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WEAK FORMS OF AMENABILITY FOR BANACH ALGEBRAS

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Abstract

In this paper, the amenability and approximate amenability of weighted ℓ^p-direct sums of Banach algebras with unit, where $1 \le p < \infty$, are completely characterized. Applications to compact groups and hypergroups are given.

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1. Introduction

The notion of approximate amenability of a Banach algebra was introduced by Ghahramani and Loy in [7]. Dales *et al.* [6] found a necessary and sufficient condition for approximate amenability of Banach algebras, and also proved that the Banach sequence algebras $\ell^p(\omega)$, $1 \le p < \infty$, $\omega \in [1, +\infty)^I$, are not approximately amenable. The present paper is a continuation of the paper by Dales *et al.* By a direct method, it is proved that for a family of nonzero Banach algebras $\{\mathfrak{A}_i\}_{i\in I}$, $\ell^p((\mathfrak{A}_i), \omega)$ is amenable (respectively, approximately amenable) if and only if I is finite, and for each $i \in I$, $\mathfrak{A}_i$ is amenable (respectively, approximately amenable). For another proof, see [5]. The organization of the paper is as follows. Section 2 is devoted to preliminaries and notations which are needed throughout the rest of the paper. Section 3 gives a complete characterization of amenability and approximate amenability for weighted ℓ^p-direct sums of Banach algebras with unit, where $1 \le p < \infty$. In Section 4 it is proved that for the matrix Banach algebra $\mathfrak{E}_p(I)$, the two notions of amenability and approximate amenability are equivalent. Moreover, applications to compact groups and hypergroups are given. As a corollary, it is proved that if G is an infinite compact group, then the convolution Banach algebra $L^2(G)$ is not approximately amenable. This is a generalization of Proposition 2.30 of [1] (see also [2]).

2. Preliminaries

Let A be a Banach algebra, and let X be a Banach A-bimodule. A *derivation* is a bounded linear map $D : A \to X$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in A).$$

For $x \in X$, set $\mathrm{ad}_x : a \mapsto a \cdot x - x \cdot a, A \to X$. Then ad_x is a derivation; these are the *inner* derivations. A derivation $D : A \to X$ is *approximately inner* if there is a net $(x_\alpha) \subseteq X$ such that

$$D(a) = \lim_\alpha a \cdot x_\alpha - x_\alpha \cdot a \quad (a \in A).$$

A Banach algebra A is *amenable* (respectively, *approximately amenable*) if every derivation from A into X^* is inner (respectively, approximately inner) for all Banach A-bimodules X. For more details see [7, 10, 12].

The following result is taken from [6, Theorem 4.2]. For the definition of $\ell^p(\omega)$, see [6] or Definition 3.1 of the present paper.

THEOREM 2.1. *The Banach sequence algebras $\ell^p(\omega)$, $1 \le p < \infty$, $\omega \in [1, +\infty)^I$, are not approximately amenable.*

Let A be a Banach algebra. The projective tensor product $A \widehat{\otimes} A$ is a Banach A-bimodule, under the operations defined by $c \cdot (a \otimes b) = ca \otimes b$ and $(a \otimes b) \cdot c = a \otimes bc$ for $a, b, c \in A$. The corresponding *diagonal operator* $\pi_A : A \widehat{\otimes} A \to A$ is defined through $\pi_A(a \otimes b) = ab$ $(a, b \in A)$. For more details, see [4].

The following result is a characterization of amenable Banach algebras, and is taken from [10]. See also the comment after Corollary 2.2 of [6].

THEOREM 2.2. *Let A be a Banach algebra. Then A is amenable if and only if there is a constant $C > 0$ such that, for each $\epsilon > 0$ and each finite subset S of A, there exists $F \in A \otimes A$ with $\|F\|_\pi \le C$ such that, for each $a \in S$:*

(i) $\|a \cdot F - F \cdot a\|_\pi < \epsilon$;
(ii) $\|a - a\pi_A(F)\| < \epsilon$.

The following characterization of approximate amenability is taken from [6, Proposition 2.1].

THEOREM 2.3. *Let A be a Banach algebra. Then A is approximately amenable if and only if, for each $\epsilon > 0$ and each finite subset S of A, there exist $F \in A \otimes A$ and $u, v \in A$ such that $\pi_A(F) = u + v$, and for each $a \in S$:*

(i) $\|a \cdot F - F \cdot a + u \otimes a - a \otimes v\|_\pi < \epsilon$;
(ii) $\|a - au\| < \epsilon$ *and* $\|a - va\| < \epsilon$.

