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ASYMPTOTIC BEHAVIOUR OF NON-AUTONOMOUS DISSIPATIVE SYSTEMS IN HILBERT SPACE

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Abstract

In this paper we discuss the asymptotic behaviour, as $t \to \infty$, of the integral solution $u(t)$ of the non-linear evolution equation $u'(t) \in A(t)u(t) + g(t)$, $t \geq s$, $u(s) = x_0 \in \overline{D(A(s))}$, where $\{A(t)\}_{t\geq 0}$ is a family of m-dissipative operators in a Hilbert space H, and $g \in L_{loc}(0, \infty; H)$. We give some sufficient conditions and some sufficient and necessary conditions to ensure that $\sigma(t) = t^{-1}\int_s^{s+t} u(\theta)\,d\theta$ and $u(t)$ are weakly convergent.

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1. Introduction and preliminaries

Let H be a real Hilbert space with inner product $(,)$ and norm $|,|$. We consider the non-linear evolution equation

$$\begin{cases} u'(t) \in A(t)u(t) + g(t), & t \geq s \\ u(s) = x_0 \end{cases} \tag{1.1}$$

where $\{A(t)\}_{t\geq 0}$ is a family of m-dissipative operators in H, $x_0 \in \overline{D(A(s))}$ and $g \in L_{loc}(0, \infty; H)$. Our objective is to study the asymptotic behaviour, as $t \to \infty$, of the integral solution $u(t)$ of (1.1). In [6,7,9] the weak convergence of the autonomous dissipative system

$$\begin{cases} u'(t) \in Au(t) \\ u(0) = x_0 \end{cases}$$

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where A is an m-dissipative operator in H, $x_0 \in \overline{D(A)}$, has been studied. In [4, 10] Morosanu and Rouhani discussed the weak convergence of the quasi-autonomous dissipative system

$$\begin{cases} u'(t) \in Au(t) + g(t) \\ u(0) = x_0 \end{cases}$$

where $g \in L(0, \infty; H)$ (or more generally, $g - g_\infty \in L(0, \infty; H)$ for some $g_\infty \in H$).

Throughout this paper we assume that $A(t)$ satisfies the following conditions:

(H_1): there exists a continuous function $f : R_+ \to H$ and a bounded (on bounded subsets) function $L : R_+ \to R_+$ such that

$$(1.2) \qquad (y_1 - y_2, x_1 - x_2) \le |f(t) - f(s)| \cdot |x_1 - x_2| \cdot L(|x_2|)$$

for all $0 \le s \le t$, $[x_1, y_1] \in A(t)$, $[x_2, y_2] \in A(s)$.

(H_2): If $t_n \uparrow t$ in $[s, +\infty]$, $x_n \in D(A(t_n))$, $x_n \to x$ in H, then $x \in \overline{D(A(t))}$.

DEFINITION 1.1. If $u(t)$ is continuous on $[s, \infty)$, $u(s) = x_0$, $u(t) \in \overline{D(A(t))}$ for $t \in [s, \infty)$ and satisfies the inequality

$$(1.3) \qquad |u(\bar{t}) - x| \le |u(t) - x| + \int_t^{\bar{t}} (\langle y + g(\theta), u(\theta) - x\rangle_+ + c|f(\theta - f(r)|)\, d\theta$$

for all $s \le t \le \bar{t}$, $r \ge s$ and $[x, y] \in A(r)$. Then $u(t)$ is called an *integral solution* to (1.1). Here $c = L(|x|)$, $\langle y, x\rangle_+ = \lim_{h\downarrow 0}(|x + hy| - |x|)/h$ and $(y, x) = |x|\langle y, x\rangle_+$.

Clearly, a strong solution $u(t)$ to (1.1) is automatically an integral solution to (1.1), and by [5] the problem (1.1) has a unique integral solution under our hypotheses, and the inequality (1.3) is equivalent to

$$(1.4) \qquad \frac{1}{2}(|u(\bar{t}) - x|^2 - |u(t) - x|^2) \le \int_t^{\bar{t}} (g(\theta) + y, u(\theta) - x)\, d\theta + L(|x|) \int_t^{\bar{t}} |u(\theta) - x| \cdot |f(\theta) - f(r)|\, d\theta$$

for all

$$s \le t \le \bar{t}, \quad r \ge s, \quad [x, y] \in A(r).$$

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2. Weak convergence of the integral solution

LEMMA 2.1. *Suppose F is a non-empty closed convex set in H, P_F is a projection on F. Then*

(2.1) $(x - P_F x, z - P_F x) \le 0, \quad \forall z \in F,\ x \in H.$

(2.2) $|P_F x - P_F y| \le |x - y| \quad \forall x, y \in H.$

(2.3) $|P_F x - z|^2 \le |x - z|^2 - |P_F x - x|^2, \quad \forall x \in H, z \in F.$

Since Lemma 2.1 is well known, its proof will be omitted.

