$ is irrational. We now claim there is a dense set of $\alpha$'s with the property that $y(t) > 0$ for $t \geq 0$, $\limsup_{t\to\infty} y(t) = \infty$ and $\liminf_{t\to\infty} y(t) = 0$.

In our construction we use the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_1 = N$ and $a_{k+1} = N^{a_k}$ for $k \geq 1$ where $N > 1$ is an integer to be selected. We will also use the number $\tau_N$ where $\tau_{N,n} := \sum_{k=1}^{n} a_k^{-1} \longrightarrow \tau_N$ as $n \longrightarrow \infty$. Observe that

$$(5) \qquad 0 < \tau_N - \tau_{N,n} = \sum_{k=n+1}^{\infty} a_k^{-1} \leq \frac{1}{a_{n+1}} \cdot \frac{N}{N-1}.$$

First we show that $\tau_N$ is irrational. Clearly $\tau_{N,n} = m_n/a_n$ for some positive integer $m_n$. If $\tau_N = k/\ell$ for positive integers $k$ and $\ell$, then

$$0 < \tau_N - \tau_{N,n} = \left|\frac{k}{\ell} - \frac{m_n}{a_n}\right| = \left|\frac{ka_n - \ell m_n}{\ell a_n}\right| \geq \frac{1}{\ell a_n}.$$

But this contradicts (5) for large $n$.

Now let $\alpha_0 > 0$ and $\varepsilon > 0$ be given with $\varepsilon < \alpha_0$ and $\varepsilon < 1$. Let $h(\alpha) = \beta_1(\alpha)/\beta_0(\alpha)$. Then $h$ is a continuous function of $\alpha$ and maps the interval $(\alpha_0-\varepsilon, \alpha_0+\varepsilon)$ to an interval of length $\delta > 0$ containing $h(\alpha_0)$. Now pick $N > 2/\delta$ and $N > e^{2(\alpha_0+1)}$ and an integer $M$ so that $\tau_N + M/N = h(\alpha_N)$ for some $\alpha_N \in (\alpha_0-\varepsilon, \alpha_0+\varepsilon)$. Then $\beta_1/\beta_0 = \beta_1(\alpha_N)/\beta_0(\alpha_N) = M/N + \tau_N$ is an irrational number. Let $t_n = 2\pi a_n/\beta_0$. Then from (4),

$$\begin{aligned} y(t_n) &= e^{\alpha_N t_n}(2 - \cos\beta_0 t_n - \cos\beta_1 t_n) = e^{2\pi a_n \alpha_N/\beta_0}\big(1 - \cos(2\pi a_n \beta_1/\beta_0)\big) \\ &\leq (e^{2\alpha_N})^{a_n}\Big(1 - \cos\big(2\pi a_n(M/N + \tau_{N,n})\big)\Big) \\ &\leq (e^{2(\alpha_0+1)})^{a_n}\Big(1 - \cos\big(2\pi a_n(\tau_N - \tau_{N,n})\big)\Big) \end{aligned}$$

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since $a_n\tau_{N,n}$ and $a_N/N$ are integers. Now $\cos u \geq 1 - u$ for $0 < u < 1$, so

$$\begin{aligned} y(t_n) &\leq (e^{2(\alpha_0+1)})^{a_n} 2\pi a_n(\tau_N - \tau_{N,n}) \\ &\leq 2\pi \frac{N}{N-1} a_n \Big(\frac{e^{2(\alpha_0+1)}}{N}\Big)^{a_n} \end{aligned}$$

by (5). But now by choice of $N$, $e^{2(\alpha_0+1)}/N < 1$, so $\lim_{n\to\infty} y(t_n) = 0$. Thus $y$ has the desired property as we claimed.

This class of counterexamples shows that Conjecture 2 is false in general. In the notation of NDDE (1), $\sigma = 1 = \tau$, $p = e^{\alpha}$ and $-q = 2\alpha e^{\alpha} = 2p\ln p$. Hence we have found a dense collection of points along the curve $-q = 2p\ln p$ for which Conjecture 2(ii) fails. A similar construction for the equation

$$\frac{d}{dt}[y(t) + e^{\alpha} y(t-1)] = 2\alpha y(t)$$

furnishes a dense collection of examples along the curve $-q = 2\ln p$ for which Conjecture 2(ii) also fails. Here $\tau = 1 > \sigma = 0$.

Recently there have been several papers written on linear generalizations of (1) obtained by replacing

$$py(t-\tau) \text{ by } \sum_{i=1}^{k} p_i y(t-\tau_i) \text{ or}$$

$$qy(t-\sigma) \text{ by } \sum_{i=1}^{m} q_i y(t-\sigma_i).$$

See references [1], [2], [5], [6] and [7]. We offer here two examples to show that these more general equations may also have positive solutions with lim sup $= \infty$ and lim inf $=$ 0 at $\infty$.

The examples are:

$$\frac{d}{dt}[y(t) + 2e^{\alpha\tau} y(t-\tau) - e^{\alpha\sigma} y(t-\sigma)] = 2\alpha e^{-\alpha\rho} y(t+\rho) \tag{6}$$

where $\alpha > 0, \tau > 0, \sigma = (1 + \frac{2\ell}{2k+1})\tau$ for some positive integers $\ell$ and $k$, $\rho \in (\rho_0 - \varepsilon, \rho_0 + \varepsilon)$ where $\rho_0 \in [-\sigma, \infty)$ and $\varepsilon > 0$, and

$$\frac{d}{dt}[y(t) + e^{\alpha t} y(t-\tau)] = (2+\varepsilon)\alpha e^{\alpha\sigma} y(t-\sigma) - \varepsilon\alpha e^{\alpha\rho} y(t-\rho) \tag{7}$$

where $\alpha > 0, \tau > 0, \beta > 0$ and $(\beta \sin\beta\tau)/(1-\cos\beta\tau) = \alpha$, $\sigma = 2k\pi/\beta_0$, $\rho = 2n\pi/\beta_0$ where $n$ and $k$ are integers with $n > k$ and $n > \beta\tau/(2\pi)$, and $\varepsilon \in \big(0, 2/(-1+e^{2(n-k)\pi\alpha/\beta})\big)$. For both (6) and (7) the assumptions on the parameters guarantee that the characteristic equations have roots $\alpha$, $\alpha \pm i\beta$ and $\gamma < 0$. Thus

$$y(t) = e^{\gamma t} + e^{\alpha t}(1 + \cos\beta t)$$

is a solution having the desired properties.

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