3. Amenability and approximate amenability of $\ell^p((\mathfrak{A}_i), \omega)$ $(1 \le p < \infty)$

Our starting point in this section is the following definition.

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Definition 3.1. Given a set I, a family $\{\mathfrak{A}_i\}_{i\in I}$ of Banach algebras, and $\omega = (\omega_i) \in [1, +\infty)^I$, define, for $1 \le p < \infty$,

$$\ell^p((\mathfrak{A}_i), \omega) = \Big\{(\mathfrak{a}_i) : \mathfrak{a}_i \in \mathfrak{A}_i, \sum_{i\in I} \omega_i \|\mathfrak{a}_i\|^p_{\mathfrak{A}_i} < \infty\Big\}.$$

It is easy to check that $\ell^p((\mathfrak{A}_i), \omega)$ is a Banach algebra with pointwise multiplication and the norm

$$\|(\mathfrak{a}_i)\|_{p,\omega} = \Big(\sum_{i\in I} \omega_i \|\mathfrak{a}_i\|^p_{\mathfrak{A}_i}\Big)^{1/p} \quad ((\mathfrak{a}_i) \in \ell^p((\mathfrak{A}_i), \omega)).$$

The Banach algebra $\ell^p((\mathfrak{A}_i), \omega)$ is called *the weighted l^p-direct sum* of the family $(\mathfrak{A}_i)$ with *weight* ω. If for each $i \in I$, $\mathfrak{A}_i = \mathfrak{A}$, denote $\ell^p((\mathfrak{A}_i), \omega)$ by $\ell^p(I, \mathfrak{A}, \omega)$. If for each $i \in I$, $\omega_i = 1$, denote $\ell^p(I, \mathfrak{A}, \omega)$ by $\ell^p(I, \mathfrak{A})$. Also define $\ell^p(I, \omega) = \ell^p(I, \mathbb{C}, \omega)$, $\ell^p(I) = \ell^p(I, \mathbb{C})$, and $\ell^p(\omega) = \ell^p(\mathbb{N}, \omega)$.

Lemma 3.2. *Given a set I, $1 \le p < \infty$, and $\omega \in [1, +\infty)^I$, the following assertions are equivalent:*

(i) *$\ell^p(I, \omega)$ is approximately amenable;*
(ii) *$\ell^p(I, \omega)$ is amenable;*
(iii) *I is finite.*

Proof. Let I be infinite. Then there exists an infinite countable subset $I_0 = \{i_n\}_{n\in\mathbb{N}}$ of I. The mapping

$$\ell^p(I, \omega) \to \ell^p(\omega); \quad (\lambda_i) \mapsto (\lambda_{i_n})_n,$$

is a continuous epimorphism. But, by Theorem 2.1, $\ell^p(\mathbb{N}, \omega)$ is not approximately amenable. Therefore, by [7, Proposition 2.2], $\ell^p(I, \omega)$ is not approximately amenable.

Obviously, if I is finite, then $\ell^p(I, \omega)$ is amenable. □

Lemma 3.3. *Given a set I, a family $\{\mathfrak{A}_i\}_{i\in I}$ of Banach algebras with unit, and $\omega = (\omega_i) \in [1, +\infty)^I$, let $\varpi(i) = \omega_i \|e_{\mathfrak{A}_i}\|^p_{\mathfrak{A}_i}$ $(i \in I)$. Then for $1 \le p < \infty$, $\ell^p(I, \varpi)$ is a Banach algebra, the mapping*

$$\iota : \ell^1(I, \varpi) \to \ell^p((\mathfrak{A}_i), \omega); \iota(a) = (a_i e_{\mathfrak{A}_i}) \quad (a = (a_i) \in \ell^p(I, \varpi)),$$

is well defined, and there exists a linear map Θ from $\ell^p((\mathfrak{A}_i), \omega)$ into $\ell^p(I, \varpi)$ such that:

(i) *$\|\Theta\| = 1$;*
(ii) *$\Theta(\iota(a)) = a$ $(a \in \ell^p(I, \varpi))$;*
(iii) *$a\Theta(A) = \Theta(\iota(a)A)$, $\Theta(A)a = \Theta(A\iota(a))$ $(a \in \ell^p(I, \varpi), A \in \ell^p((\mathfrak{A}_i), \omega))$;*
(iv) *for $a \in \ell^p(I, \varpi)$ and $\mathcal{F} \in \ell^p((\mathfrak{A}_i), \omega)\widehat{\otimes}\ell^p((\mathfrak{A}_i), \omega)$,*

$$a \cdot (\Theta \otimes \Theta)(\mathcal{F}) = (\Theta \otimes \Theta)(\iota(a) \cdot \mathcal{F}), (\Theta \otimes \Theta)(\mathcal{F}) \cdot a = (\Theta \otimes \Theta)(\mathcal{F} \cdot \iota(a)).$$

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Proof. Since for each $i \in I$, $\|e_{\mathfrak{A}_i}\|_{\mathfrak{A}_i} \geq 1$, we have $\varpi \in [1, +\infty)^I$. Thus, $\ell^p(I, \varpi)$ is a Banach algebra. It is easy to see that ι is well defined. Let $i \in I$. By the Hahn–Banach theorem, there exists $\theta_i \in \mathfrak{A}_i{}^*$ with $\|\theta_i\| = 1$ and $\theta_i(e_{\mathfrak{A}_i}) = \|e_{\mathfrak{A}_i}\|_{\mathfrak{A}_i}$. Define

$$\Theta : \ell^p((\mathfrak{A}_i), \omega) \to \ell^p(I, \varpi); \Theta(A) = \left(\frac{1}{\|e_{\mathfrak{A}_i}\|_{\mathfrak{A}_i}} \theta_i(\mathfrak{a}_i)\right) \quad (A = (\mathfrak{a}_i) \in \ell^p((\mathfrak{A}_i), \omega)).$$

Since $\|\theta_i\| = 1$ $(i \in I)$, Θ is well defined. The equations in (i) and (ii) are direct consequences of $\|\theta_i\| = 1$ and $\theta_i(e_{\mathfrak{A}_i}) = \|e_{\mathfrak{A}_i}\|_{\mathfrak{A}_i}$ $(i \in I)$. The equations in (iii) and (iv) are proved by an easy calculation. For example, if $a = (a_i) \in \ell^p(I, \varpi)$ and $A = (\mathfrak{a}_i) \in \ell^p((\mathfrak{A}_i), \omega)$, then

$$\begin{aligned} a\Theta(A) &= (a_i)\left(\frac{1}{\|e_{\mathfrak{A}_i}\|_{\mathfrak{A}_i}} \theta_i(\mathfrak{a}_i)\right) = \left(\frac{a_i}{\|e_{\mathfrak{A}_i}\|_{\mathfrak{A}_i}} \theta_i(\mathfrak{a}_i)\right) \\ &= \left(\frac{1}{\|e_{\mathfrak{A}_i}\|_{\mathfrak{A}_i}} \theta_i(a_i\mathfrak{a}_i)\right) = \Theta((a_i\mathfrak{a}_i)) \\ &= \Theta((a_i e_{\mathfrak{A}_i})(\mathfrak{a}_i)) = \Theta(\iota(a)A). \end{aligned}$$

It follows that, for each $B, C \in \ell^p((\mathfrak{A}_i), \omega)$,

$$\begin{aligned} a \cdot (\Theta \otimes \Theta)(B \otimes C) &= (a\Theta(B)) \otimes \Theta(C) = \Theta(\iota(a)B) \otimes \Theta(C) \\ &= (\Theta \otimes \Theta)(\iota(a)B \otimes C) = (\Theta \otimes \Theta)(\iota(a) \cdot (B \otimes C)), \end{aligned}$$

and so for each $\mathcal{F} \in \ell^p((\mathfrak{A}_i), \omega)\widehat{\otimes}\ell^p((\mathfrak{A}_i), \omega)$, $a \cdot (\Theta \otimes \Theta)(\mathcal{F}) = (\Theta \otimes \Theta)(\iota(a) \cdot \mathcal{F})$. □

Given a set I and a family $\{\mathfrak{A}_i\}_{i\in I}$ of Banach algebras with unit, for the subset I_0 of I let

$$c_{00}^{I_0}((\mathfrak{A}_i)) = \{(\mathfrak{a}_i) : \mathfrak{a}_i \in \mathfrak{A}_i, \mathfrak{a}_i = 0 \text{ for } i \notin I_0\},$$

and define $E_{I_0} \in c_{00}^{I_0}((\mathfrak{A}_i))$ through $(E_{I_0})_i = e_{\mathfrak{A}_i}$ $(i \in I_0)$. These notations are used in the following lemma.