LEMMA 2.2. *Suppose $u(t)$ is an integral solution to* (1.1). *If there are $r_0 \ge s$ and $g_\infty \in H$ such that $f - f(r_0) \in L(0, \infty; H)$ and $g - g_\infty \in L(0, \infty; H)$, then $u(t)$ is bounded on $[s, \infty)$ if and only if $A^{-1}(r_0)(-g_\infty) \ne \emptyset$.*

PROOF. Firstly, we suppose that $u(t)$ is bounded on $[s, \infty)$. Since $u(t)$ is an integral solution of (1.1), then for all $t \ge s \ge 0$ and $[x, y] \in A(r_0)$ we have

$$\frac{1}{2}(|u(t) - x)|^2 - |u_0 - x|^2) \le \int_s^t (g(\theta) + y, u(\theta) - x)\, d\theta + L(|x|) \int_s^t |u(\theta) - x| \cdot |f(\theta) - f(r_0)|\, d\theta. \tag{2.4}$$

Dividing by $t - s > 0$, we obtain

$$\frac{1}{2(t-s)}(|u(t) - x|^2 - |u_0 - x|^2) \le \frac{1}{t-s} \int_s^t (g(\theta) - g_\infty, u(\theta) - x)\, d\theta + (y + g_\infty, \sigma(t) - x) + \frac{L(|x|)}{t-s} \int_s^t |u(\theta) - x| \cdot |f(\theta) - f(r_0)|\, d\theta \tag{2.5}$$

for all $t > s \ge 0$, $[x, y] \in A(r_0)$, where $\sigma(t) = (t-s)^{-1} \int_s^t u(\theta)\, d\theta$ is bounded on $[s, \infty)$. Therefore there exists a sequence $t_n \to \infty$ such that $\sigma(t_n)$ converges weakly to $p \in H$. If we take $t = t_n$ in (2.5) and let $n \to \infty$, then

$$(y + g_\infty, x - p) \le 0 \quad \text{for all } [x, y] \in A(r_0). \tag{2.6}$$

The maximality of $A(r_0)$ implies that $[p, -g_\infty] \in A(r_0)$, that is, $A^{-1}(r_0)(-g_\infty)$ is non-empty.

Conversely, if $A^{-1}(r_0)(-g_\infty) \ne \emptyset$, then there exists an element $x \in D(A(r_0))$ such that $-g_\infty \in A(r_0)x$. We take $y = -g_\infty$ in (2.4) and by a variant of Gronwall's Lemma (see [2, p. 157]) we deduce that $u(t)$ is bounded on $[s, \infty)$. The proof is complete.

THEOREM 2.3. *Suppose $u(t)$ is an integral solution to (1.1). If there are f_∞, $g_\infty \in H$ and $\tau_n \to \infty$ such that $f(\tau_n) = f_\infty$ $(n \in N)$, $F = \bigcap_{n=1}^{\infty} A^{-1}(\tau_n)(-g_\infty) \neq \emptyset$, then there is $p \in F$ such that*

$$P_F u(t) \xrightarrow{s} p \quad \text{and} \quad \sigma(t) \xrightarrow{w} p \ (t \to \infty).$$

PROOF. We may assume $g_\infty = 0$, $F = \bigcap_{n=1}^{\infty} A^{-1}(\tau_n)(0)$ (without loss of generality). Since $A^{-1}(t)$ is maximal dissipative, then F is a closed convex subset in H. Take $x \in F$, $r = \tau_n$. By the 'if' part of Lemma 2.2, $u(t)$ is bounded on $[s, \infty)$ and for all $x \in F$, $\bar{t} \geq t \geq s \geq 0$

$$|u(\bar{t}) - x| - |u(t) - x| \leq \int_s^{\bar{t}} (|g(\theta)| + L(|x|)|f(\theta) - f_\infty|)\, d\theta. \tag{2.7}$$

Hence, for every $x \in F$, the function $t \to |u(t) - x| - \int_0^t (g(\theta) - L(|x|)|f(\theta) - f_\infty|)\, d\theta$ is non-increasing and bounded on $[s, \infty)$. Since g, $f - f_\infty \in L(0, \infty; H)$ we conclude that there exists a limit

$$\lim_{t\to\infty} |u(t) - x| = \alpha(x) \quad \text{for every } x \in F. \tag{2.8}$$

We set $v(t) = P_F u(t)$. According to Lemma 2.1 (ii), $v(t)$ is bounded on $[s, \infty)$. Let $C_1 = \sup_{t \geq s} L(|v(t)|)$; for fixed $t \geq s$ we denote $y_t(h) = u(t+h)$, $h \geq 0$. Then $y_t(h)$ is an integral solution of the following equation:

$$\begin{cases} \dfrac{dy_t(h)}{dh} \in A(t+h)y_t(h) + g(t+h) \\ y_t(0) = u(t). \end{cases} \tag{2.9}$$

By the same argument above we obtain the function

$$h \to |y_t(h) - v(t)| - \int_0^h (|g(\theta + t)| + C_1|f(\theta + t) - f_\infty|)\, d\theta$$

is non-increasing. Hence $\forall t \geq s$, $h \geq 0$,

$$|u(t+h) - v(t)| - \int_t^{t+h} (|g(\theta)| + C_1|f(\theta) - f_\infty|)\, d\theta \leq |u(t) - v(t)|. \tag{2.10}$$

This implies that for all $t \geq s$, $h \geq 0$,

$$\begin{aligned} |u(t+h) - v(t+h)| &- \int_s^{t+h} (|g(\theta)| + C_1|f(\theta) - f_\infty|)\, d\theta \\ &\leq |u(t+h) - v(t)| - \int_s^{t} (|g(\theta)| + C_1|f(\theta) - f_\infty|)\, d\theta \\ &\quad - \int_t^{t+h} (|g(\theta)| + C_1|f(\theta) - f_\infty|)\, d\theta \\ &\leq |u(t) - v(t)| - \int_s^{t} (|g(\theta)| + C_1|f(\theta) - f_\infty|)\, d\theta. \end{aligned}$$

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Thus the function $t \to |u(t)-v(t)| - \int_s^t (|g(\theta)| + C_1|f(\theta) - f_\infty|)\,d\theta$ is non-increasing on $[s, +\infty)$ and there exists $\lim_{t\to\infty} |u(t) - v(t)|$.

Next, by Lemma 2.1 (iii)

$$(2.11)\qquad |v(t+h) - v(t)|^2 \le |u(t+h) - v(t)|^2 - |v(t+h) - u(t+h)|^2.$$

From (2.10) and (2.11) one obtains

$$\begin{aligned}|v(t+h) - v(t)|^2 \le |u(t) - v(t)|^2 &- |u(t+h) - v(t+h)|^2 \\ &+ 2|u(t) - v(t)| \cdot \int_t^{t+h} (|g(\theta)| + C_1|f(\theta) - f_\infty|)\,d\theta \\ &+ \left[\int_t^{t+h} (|g(\theta)| + C_1|f(\theta) - f_\infty|)\,d\theta\right]^2.\end{aligned}$$

This implies that there exists $\lim_{t\to\infty} v(t) = p$ and $p \in F$.

Now suppose $\sigma(t_k) \xrightarrow{w} y$ $(t_k \to \infty)$. By the 'only if' part of Lemma 2.2 for every $n \in \mathcal{N}$ we have $y \in F_n = A^{-1}(\tau_n)(-g_\infty)$; thus $y \in F$. According to Lemma 2.1 (i) we have

$$(u(t) - v(t), z - v(t)) \le 0, \quad \forall z \in F,$$

$$(2.12)\qquad \frac{1}{t_k}\int_s^{t_k+s} (u(\theta) - v(\theta), z - v(\theta))\,d\theta \le 0, \quad \forall z \in F.$$

Letting $t_k \to \infty$ in (2.12), one obtains

$$(y - p, z - p) \le 0, \quad \forall z \in F.$$

This implies that $y = p$ and $\sigma(t) \xrightarrow{w} p$ $(t \to \infty)$. The proof is complete.