Lemma 3.4. *Given a set I, $1 \leq p < \infty$, a family $\{\mathfrak{A}_i\}_{i\in I}$ of Banach algebras with unit, and $\omega \in [1, +\infty)^I$, let $\ell^p((\mathfrak{A}_i), \omega)$ be approximately amenable, $\epsilon > 0$, and S be a finite subset of $\ell^p((\mathfrak{A}_i), \omega)$. Then there exist a finite subset I_ϵ of I, and $B^1, \ldots, B^m, C^1, \ldots, C^m, U, V \in c_{00}^{I_\epsilon}((\mathfrak{A}_i))$ such that, if $\mathcal{F} = \sum_{n=1}^m B^n \otimes C^n$, then $\pi_{\ell^p((\mathfrak{A}_i),\omega)}(\mathcal{F}) = U + V$, and moreover, for each $A \in S$:*

(i) $\|A \cdot \mathcal{F} - \mathcal{F} \cdot A + U \otimes A - A \otimes V\|_\pi < \epsilon$;
(ii) $\|A - AU\|_{p,\omega} < \epsilon$ *and* $\|A - VA\|_{p,\omega} < \epsilon$.

Proof. By Theorem 2.3, there exists $\overline{\mathcal{F}} = \sum_{n=1}^m \overline{B}_n \otimes \overline{C}_n \in \ell^p((\mathfrak{A}_i), \omega) \otimes \ell^p((\mathfrak{A}_i), \omega)$, such that $\pi_{\ell^p((\mathfrak{A}_i),\omega)}(\overline{\mathcal{F}}) = \overline{U} + \overline{V}$, and for each $A \in S$:

(i′) $\|A \cdot \overline{\mathcal{F}} - \overline{\mathcal{F}} \cdot A + \overline{U} \otimes A - A \otimes \overline{V}\|_\pi < \epsilon/2$;
(ii′) $\|A - A\overline{U}\|_{p,\omega} < \epsilon/2$ and $\|A - \overline{V}A\|_{p,\omega} < \epsilon/2$.

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Let $\epsilon_1 = \epsilon/(8\max_{A\in S}(\|A\|_{p,\omega}+1))$. By continuity of the tensor product and the definition of $\|\cdot\|_{p,\omega}$, there exists a finite subset I_ϵ of I such that

$$\Big\|\sum_{n=1}^{m}(\overline{B}_n E_{I_\epsilon})\otimes(\overline{C}_n E_{I_\epsilon}) - \sum_{n=1}^{m}\overline{B}_n\otimes\overline{C}_n\Big\|_\pi < \epsilon_1$$

and

$$\|\overline{U}E_{I_\epsilon}-\overline{U}\|_{p,\omega},\ \|\overline{V}E_{I_\epsilon}-\overline{V}\|_{p,\omega} < \epsilon_1.$$

Let $B_n = \overline{B}_n E_{I_\epsilon}$, $C_n = \overline{C}_n E_{I_\epsilon}$ $(1\le n\le m)$, $\mathcal{F} = \sum_{n=1}^m B_n\otimes C_n$, $U = \overline{U}E_{I_\epsilon}$, and $V = \overline{V}E_{I_\epsilon}$. Then (i′) and (ii′) give (i) and (ii). □

Proposition 3.5. *Given a set I, a family $\{\mathfrak{A}_i\}_{i\in I}$ of Banach algebras with unit, and $\omega = (a_i)\in[1,+\infty)^I$, if the Banach algebra $\ell^p((\mathfrak{A}_i),\omega)$ is approximately amenable, then I is finite.*

Proof. The notations of Lemmas 3.3 and 3.4 are used. Let $\epsilon > 0$ and S be a finite subset of $\ell^p(I,\varpi)$. Since $\iota(S)$ is a finite subset of $\ell^p((\mathfrak{A}_i),\omega)$, there exist by Lemma 3.4 a finite subset I_ϵ of I, and $B_1,\ldots,B^m, C^1,\ldots,C^m, U, V\in c_{00}^{I_\epsilon}((\mathfrak{A}_i))$ such that, if $\mathcal{F} = \sum_{n=1}^m B^n\otimes C^n$, then $\pi_{\ell^p((\mathfrak{A}_i),\omega)}(\mathcal{F}) = U+V$, and for each $a\in S$:

(i) $\|\iota(a)\cdot\mathcal{F} - \mathcal{F}\cdot\iota(a) + U\otimes\iota(a) - \iota(a)\otimes V\|_\pi < \epsilon$;

(ii) $\|\iota(a)-\iota(a)U\|_{p,\omega} < \epsilon$ and $\|\iota(a) - V\iota(a)\|_{p,\omega} < \epsilon$.