REMARK 2.4. If $A(t) \equiv A$, $s = 0$ and $F = A^{-1}(-g_\infty) \neq \emptyset$, then from Theorem 2.3 we may obtain respectively the Ergodic Theorem of autonomous systems and quasi-autonomous dissipative systems in [4, 10, 5].

LEMMA 2.5. *Suppose $u(t)$ is an integral solution to* (1.1). *Then for all $T > 0$, $h > 0$, $r \ge \tau \ge s \ge 0$ and $r + h \le T$, we have*

$$(2.13)\qquad \begin{aligned}|u(r+h) - u(\tau+h)| \le{}& |u(r) - u(t)| \\ &+ \int_\tau^{\tau+h} (C_2|f(\theta + (r-\tau)) - f(\theta)| + |g(\theta + (r-\tau)) - g(\theta)|)\,d\theta\end{aligned}$$

where $C_2 = \sup\{L(t) : 0 \le t \le \sup\{|u(\theta)| : s \le \theta \le T + (r-\tau)\} + 1\}$.

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PROOF. From Theorem 1 (ii) in [11] we get

$$|u(t+h_1)-u(t)| \le |u(\tau+h_1)-u(\tau)| + \int_\tau^t (\tilde{C}|f(\theta+h_1)-f(\theta)| + |g(\theta+h_1)-g(\theta)|)\,d\theta \tag{2.14}$$

for all $s \le \tau \le t \le T$ and $h_1 > 0$, where $\tilde{C} = \sup\{L(t) : 0 \le t \le \sup\{|u(\theta+h_1)| : s \le \theta \le T\} + 1\}$.

For $h > 0,\ r \ge \tau \ge s$, let $t = \tau + h$ and $r - \tau = h_1$ in (2.14). One obtains (2.13). The proof is complete.

THEOREM 2.6. *Suppose $u(t)$ is an integral solution to* (1.1). *If there are $r_0 \ge s$ and $g_\infty \in H$ such that $-g_\infty \in R(A(r_0))$, $f - f(r_0) \in L(0,\infty; H)$ and $g - g_\infty \in L(0,\infty; H)$, then there exists $p \in A^{-1}(r_0)(-g_\infty)$ such that w-$\lim_{t\to\infty}\sigma(t) = p$.*

PROOF. Firstly, by Lemma 2.2, $\sup_{t\ge s}|u(t)| = M < \infty$. We set $\epsilon_1(r,\tau) =$

$$\begin{cases} \int_\tau^\infty M|f(\theta+(r-\tau))-f(\theta)|\,d\theta + \int_\tau^\infty |g(\theta+(r-\tau))-g(\theta)|\,d\theta, & r \ge \tau, \\ \int_r^\infty M|f(\theta+(r-\tau))-f(\theta)|\,d\theta + \int_r^\infty |g(\theta+(r-\tau))-g(\theta)|\,d\theta, & r < \tau. \end{cases}$$

Then $\lim_{r,\tau\to\infty}\epsilon_1(r,\tau) = 0$. By Lemma 2.5 and Definition 3.1 in [10] we know that the curve $(u(t))_{t\ge s}$ is almost non-expansive in H. Hence by Theorem 3.8 in [10] and the 'only if' part of Lemma 2.2 there exists w-$\lim_{t\to\infty}\sigma(t) = p$ and $p \in A^{-1}(r_0)(-g_\infty)$.

COROLLARY 2.7. *Suppose $u(t)$ is an integral solution to* (1.1). *If there are $f_\infty, g_\infty \in H$ and $T > 0$ such that $f - f_\infty \in L(0,\infty; H)$, $g - g_\infty \in L(0,\infty; H)$ and $F = \bigcap_{t\ge T} A^{-1}(t)(-g_\infty) \ne \emptyset$, then $\sigma(t)$ is weakly convergent as $t \to \infty$.*

THEOREM 2.8. *Suppose $u(t)$ is an integral solution to* (1.1). *If there exist $r_0 \ge s$ and $g_\infty \in H$ such that $f - f(r_0) \in L(0,\infty; H)$ and $g - g_\infty \in L(0,\infty; H)$, then there exists w-$\lim_{t\to\infty} u(t)$ if and only if $F = A^{-1}(r_0)(-g_\infty) \ne \emptyset$ and $\omega_w(x_0) \subset F$, where $\omega_w(x_0)$ is the set of weak cluster points of $\{u(t) : t \ge s\}$.*

PROOF. 'Only if' part: Suppose w-$\lim_{t\to\infty}u(t) = p$. This implies that w-$\lim_{t\to\infty}\sigma(t) = p$. From (2.5) it follows that $p \in F$.