For $i\in I$, let Θ_i be the ith component of Θ (that is, in the notation of the proof of Lemma 3.3, $\Theta_i = (1/\|e_{\mathfrak{A}_i}\|_{\mathfrak{A}_i})\theta_i$). Let

$$\lambda_{n,i} = \Theta_i(B^n{}_iC^n{}_i) - \Theta_i(B^n{}_i)\Theta_i(C^n{}_i)\quad (1\le n\le m,\ i\in I_\epsilon)$$

and

$$F = (\Theta\otimes\Theta)(\mathcal{F}) + \sum_{n=1}^{m}\sum_{i\in I_\epsilon}\lambda_{n,i}\delta_i\otimes\delta_i,$$

where $\delta_i\in\ell^p(I,\varpi)$ is defined by $\delta_i(i)=1$ and $\delta_i(j)=0$ $(j\ne i)$. Obviously, $F\in\ell^p(I,\varpi)\otimes\ell^p(I,\varpi)$. Let $u=\Theta(U)$ and $v=\Theta(V)$. It is clear that

$$a\cdot(\delta_i\otimes\delta_i) = (\delta_i\otimes\delta_i)\cdot a\quad (a\in\ell^p(I,\varpi),\ i\in I),$$

and so by Lemma 3.3(iv), for each $a\in\ell^p(I,\varpi)$,

$$\begin{aligned} a\cdot F - F\cdot a &= a\cdot(\Theta\otimes\Theta)(\mathcal{F}) - (\Theta\otimes\Theta)(\mathcal{F})\cdot a\\ &= (\Theta\otimes\Theta)(\iota(a)\cdot\mathcal{F} - \mathcal{F}\cdot\iota(a)). \end{aligned}$$

Thus, by (i) in this proof and Lemma 3.3(ii) and (i), for each $a\in S$,

$$\begin{aligned} \|a\cdot F - F\cdot a + u\otimes a - a\otimes v\|_\pi &= \|(\Theta\otimes\Theta)(\iota(a)\cdot\mathcal{F} - \mathcal{F}\cdot\iota(a) - U\otimes\iota(a) - \iota(a)\otimes V)\|_\pi\\ &\le \|\iota(a)\cdot\mathcal{F} - \mathcal{F}\cdot\iota(a) - U\otimes\iota(a) - \iota(a)\otimes V\|_\pi < \epsilon. \end{aligned}$$

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Also, by (ii) and Lemma 3.3(i),

$$\|a - au\|_{p,\varpi} = \|\Theta(\iota(a) - \iota(a)U)\|_{p,\varpi} \le \|\iota(a) - \iota(a)U\|_{p,\omega} < \epsilon,$$

and similarly $\|a - va\|_{p,\varpi} < \epsilon$. Moreover,

$$\begin{aligned}\pi_{\ell^p(I,\varpi)}(F) &= \sum_{n=1}^{m} \Theta(B^n)\Theta(C^n) + \sum_{n=1}^{m}\sum_{i\in I_\epsilon} \lambda_{n,i}\delta_i\delta_i \\ &= \sum_{n=1}^{m}\sum_{i\in I_\epsilon} \Theta_i(B^n{}_i)\Theta_i(C^n{}_i)\delta_i + \sum_{n=1}^{m}\sum_{i\in I_\epsilon} \lambda_{n,i}\delta_i \\ &= \sum_{n=1}^{m}\sum_{i\in I_\epsilon} \Theta_i(B^n{}_i C^n{}_i)\delta_i = \sum_{n=1}^{m} \Theta(B^n C^n) \\ &= \Theta(\pi_{\ell^p((\mathfrak{A}_i),\omega)}(\mathcal{F})) = \Theta(U + V) = u + v.\end{aligned}$$

Therefore, by Theorem 2.3, $\ell^p(I, \varpi)$ is approximately amenable. Hence, by Lemma 3.2, I is finite. □