'If' part: Since $F \ne \emptyset$ and $\omega_w(x_0) \subset F$, according to Lemma 2.2, $\omega_w(x_0) \ne \emptyset$. Let p, q be arbitrary in $\omega_w(x_0) \subset F$. We have

$$|u(t)-p|^2 = |u(t)-q|^2 + 2(u(t)-q, q-p) + |q-p|^2, \quad t \ge s. \tag{2.15}$$

Now for all $\bar{t} \geq t \geq s,\ x \in F$ we have

$$|u(\bar{t}) - x| - |u(t) - x| \leq \int_t^{\bar{t}} (|g(\theta) - g_\infty| + L(|x|)|f(\theta) - f(r_0)|)\, d\theta.$$

Thus the function

$$t \to |u(t) - x| + \int_0^t (|g(\theta) - g_\infty| + L(|x|)|f(\theta) - f(r_0)|)\, d\theta$$

is non-increasing on $[s, \infty)$ and there exists $\lim_{t\to\infty} |u(t)-x| = \alpha(x)$. Now $p, q \in F$; then from (2.15) we get

$$\alpha^2(p) - \alpha^2(q) = |q - p|^2 \tag{2.16}$$

and

$$\alpha^2(q) - \alpha^2(p) = |p - q|^2. \tag{2.17}$$

Hence $p = q$, $\omega_w(x_0)$ contains only one element and $w\text{-}\lim_{t\to\infty} u(t) = p$. The proof is complete.

LEMMA 2.9. *Suppose $u(t)$ is an integral solution to* (1.1) *with $g(t) \equiv 0$ and $x_0 = x$ and F is a closed subset of H. If $\omega_w(x) \subset F$ for all $x \in D(A(s))$ then $\omega_w(x) \subset F$ for all $x \in \overline{D(A(s))}$.*

PROOF. Let $x \in \overline{D(A(s))}$ and let $x_n \to x$ with $x_n \in D(A(s))$. If $y \in \omega_w(x)$ then there exists a sequence $t_k \to \infty$ such that $u(t_k) = U(t_k, s)x \overset{w}{\to} y$, where $U(t, s)$ is an evolution operator generated by $A(t)$. For every fixed n the sequence $|U(t_k, s)x_n|$ is bounded and therefore $U(t_k, s)x_n$ has a weakly convergent subsequence $U(t_{k_j}, s)x_n \overset{w}{\to} y_n$. Clearly $y_n \in \omega_w(x_n) \subset F$ and

$$|y_n - y| \leq \varliminf_{j\to\infty} |U(t_{k_j}, s)x_n - U(t_{k_j}, s)x| \leq |x_n - x|.$$

Thus $y_n \to y$ and $y \in F$. The proof is complete.

THEOREM 2.10. *Suppose $u(t)$ is an integral solution to* (1.1) *with $g(t) \equiv 0$ and $x_0 = x$, the function $f(t)$ in the condition (H_1) is of bounded variation on $[s, T]$ and $\bigvee_s^T(f) = M_T \leq M_0 < \infty$ for all $T > s$. If there exist $T_0 > s$ and $f_\infty \in H$ such that $F = \bigcap_{t\geq T_0} A^{-1}(t)(0) \neq \emptyset$, $f - f_\infty \in L(0, \infty; H)$ and satisfying the condition*

(H_3): *There exists $x_0 \in F$ such that $x_n \overset{w}{\to} x$, $y_n \in A(t_n)x_n$ $(t_n \to \infty)$ and $\lim_{n\to\infty}(y_n, x_n - x_0) = 0$ imply $x \in F$.*

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Then $u(t) = U(t,s)x$ is weakly convergent as $t \to \infty$.

PROOF. Since F is a closed convex subset of H, by Lemma 2.9 it is sufficient to prove that $\omega_w(x) \subset F$ for every $x \in D(A(s))$. Let $x \in D(A(s))$ and $y \in \omega_w(x)$ be such that $u(t_k) = U(t_k, s)x \xrightarrow{w} y$ ($t_k - t_{k-1} > 1, t_k \to \infty$). Set

$$\check{D}(A(s)) = \{x \in \overline{D(A(s))} : L(s,x) = \varliminf_{h\to 0^+} h^{-1}|U(h+s,s)x - x| < \infty\}.$$

Then $D(A(s)) \subset \check{D}(A(s)) \subset \overline{D(A(s))}$ and for $x \in D(A(s))$ we have

$$|U(\bar{t}+s,s)x - U(t+s,s)x| \le \omega^{-1}(e^{\omega\bar{t}} - e^{\omega t})[L(s,x) + M_T], \quad \forall\omega > 0 \tag{2.18}$$

(see [5, p. 25]). Since for $x \in D(A(s))$, $u(t) = U(t,s)x$ is a strong solution to (1.1) with $g(t) \equiv 0$ and $x_0 = x$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u(t) - x_0|^2 = (u'(t), u(t) - x_0), \quad \text{a.e. } t \ge s.$$

Analogously to Theorem 2.8 we can prove that there exists $\lim_{t\to\infty}|u(t) - x_0|$ for $x_0 \in F$. Thus $h(t) = (u'(t), u(t) - x_0) \in L(s, +\infty)$. We shall now prove that there exists a sequence τ_j such that $\tau_j \to \infty$, $h(\tau_j) \to 0$ and $U(\tau_j, s)x \xrightarrow{w} y$. For every $\epsilon > 0$ ($\epsilon < 1/2$) let $Q_\epsilon = \{t \ge s : h(t) \ge \epsilon\}$. The measure of Q_ϵ is finite since $h(t) \in L(s, +\infty)$ and therefore Q_ϵ can contain at most a finite number of the intervals $(t_k - \epsilon, t_k)$. It follows that there exists a τ large enough such that $h(\tau) < \epsilon$ and $0 < t_k - \tau < \epsilon$ for some t_k large enough. Therefore, we can choose a sequence τ_j such that $\tau_j \to \infty$, $0 < t_{k_j} - \tau_j < 1/j$ and $h(\tau_j) < 1/j$. By (2.18) we have

$$|U(t_{k_j}, s)x - U(\tau_j, s)x| \le \frac{1}{j}(L(s,x) + M_0),$$
$$u(\tau_j) = U(\tau_j, s)x \xrightarrow{w} y.$$

Since $u'(\tau_j) \in A(\tau_j)u(\tau_j)$, $\lim_{j\to\infty} h(\tau_j) = (u'(\tau_j), u(\tau_j) - x_0) = 0$. By the condition (H_3) one obtains $y \in F$. The proof is complete.

Next, we shall consider the quasi-autonomous dissipative system

$$\begin{cases} u'(t) \in Au(t) + f(t), & t > 0 \\ u(0) = x, & x \in \overline{D(A)} \end{cases} \tag{2.19}$$

where A is an m-dissipative operator and $f \in L(0, \infty; H)$.

DEFINITION 2.11. A dissipative set A is 3-*dissipative* if $\forall u_1, u_2, u_3 \in D(A)$

$$(Au_1, u_1 - u_2) + (Au_2, u_2 - u_3) + (Au_3, u_3 - u_1) \le 0. \tag{2.20}$$

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THEOREM 2.12. *Suppose $u(t)$ is an integral solution to* (2.19), $F = A^{-1}(0) \neq \emptyset$. *If A is 3-dissipative, then for every $x \in \overline{D(A)}$, $\omega_w(x) \subset F$ and $u(t)$ is weakly convergent as $t \to \infty$*

PROOF. Firstly, suppose $f(t)$ is of continuous bounded variation on $[0, T]$, $\bigvee_0^T(f) = M_T \leq M_0 < \infty$ $(\forall T > 0)$, and there exists $T_0 > s$ such that $f(t) = 0$ for $t \geq T_0$. Set $A(t)x = Ax + f(t)$ for all $x \in D(A)$ and $t \geq 0$. Then the equation (2.19) is equivalent to the evolution equation

$$(2.21) \qquad \begin{cases} u'(t) \in A(t)u(t), & t > 0 \\ u(0) = x, & x \in \overline{D(A(0))} = \overline{D(A)} \end{cases}$$