Remark 3.6. If, for each $i \in I$, $\mathfrak{A}_i$ has a nonzero character ϕ_i, then there is a simple proof for the above proposition. To see this, suppose that $\ell^p((\mathfrak{A}_i), \omega)$ is approximately amenable. Define

$$\Theta : \ell^p((\mathfrak{A}_i), \omega) \to \ell^p(I, \varpi);\ (\mathfrak{a}_i) \mapsto (\phi_i(\mathfrak{a}_i)),$$

where $\varpi_i = \omega_i/\|\phi_i\|^p$ $(i \in I)$. Note that for each $i \in I$, $\|\phi_i\| \le 1$ (see [4, Section 16]), and so $\varpi_i \ge 1$. Clearly Θ is a bounded linear map. For each $i \in I$, there is $\mathfrak{a}_i^0 \in \mathfrak{A}_i$ with $\|\mathfrak{a}_i^0\|_{\mathfrak{A}_i} = 1$, such that $|\phi_i(\mathfrak{a}_i^0)| \ge \frac{1}{2}\|\phi_i\|$. Let $a := (\lambda_i) \in \ell^p(I, \varpi)$. Then it is easy to show that if $A = ((\lambda_i/\phi_i(\mathfrak{a}_i^0))\mathfrak{a}_i^0)$, then $A \in \ell^p((\mathfrak{A}_i), \omega)$, and $\Phi(A) = a$. It follows that Φ is a continuous epimorphism. Hence, by [7, Proposition 2.2], $\ell^p(I, \varpi)$ is approximately amenable, and so by Lemma 3.2, I is finite.

Lemma 3.7. *Given a set I, a family $\{\mathfrak{A}_i\}_{i\in I}$ of Banach algebras, and $\omega \in [1, +\infty)^I$, if $1 \le p < \infty$, and $\ell^p((\mathfrak{A}_i), \omega)$ is amenable (respectively, approximately amenable), then, for each $i \in I$, $\mathfrak{A}_i$ is amenable (respectively, approximately amenable).*

Proof. For each $i \in I$, the mapping $\pi_i : \ell^p((\mathfrak{A}_i), \omega) \to \mathfrak{A}_i; (\mathfrak{a}_i) \mapsto \mathfrak{a}_i$ is a bounded algebra homomorphism. By [12, Proposition 2.3.1] (respectively, [7, Proposition 2.2]), $\mathfrak{A}_i$ is amenable (respectively, approximately amenable). □

The following result is the main theorem of the present paper.

Theorem 3.8. *Given a set I, a family $\{\mathfrak{A}_i\}_{i\in I}$ of Banach algebras with unit, and $\omega = (a_i) \in [1, +\infty)^I$, if $1 \le p < \infty$, then the following statements are equivalent.*

(i) *$\ell^p((\mathfrak{A}_i), \omega)$ is amenable (respectively, approximately amenable).*
(ii) *The set I is finite, and, for each $i \in I$, $\mathfrak{A}_i$ is amenable (respectively, approximately amenable).*

Proof. (i) $\Rightarrow$ (ii) is a consequence of Proposition 3.5 and Lemma 3.7.

(ii) $\Rightarrow$ (i) follows from [12, Corollary 2.3.19] (where, for each $i \in I$, $\mathfrak{A}_i$ is amenable), and [7, Proposition 2.7] (where, for each $i \in I$, $\mathfrak{A}_i$ is approximately amenable). □

4. Applications to compact groups and hypergroups

Let H be an n-dimensional Hilbert space and suppose that $B(H)$ is the space of all linear operators on H. For $E \in B(H)$, let $(\lambda_1, \ldots, \lambda_n)$ be the sequence of eigenvalues of the operator $|E|$, written in any order. Define $\|E\|_{\varphi_p} = (\sum_{i=1}^n |\lambda_i|^p)^{1/p}$ $(1 \le p < \infty)$. For more details, see [8, Definition D.37 and Theorem D.40].

Let I be an arbitrary index set. For each $i \in I$, let H_i be a finite-dimensional Hilbert space of dimension d_i, and let $a_i \ge 1$ be a real number. Define

$$\mathfrak{E}_p(I) = \ell^p(((B(H_i), \|\cdot\|_{\varphi_p})), (a_i)) \quad (1 \le p < \infty).$$

This definition is taken from [8, Section 28], using the notation of Definition 3.1.