where $A(t)$ is m-dissipative, $F = \bigcap_{t \geq T_0} A^{-1}(t)(0) = A^{-1}(0) \neq \emptyset$ and satisfies the conditions (H_1) and (H_2). Take $x_0 \in F$, let $x_n \xrightarrow{w} x$, $y_n \in A(t_n)x_n = Ax_n + f(t_n)$ $(t_n \to \infty)$ and $(y_n, x_n - x_0) \to 0$. By Definition 2.11, for $u \in D(A)$ and $v \in Au$ we have

$$\begin{aligned} 0 &\geq (Ax_n, x_n - x_0) + (A^0x_0, x_0 - u) + (v, u - x_n) \\ &= (y_n, x_n - x_0) + (A^0x_0, x_0 - u) + (v, u - x_n) - (f(t_n), x_n - x_0). \end{aligned}$$

Letting $n \to \infty$, one obtains $(v, u - x) \leq 0$, $\forall [u, v] \in A$.

Thus $x \in F$ and the condition (H_3) is valid. By Theorem 2.10, for every $x \in \overline{D(A)}$, $\omega_w(x) \subset F$ and there is $w\text{-}\lim_{t\to\infty} u(t)$.

For $f \in L(0, \infty; H)$ there exists $f_n \in C_0^\infty(0, \infty; H)$ such that $f_n \to f$ (in $L(0, \infty; H)$). If $u_n(t)$ is an integral solution of an initial value problem

$$(2.22) \qquad \begin{cases} u_n'(t) \in Au_n(t) + f_n(t), & t > 0 \\ u_n(0) = x, & x \in \overline{D(A)} \end{cases}$$

then clearly, there exists $s - \lim_{t\to\infty} u_n(t) = u(t)$ and the limit is uniformly convergent on $t \geq 0$. Moreover, by the proof above, for every n there exists $w\text{-}\lim_{t\to\infty} u_n(t) = p_n$. This implies that there exist $s\text{-}\lim_{t\to\infty} p_n = p$ and $w\text{-}\lim_{t\to\infty} u(t) = p$. The proof is complete.

REMARK 2.13. If $f(t) \equiv 0$ in Theorem 2.12, then for every $x \in \overline{D(A)}$ there exists $w\text{-}\lim_{t\to\infty} S(t)x$, where $S(t)$ is a non-linear contraction semigroup generated by A. This implies the conclusion of Proposition 2.14 in [7].

Let $-A = \partial\varphi$ be the subdifferential of an l.s.c proper convex function. Then A is a maximal 3-dissipative operator. Hence we get the conclusion of Theorem 2.3 in [4].

COROLLARY 2.14. *Let $-A = \partial\varphi$ be the subdifferential of an l.s.c proper convex function, $f \in L(0, \infty; H)$ and $u(t)$ be an integral solution to* (2.19). *If $A^{-1}(0) \neq \emptyset$, then for every $x \in \overline{D(A)}$ there exists $w\text{-}\lim_{t\to\infty} u(t)$.*

3. Examples

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $H = L^2(\Omega)$. Let $\beta \subset \mathbb{R}^1 \times \mathbb{R}^1$ be a maximal monotone and $0 \in D(\beta)$. Then there exists an l.s.c proper convex function $j : \mathbb{R}^1 \to (-\infty, +\infty]$ such that $\beta = \partial j$.

EXAMPLE 3.1. Consider the equation

$$\text{(3.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} \in \Delta u - \beta(u(t,x)) + f(t,x), & t > 0, \text{ a.e. } x \in \Omega \\ u(t,x) = 0, & x \in \partial\Omega,\ t \geq 0 \\ u(0,x) = u_0(x), & \text{a.e. } x \in \Omega. \end{cases}$$

Assume $0 \in R(\beta)$. For example

$$\beta(x) = \begin{cases} [-e^{-1}, e^{-1}] & \text{if } x = 0 \\ e^{-1}(1+x) & \text{if } x > 0 \\ e^{-1}(x-1) & \text{if } x < 0. \end{cases}$$

Then $\beta \subset \mathbb{R}^1 \times \mathbb{R}^1$ is maximal monotone and $0 \in \beta(0)$. We set

$$\varphi(u) = \begin{cases} 2^{-1} \int_\Omega |\operatorname{grad} u|^2 dx + \int_\Omega j(u)\,dx, & u \in H_0^1(\Omega),\ j(u) \in L(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\varphi : H \to (-\infty, +\infty]$ is an l.s.c proper convex function. The subdifferential

$$\partial\varphi(u) = \{v \in L^2(\Omega) : v(x) \in \beta(u(x)) - \Delta u(x),\ \text{a.e. } x \in \Omega\}$$

and $\partial\varphi^{-1}(0) \neq \emptyset$. If $u_0 \in L^2(\Omega)$ and $f(t,x) \in L(0,\infty; H)$, by Corollary 2.14 the integral solution $u(t)$ of the problem (3.1) is weakly convergent as $t \to \infty$ in $L^2(\Omega)$.

EXAMPLE 3.2. Let

$$\beta(t)x = \begin{cases} [-e^{-1}, e^{-1}], & \text{if } x = 0 \\ e^{-1}(1+x) + e^{-t}x, & \text{if } x > 0 \\ e^{-1}(x-1) + e^{-t}x, & \text{if } x < 0. \end{cases} \qquad \text{for } t \geq 0$$

Then $\beta(t)$ is a maximal monotone set in $\mathbb{R}^1 \times \mathbb{R}^1$ for each $t \geq 0$, $0 \in D(\beta(t))$, $0 \in \beta(t)(0)$ for $t \geq 0$ and $D(\beta(t)) = \mathbb{R}^1$ is independent of t. We consider the equation

$$\text{(3.2)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} \in \Delta - \tilde{\beta}(t)u + g(t,x), & t \geq 0, \text{ a.e. } x \in \Omega \\ u(t,x) = 0 & x \in \partial\Omega,\ t \geq 0 \\ u(0,x) = u_0(x), & \text{a.e. } x \in \Omega \end{cases}$$

where $g(t,x) \in L(0,\infty;H)$ and $\tilde{\beta}(t) = \{[u,v] : u,v \in L^2(\Omega)$ and $v(x) \in \beta(t)u(x)$, a.e. $x \in \Omega\}$. Let

$$D(A(t)) = H^2(\Omega) \cap H_0^1(\Omega) \cap D(\tilde{\beta}(t)), \quad t \geq 0$$

and

$$A(t)u = \Delta u - \tilde{\beta}(t)u \quad \text{for } u \in D(A(t)).$$

Clearly, each $A(t)$ is m-dissipative in H, $D(A(t)) = \mathscr{D}$ is independent of t and $0 \in A^{-1}(t)(0)$ for all $t \geq 0$. Hence (H_2) is satisfied. Further we can prove that $A(t)$ satisfies the condition (H_1) and the conditions in Corollary 2.7 are valid. By Corollary 2.7, if $u_0(x) \in L^2(\Omega)$ and $u(t)$ is an integral solution of the problem (3.2), then $\sigma(t)$ is weakly convergent as $t \to \infty$ in $L^2(\Omega)$.

References

[1] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces* (Nordhoff, Groningen, 1976).
[2] H. Brésis, *Operateurs maximaux monotones* (North-Holland, Amsterdam, 1973).
[3] M. M. Israel Jr and S. Reich, 'Asymptotic behavior of solutions of a nonlinear evolution equation', *J. Math. Anal. Appl.* **83** (1981), 43–53.
[4] G. Morosanu, 'Asymptotic behaviour of solutions of differential equations associated to monotone operators', *Nonlinear Anal.* **3** (1979), 873–883.
[5] N. H. Pavel, *Nonlinear evolution operators and semigroups*, Lecture Notes in Math. 1260 (Springer, Berlin, 1987).
[6] A. Pazy, 'Strong convergence of semigroups of nonlinear contractions in Hilbert space', *J. Analyse Math.* **34** (1978), 1–35.
[7] ——, 'On the asymptotic behaviour of semigroups of nonlinear contractions in Hilbert space', *J. Funct. Anal.* **27** (1978), 292–307.
[8] S. Reich, 'Nonlinear evolution equations and nonlinear ergodic theorems', *Nonlinear Anal.* **1** (1977), 319–330.
[9] ——, 'Almost convergence and nonlinear ergodic theorems', *J. Approx. Theory* **24** (1978), 269–272.
[10] B. D. Rouhani, 'Asymptotic behaviour of quasi-autonomous dissipative systems in Hilbert spaces', *J. Math. Anal. Appl.* **147** (1990), 465–476.
[11] Song Guozhu and Ma Jipu, 'Asymptotic behaviour of solutions to the nonlinear evolution equation', *Sci. China Ser. A* **23** (1993), 679–686.

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