By [12, Example 2.3.16], for each $i \in I$, the Banach algebra $B(H_i)$ is amenable. Hence Theorem 3.8 yields the following result.

Proposition 4.1. *Let $1 \le p < \infty$. The following statements are equivalent.*

(i) *$\mathfrak{E}_p(I)$ is approximately amenable.*
(ii) *$\mathfrak{E}_p(I)$ is amenable.*
(iii) *I is finite.*

Let K be a compact hypergroup (as defined by Jewett [9]), and $\widehat{K}$ be the set of equivalence classes of continuous irreducible representations of K (see [3], [9, Section 11.3], and [13]). For each $\pi \in \widehat{K}$, let H_π be the representation space of π and $d_\pi = \dim\ H_\pi$. By [13, Theorem 2.2], $d_\pi < \infty$. Furthermore, by the proof of [13, Theorem 2.2], there exists a constant c_π such that for each $\xi \in H_\pi$ with $\|\xi\| = 1$,

$$\int_K |\langle \pi(x)\xi, \xi\rangle|^2 \, d\omega_K(x) = c_\pi.$$

Let $k_\pi = c_\pi^{-1}$. By [13, Theorem 2.6], $k_\pi \ge d_\pi$. Moreover, if K is a group, then $k_\pi = d_\pi$. The Banach algebras $\mathfrak{E}_p(\widehat{K})$, for $p \in [1, \infty)$, are defined with each $a_\pi = k_\pi$.

Proposition 4.2. *Let K be a compact hypergroup, and $1 \le p < \infty$. The following statements are equivalent.*

(i) *$\mathfrak{E}_p(\widehat{K})$ is approximately amenable.*
(ii) *$\mathfrak{E}_p(\widehat{K})$ is amenable.*
(iii) *K is finite.*

Proof. If $\widehat{K}$ is finite, then $\mathfrak{E}_2(\widehat{K})$ is finite-dimensional. So by [13, Theorem 3.4], $L^2(K)$ is finite-dimensional, and so is $C(K)$. From the comment on [11, p. 57] it follows that K is finite. By Proposition 4.1, the proof is complete. □

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If K is a compact hypergroup, then by [3, Theorem 1.3.28], K admits a left Haar measure. Throughout the present paper we use the normalized Haar measure ω_K on the compact hypergroup K (that is, $\omega_K(K) = 1$). Note that by [13, Theorem 3.4], the convolution Banach algebra $L^2(K)$ is isometrically algebra isomorphic with $\mathfrak{C}_2(\widehat{K})$. Thus the following result is a corollary of the above proposition.

Corollary 4.3. *Let K be a compact hypergroup. The following statements are equivalent.*

(i) *The convolution Banach algebra $L^2(K)$ is approximately amenable.*
(ii) *The convolution Banach algebra $L^2(K)$ is amenable.*
(iii) *K is finite.*

As a further corollary, the following generalization of [1, Proposition 2.30] (see also [2]) is obtained.

Corollary 4.4. *Let G be an infinite compact group. Then the convolution Banach algebra $L^2(G)$ is not approximately amenable.*

If $f \in L^1(K)$ and $\sum_{\pi\in\widehat{K}} k_\pi \|\widehat{f}(\pi)\|_{\varphi_1} < \infty$ (where $\widehat{f} \in \mathfrak{C}(\widehat{K})$ is the *Fourier transform* of f, defined by $\widehat{f_\pi} = \int_K f(x)\pi(\bar{x})\, d\omega_K(x)$ $(\pi \in \widehat{K})$), we say that f has an *absolutely convergent Fourier series*. The set of all functions with absolutely convergent Fourier series is denoted by $A(K)$ and called *the Fourier space* of K. For $f \in A(K)$ we define $\|f\|_{A(K)} = \|\widehat{f}\|_1$. By [13, Proposition 4.2], $A(K)$ with the convolution product is a Banach algebra and isometrically isomorphic with $\mathfrak{C}_1(\widehat{K})$. See also [8] for further results about compact groups. Proposition 4.1 yields the following result.

Corollary 4.5. *Let K be a compact hypergroup. The following statements are equivalent.*

(i) *The convolution Banach algebra $A(K)$ is approximately amenable.*
(ii) *The convolution Banach algebra $A(K)$ is amenable.*
(iii) *K is finite.*

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